

SUPPLEMENTARY COMBINATORIAL LEMMAS

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Situation as in §8 of Boulder E.S.¹ and notes B.²

- Λ non-singular $\iff \langle \lambda_F^i, \Lambda \rangle \neq 0$ and $\langle \mu_F^i, \Lambda \rangle \neq 0$ for all i and F .
- H non-singular $\iff \lambda_F^i(H) \neq 0$ and $\mu_F^i(H) \neq 0$ for all i and F .

If F is a subset of $\{1, \dots, p\}$, set

$$\xi_F^\Lambda(H) = \begin{cases} 1 & \text{if } \begin{cases} \lambda_F^i(H)(\mu_F^i, \Lambda) < 0 & i \in F \\ \mu_F^i(H)(\lambda_F^i, \Lambda) < 0 & i \notin F \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\alpha_F^\Lambda = 1 + N(F) + N\left\{ \mu_F^i \mid i \in F, (\mu_F^i, \Lambda) < 0 \right\} + N\left\{ \lambda_F^i \mid i \notin F, (\lambda_F^i, \Lambda) < 0 \right\}$$

Lemma. *If Λ and H are non-singular,*

$$\sum_F (-1)^{\alpha_F^\Lambda} \xi_F^\Lambda(H) = 0.$$

(i) The left side is independent of H . We argue as in B. Choose F_1, F_2, H' , and H'' as there. We have to show that

$$(*) \quad (-1)^{\alpha_{F_1}^\Lambda} \xi_{F_1}^\Lambda(H') + (-1)^{\alpha_{F_2}^\Lambda} \xi_{F_2}^\Lambda(H') = (-1)^{\alpha_{F_1}^\Lambda} \xi_{F_1}^\Lambda(H'') + (-1)^{\alpha_{F_2}^\Lambda} \xi_{F_2}^\Lambda(H'')$$

[S.2] Some observations

(a)

$$\begin{aligned} \mu^j &= \mu_{F_1}^j = \mu_{F_2}^j & 1 \leq j < k \\ \lambda^j &= \lambda_{F_2}^j = \lambda_{F_1}^j & k < j \leq p \end{aligned}$$

(b)

$$\begin{aligned} \operatorname{sgn} \lambda_{F_1}^j(H') &= \operatorname{sgn} \lambda_{F_1}^j(H'') & 1 \leq j < k \\ \operatorname{sgn} \lambda_{F_2}^j(H') &= \operatorname{sgn} \lambda_{F_2}^j(H'') & 1 \leq j < k \\ \operatorname{sgn} \mu_{F_1}^j(H') &= \operatorname{sgn} \mu_{F_1}^j(H'') & k < j \leq p \\ \operatorname{sgn} \mu_{F_2}^j(H') &= \operatorname{sgn} \mu_{F_2}^j(H'') & k < j \leq p. \end{aligned}$$

(c) $\lambda_{F_2}^k$ is a positive multiple of $\mu_{F_2}^k$.

¹Langlands, Robert P., *Eisenstein Series*, Algebraic Groups and Discontinuous subgroups, AMS, Proc. of Symp. in Pure Math., vol IX

² “Notes B” refers to “Some lemmas to be applied to the Eisenstein series.”

(d)

$$\begin{aligned}\lambda_{F_2}^j &= \lambda_{F_1}^j + c^j \lambda_{F_1}^k & 1 \leq j < k \\ \mu_{F_1}^j &= \mu_{F_2}^j + e^j \lambda_{F_2}^k = \mu_{F_2}^j + d^j \mu_{F_2}^k & k < j \leq p.\end{aligned}$$

(e) As a consequence of (b) if $1 \leq j < k$

$$\operatorname{sgn} \lambda_{F_1}^j(H') = \operatorname{sgn} \lambda_{F_2}^j(H') \iff \operatorname{sgn} \lambda_{F_1}^j(H'') = \operatorname{sgn} \lambda_{F_2}^j(H'').$$

Because of (d) one of these holds and hence both hold.

Also if $k < j \leq p$

$$\operatorname{sgn} \mu_{F_1}^j(H') = \operatorname{sgn} \mu_{F_2}^j(H') \iff \operatorname{sgn} \mu_{F_1}^j(H'') = \operatorname{sgn} \mu_{F_2}^j(H'')$$

and because of (d) one of these holds. **[S.3]**(f) If $1 \leq j < k$

$$\operatorname{sgn} \lambda_{F_1}^j(H')(\mu_{F_1}^j, \Lambda) = \operatorname{sgn} \lambda_{F_2}^j(H')(\mu_{F_2}^j, \Lambda) = \operatorname{sgn} \lambda_{F_2}^j(H'')(\mu_{F_2}^j, \Lambda) = \operatorname{sgn} \lambda_{F_1}^j(H'')(\mu_{F_1}^j, \Lambda)$$

If $k < j \leq p$

$$\operatorname{sgn} \mu_{F_1}^j(H')(\lambda_{F_1}^j, \Lambda) = \operatorname{sgn} \mu_{F_2}^j(H')(\lambda_{F_2}^j, \Lambda) = \operatorname{sgn} \mu_{F_2}^j(H'')(\lambda_{F_2}^j, \Lambda) = \operatorname{sgn} \mu_{F_1}^j(H'')(\lambda_{F_1}^j, \Lambda)$$

If any of the numbers in (f) are positive then all terms of (*) are zero and we are done. Suppose they are all negative. Interchanging H' and H'' if necessary suppose that $\lambda_{F_1}^k(H')(\mu_{F_1}^k, \Lambda) < 0$. Then $\lambda_{F_1}^k(\mu_{F_1}^k, \Lambda) > 0$. There are two possibilities.

(A) $\operatorname{sgn}(\mu_{F_1}^k, \Lambda) = \operatorname{sgn}(\lambda_{F_2}^k, \Lambda)$. Thus the right side is zero. The left side is

$$(-1)^{\alpha_{F_1}^\Lambda} + (-1)^{\alpha_{F_2}^\Lambda} = 0.$$

(B) $\operatorname{sgn}(\mu_{F_1}^k, \Lambda) = -\operatorname{sgn}(\lambda_{F_2}^k, \Lambda)$. Then the questionable equality is

$$(-1)^{\alpha_{F_1}^\Lambda} = (-1)^{\alpha_{F_2}^\Lambda}.$$

which is valid since $\alpha_{F_1}^\Lambda - \alpha_{F_2}^\Lambda = 0$ or 2 .

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