SUPPLEMENTARY COMBINATORIAL LEMMAS

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Situation as in §8 of Boulder E.S.[1] and notes B.[2]

- Λ non-singular $\iff$ $\langle \lambda'_F, \Lambda\rangle \neq 0$ and $\langle \mu'_F, \Lambda\rangle \neq 0$ for all i and F.
- H non-singular $\iff$ $\lambda^i_F(H) \neq 0$ and $\mu'_F(H) \neq 0$ for all i and F.

If F is a subset of $\{1, \dots, p\}$, set

$$\xi^\Lambda_F(H) = \begin{cases} 1 & \text{if } \begin{cases} \lambda^i_F(H)(\mu^i_F, \Lambda) < 0 & i \in F \\ \mu^i_F(H)(\lambda^i_F, \Lambda) < 0 & i \notin F \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\alpha^\Lambda_F = 1 + N(F) + N\Big\{\, \mu^i_F \;\Big|\; i \in F,\ (\mu^i_F, \Lambda) < 0 \,\Big\} + N\Big\{\, \lambda^i_F \;\Big|\; i \notin F, (\lambda^i_F, \Lambda) < 0 \,\Big\}$$

Lemma. *If Λ and H are non-singular,*

$$\sum_F (-1)^{\alpha^\Lambda_F} \xi^\Lambda_F(H) = 0.$$

(i) The left side is independent of H. We argue as in B. Choose F_1, F_2 H', and H'' as there. We have to show that

$$(*) \qquad (-1)^{\alpha^\Lambda_{F_1}} \xi^\Lambda_{F_1}(H') + (-1)^{\alpha^\Lambda_{F_2}} \xi^\Lambda_{F_2}(H') = (-1)^{\alpha^\Lambda_{F_1}} \xi^\Lambda_{F_1}(H'') + (-1)^{\alpha^\Lambda_{F_2}} \xi^\Lambda_{F_2}(H'')$$

[S.2] Some observations

(a)

$$\begin{aligned} \mu^j &= \mu^j_{F_1} = \mu^j_{F_2} && 1 \leqslant j < k \\ \lambda^j &= \lambda^j_{F_2} = \lambda^j_{F_2} && k < j \leqslant p \end{aligned}$$

(b)

$$\begin{aligned} \operatorname{sgn} \lambda^j_{F_1}(H') &= \operatorname{sgn} \lambda^j_{F_1}(H'') && 1 \leqslant j < k \\ \operatorname{sgn} \lambda^j_{F_2}(H') &= \operatorname{sgn} \lambda^j_{F_2}(H'') && 1 \leqslant j < k \\ \operatorname{sgn} \mu^j_{F_1}(H') &= \operatorname{sgn} \mu^j_{F_1}(H'') && k < j \leqslant p \\ \operatorname{sgn} \mu^j_{F_2}(H') &= \operatorname{sgn} \mu^j_{F_2}(H'') && k < j \leqslant p. \end{aligned}$$

(c) $\lambda^k_{F_2}$ is a positive multiple of $\mu^k_{F_2}$.

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[1] Langlands, Robert P., *Eisenstein Series,* Algebraic Groups and Discontinuous subgroups, AMS, Proc. of Symp. in Pure Math., vol IX

[2] "Notes B" refers to "Some lemmas to be applied to the Eisenstein series."

(d)

$$\lambda^j_{F_2} = \lambda^j_{F_1} + c^j\lambda^k_{F_1} \qquad\qquad 1 \leqslant j < k$$
$$\mu^j_{F_1} = \mu^j_{F_2} + e^i\lambda^k_{F_2} = \mu^j_{F_2} + d^j\mu^k_{F_2} \qquad\qquad k < j \leqslant p.$$

(e) As a consequence of (b) if $1 \leqslant j < k$

$$\operatorname{sgn}\lambda^j_{F_1}(H') = \operatorname{sgn}\lambda^j_{F_2}(H') \iff \operatorname{sgn}\lambda^j_{F_1}(H'') = \operatorname{sgn}\lambda^j_{F_2}(H'').$$

Because of (d) one of these holds and hence both hold.

Also if $k < j \leqslant p$

$$\operatorname{sgn}\mu^j_{F_1}(H') = \operatorname{sgn}\mu^j_{F_2}(H') \iff \operatorname{sgn}\mu^j_{F_1}(H'') = \operatorname{sgn}\mu^j_{F_2}(H'')$$

and because of (d) one of these holds. **[S.3]**

(f) If $1 \leqslant j < k$

$$\operatorname{sgn}\lambda^j_{F_1}(H')(\mu^j_{F_1},\Lambda) = \operatorname{sgn}\lambda^j_{F_2}(H')(\mu^j_{F_2},\Lambda) = \operatorname{sgn}\lambda^j_{F_2}(H'')(\mu^j_{F_2},\Lambda) = \operatorname{sgn}\lambda^j_{F_1}(H'')(\mu^j_{F_1},\Lambda)$$

If $k < j \leqslant p$

$$\operatorname{sgn}\mu^j_{F_1}(H')(\lambda^j_{F_1},\Lambda) = \operatorname{sgn}\mu^j_{F_2}(H')(\lambda^j_{F_2},\Lambda) = \operatorname{sgn}\mu^j_{F_2}(H'')(\lambda^j_{F_2},\Lambda) = \operatorname{sgn}\mu^j_{F_1}(H'')(\lambda^j_{F_1},\Lambda)$$

If any of the numbers in (f) are positive then all terms of $(*)$ are zero and we are done. Suppose they are all negative. Interchanging H' and H'' if necessary suppose that $\lambda^k_{F_1}(H')(\mu^k_{F_1},\Lambda) < 0$. Then $\lambda^k_{F_1}(\mu^k_{F_1},\Lambda) > 0$. There are two possibilities.

(A) $\operatorname{sgn}(\mu^k_{F_1},\Lambda) = \operatorname{sgn}(\lambda^k_{F_2},\Lambda)$. Thus the right side is zero. The left side is

$$(-1)^{\alpha^\Lambda_{F_1}} + (-1)^{\alpha^\Lambda_{F_2}} = 0.$$

(B) $\operatorname{sgn}(\mu^k_{F_1},\Lambda) = -\operatorname{sgn}(\lambda^k_{F_2},\Lambda)$. Then the questionable equality is

$$(-1)^{\alpha^\Lambda_{F_1}} = (-1)^{\alpha^\Lambda_{F_2}}.$$

which is valid since $\alpha^\Lambda_{F_1} - \alpha^\Lambda_{F_2} = 0$ or 2.

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