

# SUPPLEMENTARY COMBINATORIAL LEMMAS

ROBERT P. LANGLANDS

Situation as in §8 of Boulder E.S.<sup>1</sup> and notes B.<sup>2</sup>

- $\Lambda$  non-singular  $\iff \langle \lambda'_F, \Lambda \rangle \neq 0$  and  $\langle \mu'_F, \Lambda \rangle \neq 0$  for all  $i$  and  $F$ .
- $H$  non-singular  $\iff \lambda_F^i(H) \neq 0$  and  $\mu_F^i(H) \neq 0$  for all  $i$  and  $F$ .

If  $F$  is a subset of  $\{1, \dots, p\}$ , set

$$\xi_F^\Lambda(H) = \begin{cases} 1 & \text{if } \begin{cases} \lambda_F^i(H)(\mu_F^i, \Lambda) < 0 & i \in F \\ \mu_F^i(H)(\lambda_F^i, \Lambda) < 0 & i \notin F \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\alpha_F^\Lambda = 1 + N(F) + N\left\{ \mu_F^i \mid i \in F, (\mu_F^i, \Lambda) < 0 \right\} + N\left\{ \lambda_F^i \mid i \notin F, (\lambda_F^i, \Lambda) < 0 \right\}$$

**Lemma.** *If  $\Lambda$  and  $H$  are non-singular,*

$$\sum_F (-1)^{\alpha_F^\Lambda} \xi_F^\Lambda(H) = 0.$$

(i) The left side is independent of  $H$ . We argue as in B. Choose  $F_1, F_2, H'$ , and  $H''$  as there. We have to show that

$$(*) \quad (-1)^{\alpha_{F_1}^\Lambda} \xi_{F_1}^\Lambda(H') + (-1)^{\alpha_{F_2}^\Lambda} \xi_{F_2}^\Lambda(H') = (-1)^{\alpha_{F_1}^\Lambda} \xi_{F_1}^\Lambda(H'') + (-1)^{\alpha_{F_2}^\Lambda} \xi_{F_2}^\Lambda(H'')$$

**[S.2]** Some observations

(a)

$$\begin{aligned} \mu^j &= \mu_{F_1}^j = \mu_{F_2}^j & 1 \leq j < k \\ \lambda^j &= \lambda_{F_2}^j = \lambda_{F_1}^j & k < j \leq p \end{aligned}$$

(b)

$$\begin{aligned} \operatorname{sgn} \lambda_{F_1}^j(H') &= \operatorname{sgn} \lambda_{F_1}^j(H'') & 1 \leq j < k \\ \operatorname{sgn} \lambda_{F_2}^j(H') &= \operatorname{sgn} \lambda_{F_2}^j(H'') & 1 \leq j < k \\ \operatorname{sgn} \mu_{F_1}^j(H') &= \operatorname{sgn} \mu_{F_1}^j(H'') & k < j \leq p \\ \operatorname{sgn} \mu_{F_2}^j(H') &= \operatorname{sgn} \mu_{F_2}^j(H'') & k < j \leq p. \end{aligned}$$

(c)  $\lambda_{F_2}^k$  is a positive multiple of  $\mu_{F_2}^k$ .

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<sup>1</sup>Editorial comment: Langlands, Robert P., *Eisenstein Series*, Algebraic Groups and Discontinuous subgroups, AMS, Proc. of Symp. in Pure Math., vol IX.

<sup>2</sup>Editorial comment: “Notes B” refers to “Some lemmas to be applied to the Eisenstein series.”

(d)

$$\begin{aligned}\lambda_{F_2}^j &= \lambda_{F_1}^j + c^j \lambda_{F_1}^k & 1 \leq j < k \\ \mu_{F_1}^j &= \mu_{F_2}^j + e^j \lambda_{F_2}^k = \mu_{F_2}^j + d^j \mu_{F_2}^k & k < j \leq p.\end{aligned}$$

(e) As a consequence of (b) if  $1 \leq j < k$ 

$$\operatorname{sgn} \lambda_{F_1}^j(H') = \operatorname{sgn} \lambda_{F_2}^j(H') \iff \operatorname{sgn} \lambda_{F_1}^j(H'') = \operatorname{sgn} \lambda_{F_2}^j(H'').$$

Because of (d) one of these holds and hence both hold.

Also if  $k < j \leq p$ 

$$\operatorname{sgn} \mu_{F_1}^j(H') = \operatorname{sgn} \mu_{F_2}^j(H') \iff \operatorname{sgn} \mu_{F_1}^j(H'') = \operatorname{sgn} \mu_{F_2}^j(H'')$$

and because of (d) one of these holds. **[S.3]**(f) If  $1 \leq j < k$ 

$$\operatorname{sgn} \lambda_{F_1}^j(H')(\mu_{F_1}^j, \Lambda) = \operatorname{sgn} \lambda_{F_2}^j(H')(\mu_{F_2}^j, \Lambda) = \operatorname{sgn} \lambda_{F_2}^j(H'')(\mu_{F_2}^j, \Lambda) = \operatorname{sgn} \lambda_{F_1}^j(H'')(\mu_{F_1}^j, \Lambda)$$

If  $k < j \leq p$ 

$$\operatorname{sgn} \mu_{F_1}^j(H')(\lambda_{F_1}^j, \Lambda) = \operatorname{sgn} \mu_{F_2}^j(H')(\lambda_{F_2}^j, \Lambda) = \operatorname{sgn} \mu_{F_2}^j(H'')(\lambda_{F_2}^j, \Lambda) = \operatorname{sgn} \mu_{F_1}^j(H'')(\lambda_{F_1}^j, \Lambda)$$

If any of the numbers in (f) are positive then all terms of (\*) are zero and we are done. Suppose they are all negative. Interchanging  $H'$  and  $H''$  if necessary suppose that  $\lambda_{F_1}^k(H')(\mu_{F_1}^k, \Lambda) < 0$ . Then  $\lambda_{F_1}^k(\mu_{F_1}^k, \Lambda) > 0$ . There are two possibilities.

(A)  $\operatorname{sgn}(\mu_{F_1}^k, \Lambda) = \operatorname{sgn}(\lambda_{F_2}^k, \Lambda)$ . Thus the right side is zero. The left side is

$$(-1)^{\alpha_{F_1}^\Lambda} + (-1)^{\alpha_{F_2}^\Lambda} = 0.$$

(B)  $\operatorname{sgn}(\mu_{F_1}^k, \Lambda) = -\operatorname{sgn}(\lambda_{F_2}^k, \Lambda)$ . Then the questionable equality is

$$(-1)^{\alpha_{F_1}^\Lambda} = (-1)^{\alpha_{F_2}^\Lambda}.$$

which is valid since  $\alpha_{F_1}^\Lambda - \alpha_{F_2}^\Lambda = 0$  or 2.

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