A THEOREM CONCERNING THE DIFFERENTIAL EQUATIONS SATISFIED BY NORMAL FUNCTIONS ASSOCIATED TO ALGEBRAIC CYCLES

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In this paper we shall give a general discussion of normal functions culminating in a result characterizing those normal functions arising from algebraic cycles as being solutions to a family of ordinary differential equations parametrized by the hypersurfaces of large degree passing through the given cycle or through one homologous to it. In Section 1 the theorem will be informally discussed for the case of curves on a surface where a minimum of technical machinery is necessary. Along the way we give proofs of the main classical results in the theory of normal functions and find some new information on the Hodge bundles arising from a Lefschetz pencil of curves. Then in Section 2 we turn to higher dimensions. Following a discussion of the definition and basic properties of normal functions we analyze the Hodge bundles arising from the cotangent spaces to the intermediate Jacobians in a Lefschetz pencil (c.f. (2.13c) and (2.14c)), and then shall reprove the result (c.f. (2.9) for the statement) characterizing the fundamental classes of normal functions by their Hodge type. Finally, after some general observations on Picard-Fuchs equations we formulate and prove our main result Theorem 2.2, and then conclude the paper with some observations concerning the problem of constructing algebraic cycles.

Unless otherwise specified, homology will be with **Z**-coefficients and cohomology with **C**-coefficients. Hopefully the other notations and terminology are standard.

1. Discussion of the Theorem for Curves on a Surface.

a) Let S be a smooth algebraic surface and $L \to S$ a very ample line bundle, which we shall think of as the hyperplane bundle for the embedding $S \hookrightarrow \mathbf{P}^N$ induced by the complete linear system $|L|$. Recall

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that a Lefschetz pencil is a pencil of curves $|C_t|$ selected from $|L|$ such that the generic curve C_t is smooth and the singular curves $C_{t_1}, \ldots, C_{t_N}$ are irreducible with one ordinary double point not in the base locus $B = C_0 \cdot C_\infty$ of the pencil. A general pencil is obtained by selecting a line in the dual projective space $\mathbf{P}^{N*}$ parametrizing the curves in $|L|$, and the pencil is Lefschetz when this line meets the hypersurface $S^* \subset \mathbf{P}^{N*}$ dual to S transversely at smooth points. It is classical how one understands the topology of S in terms of the monodrony on a general $H_1(C_t)$ where t varies over $\mathbf{P}^1 - \{t_1, \ldots, t_N\}$, and we shall review this below as needed. At the moment we shall discuss the Hodge bundle associated to the pencil.

Over the $\mathbf{P}^1$ parametrizing the curves in the pencil there is a vector bundle $E \to \mathbf{P}^1$, the Hodge bundle, whose fibre over $t \in \mathbf{P}^1 - \{t_1, \ldots, t_N\}$ is the vector space $H^0(\Omega_{C_t}{}^1)$ of holomorphic differentials on the smooth curve C_t. Each singular curve C_{t_α} has a normalization $\tilde{C}_{t_\alpha}$, a smooth curve of genus one less than that of a generic C_t and having on it two marked points which when identified give back the singular curve C_{t_α}. The fibre E_{t_α} then consists of the meromorphic differentials on $\tilde{C}_{t_\alpha}$ having at most logarithmic singularities at the marked points.

Alternatively, we may describe the sections of the Hodge bundle by using residues. Given an irreducible curve $C \in |L|$ defined by a section $s \in H^0(\mathcal{O}_S(L))$, there is the Poincaré residue sequence

$$0 \longrightarrow \Omega_S{}^2 \xrightarrow{\ s\ } \Omega_S{}^2(L) \xrightarrow{\ \mathrm{Res}\ } \omega_C \longrightarrow 0. \tag{1.1}$$

When C is smooth, $\omega_C = \Omega_C{}^1$ is the usual sheaf of holomorphic differentials; when C has a double point ω_C exactly corresponds to meromorphic 1-forms on $\tilde{C}$ having at most logarithmic singularities over the double point. For all t, then, the fibre of the Hodge bundle is

$$E_t = H^0(\omega_{C_t}).$$

Denoting by $\eta \in H^1(\Omega_S{}^1)$ the Chern class of $L \to S$ and using $h^1(\Omega_S{}^2(L)) = 0$, the exact cohomology sequence of (1.1) gives a commutative diagram

$$\begin{array}{ccccccccc} 0 \to H^0(\Omega_S{}^2) & \to & H^0(\Omega_S{}^2(L)) & \to & H^0(\omega_C) & \to & H^1(\Omega_S{}^2) & \to & 0. \\ & & & & \big\uparrow & \nearrow_{\eta} & & & \\ & & & & H^0(\Omega_S{}^1) & & & & \end{array} \tag{1.2}$$

By the Hard Lefschetz theorem, η induces an isomorphism, and the image of

$$H^0(\Omega_S{}^1) \to H^0(\omega_{C_t})$$

may be thought of as the fixed part of the variation of Hodge structure defined by the curves C_t; this image therefore defines a trivial summand of the Hodge bundle $E \to \mathbf{P}^1$.

To analyze the variable part of E we choose sections s_0 and $s_\infty \in H^0(\mathcal{O}_S(L))$ defining C_0 and C_∞ (assumed smooth) and set $s_t = s_0 + ts_\infty$. For $\omega \in H^0(\Omega_S{}^2(L))$ and $t' = 1/t$, since

$$\operatorname{Res}\left(\frac{\omega}{s_0 + ts_\infty}\right) = t' \operatorname{Res}\left(\frac{\omega}{t's_0 + s_\infty}\right)$$

we deduce that the exact sequence in (1.2) globalizes over $\mathbf{P}^1$ to give

$$0 \to H^0(\Omega_S{}^2) \to H^0(\Omega_S{}^2(L)) \otimes \mathcal{O}(1) \to E \to H^1(\Omega_S{}^2) \to 0. \tag{1.3}$$

Here the vector spaces $H^q(\Omega_S{}^2)$ and $H^0(\Omega_S{}^2(L))$ are thought of as trivial bundles over $\mathbf{P}^1$, $\mathcal{O}(1)$ has the usual meaning, and we are retaining E to denote the corresponding sheaf $\mathcal{O}_{\mathbf{P}^1}(E)$. With the usual characters

$$\begin{cases} g = h^{1,0}(C_t) \\ q = h^{1,0}(S) \\ p_g = h^{2,0}(S), \end{cases}$$

we infer from (1.3) that the Grothendieck decomposition of E is given by given by

$$E \cong \mathcal{O}^{(q)} \oplus \left(\bigoplus_{i=1}^{g-q} \mathcal{O}(k_i + 1)\right), \qquad k_i \geq 0, \tag{1.4}$$

so that the variable part $E_v = \oplus\ \mathcal{O}(k_i + 1)$ of the Hodge bundle is positive. By tensoring (1.3) with $\mathcal{O}(-2)$ and taking cohomology, using $H^1(\mathbf{P}^1, \mathcal{O}(-2)) \cong \mathbf{C}$ we deduce the isomorphism

$$H^0(\mathbf{P}^1, E \otimes \mathcal{O}(-2)) \cong H^0(\Omega_S{}^2), \tag{1.5}$$

a fact which may be seen directly as follows: Each holomorphic 2-form ω on S may be uniquely written

$$\omega = \varphi(t) \wedge dt,$$

where easy local considerations show that $\varphi(t) \in H^0(\omega_{C_t})$. Since dt has a pole of order 2 at $t = \infty$ we infer that $\varphi(t) \in H^0(\mathbf{P}^1, E \otimes \mathcal{O}(-2))$ vanishes to 2nd order there.

We remark that the exact nature of the numbers k_i in (1.4) has to do with the linear series $|K - \mu L|$ for $\mu = 0, 1, \ldots$. For example, from

$$p_g = \Sigma\, k_i$$

we see that the canonical series is empty exactly when all $k_i = 0$. Similarly, if $|K - L|$ is empty then the k_i are all either zero or one, and so (1.4) becomes

$$E \cong \mathcal{O}^{(q)} \oplus \mathcal{O}(1)^{(g-q-p_g)} \oplus \mathcal{O}(2)^{p_g},$$

a situation which may always be achieved by choosing L sufficiently ample.

b) To discuss how the topology of S is described by the Lefschetz pencil it is useful to take two points of view. One is intuitive as in the Borel tract [6] or in the beautiful exposition [9], and the other is by applying Morse theory and the Leray spectral sequence as in Andreotti-Frankel [1].

In either case some local analysis around the critical values is necessary, and for this the essential points are the following: In the complex t-plane mark a non-critical reference point t_0 and draw non-intersecting paths γ_α from t_0 to t_α. Then there is for each α a vanishing cycle $\delta_\alpha \in H_1(C_{t_0})$ such that as t moves along γ_α the displaced cycle $\delta_\alpha(t)$ on C_t shrinks to the double point as $t \to t_\alpha$. The locus of $\delta_\alpha(t)$ along this path then gives a cone $\Delta_\alpha \in H_2(S, C_{t_0})$ with $\partial\Delta_\alpha = \delta_\alpha$. If $\gamma \in H_1(C_{t_0})$ is any other cycle and we displace γ along a closed loop turning once around t_α, the change in γ is measured by the Picard-Lefschetz formula

$$\gamma \to \gamma + (\gamma, \delta_\alpha)\, \delta_\alpha. \tag{1.6}$$

To give what intuitive remarks about topology we shall require we recall that the primitive cohomology $H_{\text{prim}}^2(S, \mathbf{Q})$ is represented via Poincaré duality (and ignoring torsion) by 2-cycles $\Gamma \in H_2(S)$ such that the intersection cycle $\Gamma \cdot C_{t_0} = 0$ in $H_0(C_{t_0})$. For t in the punctured t-plane $\mathbf{P}^1 - \{t_1, \ldots, t_N\}$ we locally write

$$\Gamma \cdot C_t = \partial \gamma_t$$

where $\gamma_t \in H_1(C_t, C_t \cap \Gamma_t)$ is a path on C_t which maps to $\Gamma \cdot C_t$ in $H_0(C_t \cap \Gamma_t)$. An obvious extension of (1.6) to the relative case gives that as t turns once around t_α,

$$\gamma_t \to \gamma_t + \lambda_\alpha \delta_\alpha$$

where $\lambda_\alpha = (\gamma_t, \delta_\alpha)$ is an integer. Since γ_t is invariant when t transverses a sufficiently large circle (we assume that $t = \infty$ is not a critical value)

$$\sum_\alpha \lambda_\alpha \delta_\alpha = 0 \quad \text{in} \quad H_1(C_{t_0}). \tag{1.7}$$

By virtue of this relation, the relative 2-cycle

$$\sum_\alpha \lambda_\alpha \Delta_\alpha \in H_2(S, C_{t_0}) \tag{1.8}$$

is in the image of the mapping

$$H_2(S) \to H_2(S, C_{t_0}),$$

and an easy argument shows that Γ is represented by the cycle (1.8). Summarizing, the primitive homology is given by cycles (1.8) satisfying (1.7) where the coefficients λ_α are determined by considering the variation of a chain γ_t with $\partial\gamma_t = \Gamma \cdot C_t$.

We now describe the same situation in the Andreotti-Frankel terminology. By blowing up S at the base locus $B = C_{t_0} \cdot C_\infty$ of the pencil we obtain the smooth surface $X = \bigcup_{t \in \mathbf{P}^1} C_t$, which then admits an obvious fibering $f: X \to \mathbf{P}^1$. The Leray spectral sequence for the constant sheaf $\mathbf{Z}$ degenerates at E_2 (Andreotti-Frankel, loc. cit.), and the primitive part of $H^2(S, \mathbf{Q}) \subset H^2(X, \mathbf{Q})$ is easily seen to lie in the term $E_2^{1,1} = H^1(\mathbf{P}^1, R_f^1 \mathbf{Z})$.

Now then to describe $H^1(\mathbf{P}^1, R_f^1 \mathbf{Z})$ we choose the covering $\{U_0, U_1,$

$\ldots, U_N\}$ of $\mathbf{P}^1$ where $U_0 = \mathbf{P}^1 - \{t_1, \ldots, t_N\}$ and U_α is an ϵ-disc around t_α. For this covering the 1-cocycles are just

$$\prod_\alpha H^0(U_\alpha{}^*, R_f{}^1\, \mathbf{Z}),$$

which we may think of as being given by an invariant 1-cycle in each punctured disc $U_\alpha{}^*$. For a primitive class given by (1.8) the corresponding Čech class is $\Pi_\alpha \{\lambda_\alpha \cdot \delta_\alpha\}$. Similarly, a class in $H_{\text{prim}}{}^0(B)$ is represented by a zero-cycle D of degree zero supported on the base locus. On each curve C_t we may write $D = \partial\gamma_t$ for some 1-chain γ_t whose variation around $t = t_\alpha$ leads again to a Čech cocycle in $\Pi_\alpha H^0(U_\alpha{}^*, R_f{}^1\, \mathbf{Z})$. These two subspaces of $H^0(\mathbf{P}^1, R_f{}^1\, \mathbf{Z}) \subset H^2(X, \mathbf{Z})$ are orthogonal under the cup product, and again modulo torsion there is a decomposition

$$H^1(\mathbf{P}^1, R_f{}^1\, \mathbf{Z}) \cong H_{\text{prim}}{}^0(B, \mathbf{Q}) \oplus H_{\text{prim}}{}^2(S, \mathbf{Q}). \tag{1.9}$$

c) Having described the Hodge bundle and homology in our pencil $|C_t|$ we now turn to the family of Jacobians. In general, if $C \in |L|$ has only ordinary double points the (generalized) Jacobian variety $J(C)$ is defined to be

$$H^0(\omega_C)^*/H_1(C^*, \mathbf{Z})$$

where $C^* = C$-{double points}, and where $H_1(C^*, \mathbf{Z})$ is embedded as a discrete subgroup of $H^0(\omega_C)^*$ by integration. By elementary duality theory (or directly)

$$J(C) \cong H^1(\mathcal{O}_C)/H^1(C, \mathbf{Z}).$$

Returning to our pencil, it is easily verified that

$$J = \bigcup_{t \in \mathbf{P}^1} J(C_t)$$

forms naturally a complex manifold (c.f. Jambois [5]) such that $J \to \mathbf{P}^1$ is an analytic fibre space of complex Lie groups. The cross sections of this fibre space are by definition the group of *normal functions.*

The origin of the word "normal functions" may be explained this way. Given $\omega \in H^0(\Omega_C{}^2(L))$, as explained above in Section 1a) we obtain the holomorphic section

$$\omega(t) = \operatorname{Res}\left(\frac{\omega}{s_0 + ts_\infty}\right) \in H^0(\omega_{C_t}) = E_t$$

of the Hodge bundle. Since $J(C_t) = E_t^*/(\text{periods})$, we may view a normal function ν as a multi-valued holomorphic linear function on E_t for $t \in \mathbf{P}^1 - \{t_1, \ldots, t_N\}$ which has a suitable growth property (noted below) as $t \to t_\alpha$. Suppose we let

$$\nu(t) = \langle \nu, \omega(t) \rangle$$

denote this function. Analytic continuation of $\nu(t)$ around $t = t_\alpha$ adds on a period

$$\pi_\alpha(t) = \lambda_\alpha \int_{\delta_\alpha} \omega(t) \tag{1.10}$$

where $\delta_\alpha \in H_1(C_t, \mathbf{Z})$ is the vanishing cycle. In this way the sections of $J \to \mathbf{P}^1$ may be represented by suitable multi-valued analytic functions whose local monodromy has the form (1.10).

Since analytic continuation around a sufficiently large circle leaves $\nu(t)$ invariant, and since ω was arbitrary, we infer that

$$\sum_\alpha \lambda_\alpha \delta_\alpha = 0 \quad \text{in} \quad H_1(C_{t_0}, \mathbf{Z}).$$

Thus, by the discussion centered around (1.7) and (1.8), to each normal function ν we may assign the primitive cycle $\sum_\alpha \lambda_\alpha \Delta_\alpha \in H_{\text{prim}}{}^2(S, \mathbf{Q})$; this is called the *fundamental class* η_ν of the normal function.

Normal functions and their fundamental classes may also be described using the fibering $f: X \to \mathbf{P}^1$ of the blown up surface $X = \bigcup_{t \in \mathbf{P}^1} C_t$. The basic observations are that

$$R_f{}^1\mathcal{O} \cong E^* \tag{1.11}$$

is the sheaf of Lie algebras associated to the analytic fibre space $J \to \mathbf{P}^1$, and that the sheaf of discrete subgroups is

$$R_f{}^1\mathbf{Z} \subset R_f{}^1\mathcal{O}.$$

In other words, denoting still by J the sheaf of holomorphic sections of the family of Jacobians, we have the exact sheaf sequence

$$0 \to R_f{}^1\mathbf{Z} \to R_f{}^1\mathcal{O} \to J \to 0. \tag{1.12}$$

To compute the cohomology sequence we refer to (1.4) to conclude that

$$H^0(\mathbf{P}^1, R_f{}^1\mathcal{O}) \cong H^1(S, \mathcal{O}).$$

It follows that, with the non-standard notation $J(S) = H^1(S, \mathcal{O})/H^1(S, \mathbf{Z})$ for the Picard variety of S, the fixed part of the family $\{J(C_t)\}$ is $J(S)$ and the exact cohomology sequence of (1.12) leads to a commutative diagram

$$\begin{array}{ccccccccc} 0 \to J(S) \to H^0(\mathbf{P}^1, J) & \xrightarrow{\delta} & H^1(\mathbf{P}^1, R_f{}^1\mathbf{Z}) & \to & H^1(\mathbf{P}^1, R_f{}^1\mathcal{O}) & \to \cdots \\ & & \Big\downarrow \alpha & & \Big\| \beta & \\ & & H_{\text{prim}}{}^2(S, \mathbf{Q}) & \to & H^2(\mathcal{O}_S). & \end{array} \tag{1.13}$$

Here the vertical map α is given by the Leray spectral sequence according to (1.9) and the composition $\alpha\delta(\nu) = \eta_\nu$ gives the fundamental class of ν, as may be verified using the Čech covering described above. As for the vertical map β, from (1.5) and (1.11)

$$\begin{aligned} H^1(\mathbf{P}^1, R_f{}^1\mathcal{O}) &\cong H^1(\mathbf{P}^1, E^*) \\ &\cong H^0(\mathbf{P}^1, E \otimes \mathcal{O}(-2))^* \\ &\cong H^0(\Omega_S{}^2)^* \\ &\cong H^2(\mathcal{O}_S), \end{aligned}$$

where we have used Serre duality twice. This allows us to derive the basic result, due to Lefschetz, characterizing the fundamental classes of normal functions:

A class $\eta \in H_{\text{prim}}{}^2(S, \mathbf{Z})$ is the fundamental class of a normal function if, and only if, η is of Hodge type (1, 1). (1.14)

To prove this result we need only show that the mapping $H_{\text{prim}}{}^2(S, \mathbf{Q}) \to H^{0,2}(S)$ in (1.13) is that given by the Hodge decomposition, and then the theorem follows by exactness of the cohomology sequence. This verification is "just" a commutativity check.

Actually, it is quite instructive to give the classical global argument, for which the basic formula is the following: If ν is a normal function with fundamental class $\Sigma_\alpha \lambda_\alpha \Delta_\alpha$, and if $\omega(t) = \text{Res}\,(\omega/s_0 - ts_\infty)$ is a holomorphic section of the Hodge bundle, then we claim that

$$\nu(t) = \sum_{\alpha=1}^{N} \frac{\lambda_\alpha}{2\pi\sqrt{-1}} \int_{t_0}^{t_\alpha} \frac{\pi_\alpha(s)\, ds}{t - s} \tag{1.15}$$

where $\pi_\alpha(s)$ is the period (1.10) of $\omega(s)$ over the vanishing cycle δ_α.

Proof. On the complex t-sphere minus the cuts γ_α from t_0 to t_α we may view the normal function $\nu(t)$ as being a single-valued holomorphic function having the following properties:

i) near the critical point t_α the difference

$$\nu(t) - \frac{\lambda_\alpha}{2\pi\sqrt{-1}} \pi_\alpha(t) \log\,(t - t_\alpha)$$

is holomorphic and single-valued (thus $\nu(t)$ has the jump $\lambda_\alpha \pi_\alpha(t)$ across the cut γ_α), and

ii) $\nu(t)$ vanishes to the same order $k + 1$ ($k \geq 0$) at $t = \infty$ as the section $\omega(t)$ of $E \to \mathbf{P}^1$.

On the other hand, suppose we consider the potential-theoretic integral on the right hand side of (1.15). This integral has the jump $\lambda_\alpha \pi_\alpha(t)$ across γ_α, and the difference

$$\frac{\lambda_\alpha}{2\pi\sqrt{-1}} \int_{t_0}^{t_\alpha} \frac{\pi_\alpha(s)\, ds}{s - t} - \frac{\lambda_\alpha}{2\pi\sqrt{-1}} \pi_\alpha(t) \log\,(t - t_\alpha)$$

is single-valued and holomorphic at $t = t_\alpha$. Moreover, using (1.7) which implies that $\Sigma_\alpha\, \lambda_\alpha \pi_\alpha(t) \equiv 0$, the integral is single-valued outside a large circle. To examine its behavior near infinity we set $t' = 1/t$ and write it as

$$t'\left(\sum_{\alpha=1}^{N} \frac{\lambda_\alpha}{2\pi\sqrt{-1}} \int_{t_0}^{t_\alpha} \frac{\pi_\alpha(s)\, ds}{t's - 1}\right). \tag{1.16}$$

It follows that the difference of $\nu(t)$ and the integral is single-valued and holomorphic on the t-sphere and vanishes at $t = \infty$, which then establishes (1.15).

Now (1.14) is an immediate consequence. Using (1.16) we have near $t = \infty$

$$\nu(t) = -\sum_{l=0}^{\infty} \left(\sum_{\alpha=1}^{N} \frac{\lambda_\alpha}{2\pi\sqrt{-1}} \int_{t_0}^{t_\alpha} s^l \pi_\alpha(s)\, ds\right) t'^{l+1}. \tag{1.17}$$

If $\omega(t)$ vanishes to order $k + 1$ at infinity the same must be true of $\nu(t)$, and therefore if $k \geq 1$

$$0 = \sum_{\alpha=1}^{N} \frac{\lambda_\alpha}{2\pi\sqrt{-1}} \int_{t_0}^{t_\alpha} \pi_\alpha(s)\, ds = \int_\Gamma \varphi \tag{1.18}$$

where $\Gamma = \Sigma_a \lambda_\alpha \Delta_\alpha$ is the fundamental class of ν and $\varphi = \omega(t) \wedge dt \in H^0(\Omega_S{}^2)$ (c.f. (1.15)). This exactly says that Γ has Hodge type (1, 1).

Conversely, if $\Gamma = \Sigma_\alpha \lambda_\alpha \Delta_\alpha$ is any primitive cycle on S so that (1.7) is satisfied, then we may *define* a section of $J \to \mathbf{P}^1$ outside of $t = \infty$ by the formula (1.15). The conditions that this section extend across $t = \infty$ are exactly that Γ be orthogonal to $H^0(\Omega_S{}^2)$, and this establishes (1.14).

d) We now take up the interplay between normal functions and the algebraic curves on S. If Z is a primitive algebraic 1-cycle (=divisor in this case) on S, then each intersection

$$Z_t = Z \cdot C_t = \sum_i p_i(t) - q_i(t)$$

is a divisor of degree zero on the curve C_t, and so represents a point $\nu_Z(t) \in J(C_t)$. Thinking of normal functions as multivalued sections of the Hodge bundle, the definition of $\nu_Z(t)$ is given by the usual abelian sum

$$\nu_Z(t) = \sum_i \int_{q_i(t)}^{p_i(t)} \omega(t) \tag{1.19}$$

where $\omega(t)$ is any local holomorphic section of $E \to \mathbf{P}^1$. This formula makes sense provided that Z does not pass through any of the double points of the singular C_{t_α}, a circumstance which may always be achieved by moving Z in its linear equivalence class and noting that by Abel's theorem the abelian sum (1.19) remains invariant.

We denote by ν_Z the normal function associated to the cycle Z. The basic topological property of the map {divisors} → {normal functions} is

The fundamental class of ν_Z is equal to the fundamental class of Z. (1.20)

Indeed, write $Z_t = \partial\gamma_t$ for a 1-chain γ_t so that (1.19) becomes

$$\nu_Z(t) = \int_{\gamma_t} \omega(t),$$

and then as t turns around t_α the analytic continuation

$$\nu_Z(t) \to \nu_Z(t) + \lambda_\alpha \int_{\delta_\alpha} \omega(t)$$

is exactly reflected by the transformation

$$\gamma_t \to \gamma_t + \lambda_\alpha \, \delta_\alpha .$$

Consequently, both fundamental classes are represented by the cycle $\sum_\alpha \lambda_\alpha \, \Delta_\alpha$.

The main existence theorem, which is due to Poincaré, states:

Every normal function is of the form ν_Z for some algebraic cycle. (1.21)

When combined with (1.14) and (1.20) we obtain Lefschetz's original proof of his famous (1, 1) theorem.

The existence theorem (1.21) amounts to the Jacobi inversion theorem with dependence on parameters. Namely, select one of the base points p of the pencil $|C_t|$ as a base point (different use of the word) on each Riemann surface C_t. If for generic t the point $\nu(t) \in J(C_t)$ is not in the translate of the theta divisor by $-gp$, then as ω varies over $H^0(\mathbf{P}^1, E)$ the equations

$$\nu(t) = \sum_i \int_p^{p_i(t)} \omega(t) \tag{1.22}$$

will have a unique solution

$$D_t = \sum_{i=1}^{g} p_i(t).$$

This divisor obviously varies holomorphically with t, and at points where either $t = t_\alpha$ is a critical value or else $\nu(t) \in J(C_t)$ becomes a special divisor easy arguments show that D_t remains uniquely determined. In this way we have traced out an algebraic curve $D = \cup D_t$ on S, which provides the essential step in the proof of (1.21) in this case.

In the situation where the equations (1.22) do not determine a unique D_t for a generic t, we will have dim $|D_t| = r > 0$. Then by imposing r independent base point conditions at p we generically determine a unique D_t, and the argument proceeds as before.

In concluding this discussion we note that if we vary the normal function by the image of the fixed part

$$J(S) \to H^0(\mathbf{P}^1, J)$$

in (1.13), we may construct a family of linearly *in*equivalent curves parametrized by the Picard variety of S. This was the first complete proof of this existence theorem and provided the original motivation for the introduction of normal functions.

e) Now, as indicated in the introduction and will be explained more fully below, the formal aspects of the above discussion carry over to higher dimensions with intermediate Jacobians replacing Jacobians. In particular, the analogue of Lefschetz's characterization (1.14) of the fundamental classes goes through (c.f. Section 2b)), but the fundamental existence theorem (1.21) does not generalize due to the failure of the inversion theorem for intermediate Jacobians. This suggests that we re-examine the normal functions for curves on a surface having in mind the following general philosophy: In higher dimensions a particular Hodge structure will in general not arise from a geometric situation, but a non-trivial global variation of Hodge structure should come from algebraic geometry. Now we are not able to reprove the existence theorem without using Jacobi inversion, but we are able to show by suitable differentiation that, in a certain sense, knowing the normal function ν_Z allows us to determine the equations of Z.

Our discussion of this will be facilitated by introducing some classical notation. We imagine S as being generically projected into $\mathbf{P}^3$ having there an affine equation

$$f(x, y, z) = 0$$

where f is a polynomial of degree d, and where the Lefschetz pencil is given by the plane sections $y = t$. If D is the double curve of S, then it is well known that by residues

$$\Omega_S{}^2 \cong \mathcal{O}_S(d-4)(-D)$$

and consequently $\Omega_S{}^2(L) \cong \mathcal{O}_S(d-3)(-D)$. Sections in $H^0(\Omega_S{}^2(L))$ are then given by polynomials $p(x, y, z)$ of degree $d - 3$ passing through the double curve (adjoint conditions), and the corresponding section of the Hodge bundle is

$$\omega(t) = \operatorname{Res}\left(\operatorname{Res}\left(\frac{p(x, y, z)\, dx \wedge dy \wedge dy}{(y-t) f(x, y, z)}\right)\right)$$

$$= \frac{p(x, t, z)\, dx}{\dfrac{\partial f}{\partial z}(x, t, z)}. \tag{1.23}$$

Recall that a normal function is given by a multi-valued holomorphic function $\nu(t)$ on $\mathbf{P}^1 - \{t_1, \ldots, t_N\}$ which has certain prescribed behavior at the critical values $t = t_\alpha$ and at $t = \infty$, and whose indeterminacy is given by period

$$\pi(t) = \int_\delta \omega(t), \qquad \delta \in H_1(C_t, \mathbf{Z}). \tag{1.24}$$

Now it is well-known that these periods satisfy a linear differential equation (Picard-Fuchs equation)

$$P(t, d/dt)\pi(t) = \sum_{k=0}^{l} r_k(t) \frac{d^k \pi(t)}{(dt)^k} = 0 \tag{1.25}$$

with rational functions $r_k(t)$ as coefficients, and it is suggested that we apply the Picard-Fuchs operator $P(t, d/dt)$ to the normal function $\nu(t)$ to obtain a single-valued *rational* function $r(t)$ which should then reflect properties of ν.

More precisely, we denote by D the Gauss-Manin connection for the cohomology bundle $\mathbf{H} = \cup_{t \neq t_\alpha} H_{DR}{}^1(C_t)$ over $\mathbf{P}^1 - \{t_1, \ldots, t_N\}$. Its characteristic property is that we can differentiate under the integral

$$\frac{d}{dt}\int_\delta \omega(t) = \int_\delta \frac{D\omega(t)}{dt}. \tag{1.26}$$

In fact in our case we may take

$$\frac{D\omega(t)}{dt} = \frac{\partial \omega(t)}{\partial t}$$

$$= \frac{\partial}{\partial t}\left(\frac{p(x, t, z)\, dx}{\frac{\partial f}{\partial z}(x, t, z)}\right)$$

to be the result of formally differentiating $\omega(t)$. What is important here is that $D\omega(t)/dt$ is a rational 1-form of the 2nd kind on C_t whose cohomology class in $H_{DR}{}^1(C_t)$ is well defined. If the genus of C_t is g then the classes

$$\omega(t), \frac{D\omega(t)}{dt}, \ldots, \frac{D^{2g}\omega(t)}{(dt)^{2g}}$$

will be dependent in $H_{DR}{}^1(C_t)$, and so if they generically span a subspace of dimension l there will be a minimal linear relation on C_t

$$\sum_{k=0}^{l} r_k(t)\frac{D^k\omega(t)}{(dt)^k} = d\varphi(t), \tag{1.27}$$

where φ is the restriction of a rational function on S and the $r_k(t)$ are rational functions of t. It is clear that by using (1.26) the relation (1.27) implies the differential equation (1.25).

In order to simplify the following discussion we will assume that the irregularity $q(S) = 0$ (the general case will be taken up in Section 2 below). Then there is no fixed part to the family of Jacobians $J(C_t)$, and according to Lefschetz [6] the global monodromy group acting on $H^1(C_{t_0})$ is irreducible (in general, it is irreducible on the variable part of the cohomology). It follows that the order l of the equation (1.25) is $2g$, since the classes $\{\omega, D\omega(t)/dt, D^2\omega(t)/dt^2, \cdots\}$ span an invariant subspace of $H_{DR}{}^1(C_t)$ which must then be all of this group. The general solution to (1.25) is then of the form

$$\sum_{j=1}^{2g} c_j \int_{\delta_j} \omega(t), \qquad c_j \in \mathbf{C}, \tag{1.28}$$

when $\delta_1, \ldots, \delta_{2g}$ give a basis for $H_1(C_t)$.

Now to each normal function $\nu(t)$ we associate the rational function $P(t, d/dt)\nu(t)$. A basic result is Manin's theorem [7]: *The kernel of the mapping*

$$\nu(t) \to P(t, d/dt)\nu(t) \tag{1.29}$$

consists exactly of the torsion elements in the group of normal functions.

Proof. If the right-hand side of (1.29) is zero then $\nu(t)$ has the local form (1.28). By analytic continuation around the critical value $t = t_\alpha$ we infer that

$$\lambda_\alpha \int_{\delta_\alpha} \omega(t) = \sum_j c_j(\delta_j, \delta_\alpha) \int_{\delta_j} \omega(t), \tag{1.30}$$

where the λ_α are the coefficients in the fundamental class $\Sigma\, \lambda_\alpha\, \Delta_\alpha$ of ν.

By differentiation we deduce that (1.30) holds for any class in $H^1(C_t)$; i.e.

$$\sum_j c_j(\delta_j, \delta_\alpha) = \lambda_\alpha \in \mathbf{Z}$$

for all α. Since the global monodromy group is irreducible, the $2g \times N$ matrix $(\delta_j, \delta_\alpha)$ has maximal rank $2g$. It follows that the δ_j are *rational* numbers, and hence that some multiple $m\nu = 0$. Q.E.D.

f) By virtue of (1.29) we may ask if additional information may be obtained by further differentiation, and our main result (Theorem 1.36) will be that this is indeed the case.

As a preliminary to stating this theorem we need to observe that for the pair consisting of a smooth manifold N and submanifold S the relative de Rham cohomology $H^*(N, S)$ may be computed from the complex of differential forms φ on N such that $\varphi \mid S \equiv 0$. In case N is a smooth algebraic variety and S an algebraic subvariety we may similarly define the algebraic de Rham hypercohomology and extend Grothendieck's algebraic de Rham theorem to this situation.

Returning to our consideration of curves on a surface, for an arbitrary integer $k > 0$ we let $E \in |kL|$ and assume for the moment that E does not have multiple components, so that a $E_t = E \cdot C_t$ is smooth for t in a Zariski open set $U \subset \mathbf{P}^1$. Given a section $\omega(t) \in H^0(\omega_{C_t})$ of the Hodge bundle as above, we may consider the relative class $\omega(t) \in H_{DR}{}^1(C_t, E_t)$. Denoting by D_E the Gauss-Manin connection for the relative cohomology bundle $\bigcup_{t \in U} H_{DR}{}^1(C_t, E_t)$, exactly as in the preceding discussion there will be a Picard-Fuchs operator

$$P(t, d/dt, E) = \sum_{k=0}^{l} r_k(t, E) \frac{d^k}{(dt)^k} \tag{1.31}$$

corresponding to a minimal relation

$$\sum_{k=0}^{l} r_k(t, E) \frac{D_E{}^k \omega(t)}{(dt)^k} = 0$$

in $H_{DR}{}^1(C_t, E_t)$. It is also the case that the coefficients $r_k(t, E)$ are rational functions of $t \in \mathbf{P}^1$ and of $E \in |kL|$, at least provided we normalize by requiring $r_l(t, E) = 1$.

We will now show: *For generic E and $\omega(t)$, the equation* (1.31) *has the maximal possible rank*

$$l = 2g + kd - 1 \tag{1.32}$$

where $d = c_1{}^2(L)$ is the degree of L.

Proof. We consider the exact homology sequence

$$0 \to H_1(C_t) \to H_1(C_t, E_t) \to H_0(E_t) \to \mathbf{Z} \to 0.$$

The kernel of $H_0(E_t) \to \mathbf{Z}$ corresponds to zero cycles of degree zero supported on E_t, and so the global monodromy group operating here is irreducible provided that the curve E is irreducible (it is essentially the Galois group of the branched covering $E \to \mathbf{P}^1$). Consequently, the global monodromy group acting on $H_1(C_t, E_t)$ preserves the subspace $H_1(C_t)$ and acts irreducibly on this subspace and on the quotient space. If the span of $\{\omega(t), D_E\omega(t)/dt, \cdots\}$ fails to be all of $H_{DR}{}^1(C_t, E_t)$, then writing

$$E_t = \sum_{i=1}^{kd} p_i(t)$$

and recalling that $p \in C_t$ denotes a basepoint, we will have in a small t-disc

$$0 = \sum_i c_i \int_p^{p_i(t)} \omega(t) + \int_\gamma \omega(t) \tag{1.33}$$

where γ is a constant linear combination of cycles in $H_1(C_t)$. By analytic continuation of this equation around a branch point of $E \to \mathbf{P}^1$ where, e.g., p_1 and p_2 come together and interchange, we deduce that

$$(c_1 - c_2) \int_{p_1(t)}^{p_2(t)} \omega(t) = 0.$$

Since $\omega(t)$ is assumed generic this implies that $c_1 = c_2$. Continuing in this way we infer from the irreducibility of E that all c_i are equal, and so (1.33) becomes

$$0 = c\left(\sum_i \int_q^{p_i(t)} \omega(t)\right) + \int_\gamma \omega(t).$$

By analytic continuation of this equation around the critical points $t = t_\alpha$ we deduce that $(\gamma, \delta_\alpha) = 0$ for all vanishing cycles δ_α, and hence $\gamma = 0$. Q.E.D.

We remark that, just as for ordinary polynomials, the operator $P(t, d/dt, E)$ may become reducible for special $E \in |kL|$. What is important for our purposes is the observation that if under a specialization $E \to E_0$ the coefficients $r_k(t, E)$ specialize to rational functions $r_k(t, E_0)$ which are not identically infinite, then the solutions to the O.D.E. $P(t, d/dt, E_0)\pi(t) = 0$ are obtained by specializing solutions of $P(t, d/dt, E)\pi(t) = 0$, and hence are constant linear combinations of periods of $\omega(t)$ over cycles in $H_1(C_t, E_{0_t})$.

Now suppose that Z is a primitive algebraic cycle on S with corresponding normal function ν_Z. If we want, by a linear equivalence we may think of Z as the difference of two smooth curves. Then for any $E_0 \in |kL|$ passing through Z and for which $P(t, d/dt, E_0)$ exists as a specialization—which will be the case for generic $E_0 \in |kL|$ containing Z provided that k is sufficiently large—the normal function ν_Z satisfies the relative Picard-Fuchs equation

$$P(t, d/dt, E_0)\nu_Z(t) = 0. \tag{1.34}$$

Conversely, suppose that a given normal function $\nu(t)$ satisfies (1.34) for some E_0. Then by our remark about the solutions to (1.34)

$$\nu(t) = \int_\gamma \omega(t),$$

where γ is a constant linear combination of cycles in $H_1(C_t, E_{0_t})$. By considering the monodromy around the critical values $t = t_\alpha$ we deduce that

$$(\gamma, \delta_\alpha) = \lambda_\alpha \in \mathbf{Z}$$

for all α, and then $\sum_\alpha \lambda_\alpha \Delta_\alpha \in H_2(S, C_{t_0})$ gives the fundamental class

η_ν of the normal function, which is consequently in the image of

$$H_2(E_0, \mathbf{Q}) \to H_2(S, \mathbf{Q}) \tag{1.35}$$

since the locus of the 1-chain $\gamma(t)$ in the slit plane will give a 3-chain Γ with $\partial\Gamma \equiv \Sigma_\alpha \lambda_\alpha \Delta_\alpha$ modulo E_0.

Summarizing, we have established the following:

THEOREM. *For a given normal function ν and each divisor $E \in |kL|$ we consider the equation*

$$P(t, d/dt, E)\nu(t) = 0. \tag{1.36}$$

Thinking of E as variable, those divisors for which $P(t, d/dt, E)$ is defined and (1.36) *is satisfied are exactly the divisors for which the fundamental class η_ν is in the image of the mapping* (1.35).

2. Discussion of the Theorem in Higher Dimensions.

a) We want to extend the result just given to higher dimensional varieties. For reasons stemming from the Lefschetz hyperplane theorem and Hard Lefschetz Theorem the crucial case is the middle dimension[1]—i.e., we consider primitive algebraic n-cycles Z on a smooth projective variety M of complex dimension $2n$. As in the case $n = 1$ of curves on a surface we assume given a very ample line bundle $L \to M$ and Lefschetz pencil $|V_t|$ selected from the complete linear system $|L|$. The critical values $t = t_1, \ldots, t_N$ correspond to V_{t_α} having acquired one ordinary double point. As $t \to t_\alpha$ there will be a vanishing cycle $\delta_\alpha \in H_{2n-1}(V_t)$, and the Picard-Lefschetz transformation on $H_{2n-1}(V_t)$ as t turns around t_α is

$$\gamma \to \gamma \pm (\gamma, \delta_\alpha)\,\delta_\alpha. \tag{2.1}$$

Marking a path from a fixed point t_0 to t_α the locus of the vanishing cycles δ_α along this path traces out the relative cycle $\Delta_\alpha \in H_{2m}(M, V_0)$. Ignoring questions of torsion the primitive part of $H^{2n}(M, \mathbf{Q})$ is exactly

[1] c.f. the discussion in [6] reducing the Lefschetz (1, 1) theorem in general to the case of curves on a surface.

represented by relative cycles $\Sigma_\alpha \lambda_\alpha \Delta_\alpha$ satisfying $\Sigma_\alpha \lambda_\alpha \delta_\alpha = 0$ in $H_{2n-1}(V_0)$. In fact the topological discussion in the case $n = 1$ (c.f. Section 1b)) goes over pretty much verbatim, the only real differences occurring between odd and even dimensions. Thus a primitive algebraic n-cycle Z on M satisfies either of the equivalent conditions: $Z \cdot V_t = 0$ in $H_{2n-2}(V_t; \mathbf{Q})$; or, in cohomology, the fundamental class η_Z maps to zero under the restriction mapping $H^{2n}(M, \mathbf{Q}) \to H^{2n}(V_t, \mathbf{Q})$. Given such a primitive algebraic n-cycle Z, its corresponding relative cycle $\Sigma_\alpha \lambda_\alpha \Delta_\alpha$ is obtained by writing

$$Z_t = Z \cdot V_t = \partial \gamma_t$$

when $\gamma_t \in H_{2n-1}(V_t, Z_t)$. As t turns around the critical value t_α, by an obvious extension of (2.1) we have

$$\gamma_t \to \gamma_t \pm \lambda_\alpha \delta_\alpha .$$

Then it is proved as before that these λ_α are the coefficients in the relative cycle corresponding to Z.

Before taking up normal functions we must discuss a little Hodge theory. For a smooth projective variety V we denote by $H^m(V)$ the complex cohomology and recall the Hodge decomposition

$$\begin{cases} H^m(V) = \bigoplus_{p+q=m} H^{p,q}(V), \\ H^{p,q}(V) = \overline{H^{q,p}(V)}. \end{cases}$$

It will be convenient to consider also the Hodge filtration defined by

$$F^k H^m(V) = \bigoplus_{p \geq k} H^{p,m-p}(V).$$

We may think of $H^m(V)$ as either de Rham cohomology or Čech cohomology for the constant sheaf $\mathbf{C}$. In each of these cases the Hodge filtration has a useful description which we shall now recall. Denoting by $A^{p,q}(V)$ the smooth (p, q) forms on V, by $F^k A^m(V) = \oplus_{p \geq k} A^{p,m-p}(V)$ the Hodge filtration on forms, by $Z(F^k A^m(V))$ the d-closed forms, and finally by $\hat{\Omega}_V{}^k$ the sheaf of closed holomorphic k-forms on V we have the following (c.f. [3]):

There are natural isomorphisms

$$\begin{cases} Z(F^kA^m(V))/d(F^kA^{m-1}(V)) \cong H^{m-k}(V, \hat{\Omega}_V{}^k) \\ \qquad\qquad\qquad\qquad\qquad\qquad \cong F^kH^m(V) \end{cases} \tag{2.2}$$

Now suppose that V has dimension $2n - 1$. Setting

$$\begin{cases} H^+(V) = F^nH^{2n-1}(V), \qquad \text{and} \\ H^-(V) = H^{2n-1}(V)/H^+(V) \\ \qquad\quad \cong H^+(V)^* \end{cases}$$

the intermediate Jacobian[2] is defined to be

$$\begin{aligned} J(V) &= H^-(V)/H^{2n-1}(V, \mathbf{Z}) \\ &\cong H^+(V)^*/H_{2n-1}(V, \mathbf{Z}) \end{aligned}$$

where the second isomorphism results from duality. Thus, a point $\nu \in J(V)$ is given by a linear function on $H^+(V)$ taken modulo those linear functions arising by integration over cycles, which are called periods. Each algebraic $(n - 1)$-cycle $Z \subset V$ which is homologous to zero defines a point ν_Z in $J(V)$ by the linear function

$$\omega \to \int_\gamma \omega \tag{2.3}$$

where γ is a real $(2n - 1)$ chain with $\partial\gamma = Z$ and

$$\omega \in Z(F^nA^{2n-1}(V))/d(F^nA^{2n-2}(V)) \cong H^+(V)$$

using (2.2). It is straightforward to verify that (2.3) is independent modulo periods of the choice of γ and of the choice of form representing a class in $H^+(V)$, and so represents a well-defined point in $J(V)$. When $n = 1$ we just have the familiar mapping from divisors of degree zero to the Jacobian variety of a curve.

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[2]We shall only be concerned with the intermediate Jacobian in the middle dimension.

Some properties of the mapping (2.3) which are more or less immediate from the definition are these:

i) ν_Z varies holomorphically with Z;
ii) consequently, ν maps rationally equivalent cycles to the same point, and we are free to utilize the moving lemma to put cycles in general position; and
iii) if the Hodge decomposition is skew to the rational cohomology in the sense that

$$H^{2n-1}(V, \mathbf{Q}) \cap (H^{n,n-1}(V) \oplus H^{n-1,n}(V)) = (0), \tag{2.4}$$

then ν maps algebraically equivalent cycles to the same point.

The next properties have to do with a holomorphic family $\{V_t\}_{t\in B}$ of smooth V_t's. We denote by $\mathbf{H} \to B$ the cohomology bundle with Gauss-Manin connection D; thus

$$\mathbf{H} = \bigcup_{t\in B} H^{2n-1}(V_t)$$

and the equation $D\gamma_t = 0$ defines the local condition that cycles $\gamma_t \in H^{2n-1}(V_t)$ be displaced from a fixed cycle γ_{t_0} by diffeomorphisms $V_{t_0} \xrightarrow{\sim} V_t$. We recall that the

$$F^k\mathbf{H} = \bigcup_{t\in B} F^kH^{2n-1}(V_t)$$

give holomorphic sub-bundles and $D(F^k\mathbf{H}) \subset \Omega_B{}^1(F^{k-1}\mathbf{H})$. The relevant properties of (2.3) with dependence on parameters are these:

iv) the intermediate Jacobians $J(V_t)$ vary holomorphically with t, and $J = \bigcup_{t\in B} J(V_t)$ forms in a natural way an analytic fibre space of complex tori over B;
v) if $Z_t \subset V_t$ is an algebraic $(n - 1)$-cycle which varies holomorphically with t and which is homologous to zero, then $\nu_{Z_t} \in J(V_t)$ varies holomorphically with t; and finally
vi) the section ν_Z of the fibre space $J \to B$ defined by $\nu_Z(t) = \nu_{Z_t}$ as in v) just above is *quasi-horizontal* in the sense that

$$D\nu_Z = 0 \tag{2.5}$$

where

$$D:\mathcal{O}(J) \to \Omega^1(F^{n+1}\mathbf{H})$$

is the mapping induced from the Gauss-Manin connection (c.f. [3]).

Our last properties have to do with the behavior of $J(V_t)$ as t approaches a critical value $t = t_\alpha$ in the Lefschetz pencil $|V_t|$ on M, and they are:

vii) there is naturally defined a generalized intermediate Jacobian $J(V_{t_\alpha})$ (whose precise properties will be recalled as needed below) such that

$$J = \bigcup_{t \in \mathbf{P}^1} J(V_t)$$

forms naturally a fibre space of complex Lie groups over $\mathbf{P}^1$ extending the previously defined family over $\mathbf{P}^1 - \{t_1, \ldots, t_N\}$; and

(viii) given a primitive algebraic n-cycle Z on M with $Z_t = Z \cdot V_t$, the cross-section $\nu_Z(t) \in J(V_t)$ defined for $t \neq t_1, \ldots, t_N$ by v) above extends across the critical values to give a quasi-horizontal section of $J \to \mathbf{P}^1$.

Definition. The *normal functions* associated to the Lefschetz pencil $|V_t|$ are the quasi-horizontal holomorphic cross-sections of $J \to \mathbf{P}^1$.

Intuitively we may view a normal function ν as follows: First we define the *Hodge bundle*

$$E = \bigcup_{t \in \mathbf{P}^1} T_0(J(V_t))^*$$

to be the bundle of dual spaces to the Lie algebras of the $J(V_t)$; note that for t not a critical value the fibre

$$E_t \cong F^n H^{2n-1}(V_t);$$

the remaining fibres E_{t_α} will be identified below. If $\omega(t)$ is any meromorphic section of the Hodge bundle $E \to \mathbf{P}^1$, then considering ν as a multi-valued section of E^* we obtain a multi-valued meromorphic function

$$\langle \nu, \omega(t) \rangle \tag{2.6}$$

on $\mathbf{P}^1 - \{t_1, \ldots, t_N\}$. Under analytic continuation around $t = t_\alpha$ this function changes by adding on a period $\lambda_\alpha \int_{\delta_\alpha} \omega(t)$ where δ_α is the vanishing cycle. Assuming that $t = \infty$ is not a critical value, as t describes a circle $|t| = r$ of sufficiently large radius the function (2.6) analytically continues back to itself, and consequently

$$\sum_{\alpha=1}^{N} \lambda_\alpha \int_{\delta_\alpha} \omega(t) = 0$$

for all $\omega(t)$. It follows that

$$\sum_{\alpha=1}^{N} \lambda_\alpha \delta_\alpha = 0 \qquad \text{in } H_{2n-1}(V_{t_0}),$$

and therefore $\Sigma_\alpha \lambda_\alpha \Delta_\alpha \in H_{2n}(M, V_{t_0})$ defines a primitive class $\eta_\nu \in H_{\text{prim}}{}^{2n}(M, \mathbf{Q})$, which we shall call the *fundamental class* of the normal function ν.

In more precise terms, if we let

$$X = \bigcup_{t \in \mathbf{P}^1} V_t$$

be obtained by blowing up the base locus $B = V_{t_0} \cdot V_\infty$ of the pencil, then $X \xrightarrow{f} \mathbf{P}^1$ is an analytic fibre space and it is a consequence of Lefschetz theory that (c.f. [1] and (1.9))

$$H^1(\mathbf{P}^1, R_f{}^{2n-1}\mathbf{Z}) \otimes \mathbf{Q} \cong H_{\text{prim}}{}^{2n}(M, \mathbf{Q}) \oplus H_{\text{prim}}{}^{2n-2}(B, \mathbf{Q}). \tag{2.7}$$

On the other hand there is an exact sheaf sequence

$$0 \to R_f{}^{2n-1}\mathbf{Z} \to E^* \to J \to 0,$$

of which a piece of the cohomology sequence is

$$H^0(\mathbf{P}^1, J) \xrightarrow{\delta} H^1(\mathbf{P}^1, R_f{}^{2n-1}\mathbf{Z}) \to H^1(\mathbf{P}^1, E^*). \tag{2.8}$$

The normal function ν gives a section in $H^0(\mathbf{P}^1, J)$, and the fundamental class η_ν is just the projection of $\delta(\nu)$ in $H_{\text{prim}}{}^{2n}(M, \mathbf{Q})$ using (2.7). That these descriptions agree is accomplished by computing $H^1(\mathbf{P}^1, R_f{}^{2n-1}\mathbf{Z})$ from the covering $\{U_0, U_1, \ldots, U_N\}$ of $\mathbf{P}^1$ where $U_0 = \mathbf{P}^1 - \{t_1, \ldots, t_N\}$ and U_α is a small disc around t_α, just as was done above (1.9).

Of course if the normal function ν arises from a primitive algebraic cycle Z on M then their fundamental classes are the same; this follows from the above discussion. In particular

$$\eta_\nu \in H^{n,n}(M) \cap H_{\text{prim}}{}^{2n}(M, \mathbf{Q}).$$

Conversely, it was proved by Zucker [10] that:

A cohomology class $\eta \in H_{\text{prim}}{}^{2n}(M, \mathbf{Q})$ is the fundamental class of a normal function if and only if it has Hodge type (n, n). (2.9)

This result had been previously established by Spencer Bloch and the author for sufficiently ample Lefschetz pencils, and we shall give this argument in the next section as a preliminary to discussing Picard-Fuchs equations.

The upshot is that we have a complete generalization of the Lefschetz part of the classical theory of normal functions, but of course what is missing in Poincaré's existence theorem. Now the existence theorem would undoubtedly follow from the Jacobi inversion theorem for the $J(V_t)$'s (c.f. [11]), but it is well known that for $n > 1$ this result is false. In fact, assuming for simplicity of notation that $H^{2n-1}(M) = 0$ (e.g., take $M \subset \mathbf{P}^{2n+1}$ to be a smooth hypersurface), by taking $|V_t|$ from a sufficiently ample linear system we may assume that $h^{2n-1,0}(V_{t_0}) \neq 0$. Recalling that the global monodromy action of $\pi_1(\mathbf{P}^1 - \{t_1, \ldots, t_N\})$ on $H^{2n-1}(V_{t_0}, \mathbf{Q})$ is irreducible, it follows that (2.4) is satisfied for generic t, since otherwise the left-hand side would give an invariant subspace. Therefore the group of invertible points on a generic $J(V_t)$ is a countable subgroup, one which will however be infinite provided that there is a primitive algebraic n-cycle Z on M with $\eta_Z \neq 0$ in $H^{2n}(M, \mathbf{Q})$. In a sense the basic difficulty here is that for higher weight not all Hodge structures come from geometry, even in the broadest motiv-theoretic sense. However, since a non-trivial variation of Hodge structure is in general supposed to come from geometry, and in fact since it seems that the family of determinantal varieties constructed from the differential of the variation of Hodge structure obtained by looking at the linear maps of different ranks must be non-trivial, what is suggested is that we investigate further the infinitesimal properties of normal functions.

b) Before taking up Picard-Fuchs equations it will be useful to analyze the Hodge bundle $E \to \mathbf{P}^1$. We recall (c.f. Schmid [8]) that the general fibre is $E_t = H^+(V_t)$, and those sections of E defined in a punctured disc $0 < |t - t_\alpha| < \epsilon$ around the critical value t_α and which extend are defined by the growth condition

$$\left| \int_\gamma \omega(t) \right| = 0(-\log |t - t_\alpha|). \tag{2.10}$$

In order to isolate the essential points we shall consider the case where $\dim M = 4$ and the line bundle $L \to M$ is chosen sufficiently ample to insure that

$$H^p(M, \Omega_M{}^q(kL)) = 0 \quad \text{for} \quad p, k > 0. \tag{2.11}$$

Along the way we shall then simply state the general conclusions which may be drawn from similar arguments (for details, c.f. Zucker's paper [10]).

To begin, by the Lefschetz hyperplane theorem we may consider $H^3(M)$ as a subgroup of $H^3(V_t)$ $(t \neq t_1, \ldots, t_N)$, and in fact we may write

$$H^3(V_t) = H^3(M) \oplus H_{\text{var}}{}^3(V_t)\,^3 \tag{2.12}$$

where $H^3(M)$ represents the fixed part of the variation of Hodge structure defined by the $\{F^k H^3(V_t)\}$, and where $H_{\text{var}}{}^3(V_t)$ is the orthogonal complement of this fixed part relative to the cup product on $H^3(V_t)$. We shall represent $H_{\text{var}}{}^3(V_t)$ by residues of meromorphic forms on M having poles along V_t and where the filtration by order of pole corresponds to the Hodge filtration, and then it will be easy to see how this representation varies with t.

Suppose then that $V \in |L|$ is a smooth divisor, denote by $\Omega_M{}^p(kV)$ the sheaf on M of meromorphic p-forms having a pole of order $\leq k$

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[3] This decomposition is defined on rational cohomology.

along V and by $\hat{\Omega}_M{}^p(kV)$ the subsheaf of closed forms. We claim then that there are a pair of exact sequences[4]

$$0 \to \Omega_M{}^4 \to \Omega_M{}^4(V) \xrightarrow{\text{Res}} \Omega_V{}^3 \to 0 \tag{2.13}$$

$$\begin{cases} 0 \to \hat{\Omega}_M{}^3(V) \to \Omega_M{}^3(V) \xrightarrow{d} \Omega_M{}^4(2V) \to 0 \\ 0 \to \hat{\Omega}_M{}^3 \to \hat{\Omega}_M{}^3(V) \xrightarrow{\text{Res}} \hat{\Omega}_V{}^2 \to 0 \end{cases} \tag{2.14}$$

(recall that $\Omega^4 = \hat{\Omega}^4$ since dim $M = 4$), where in both cases "Res" is the Poincaré residue operator.

Proof. In local coordinates x, y_1, y_2, y_3 where V is given by $x = 0$, a section of $\Omega_M{}^4(2V)$ is

$$\psi = \frac{f(x, y)\, dx \wedge dy_1 \wedge dy_2 \wedge dy_3}{x^2}.$$

Setting

$$\varphi = -\frac{f(x, y)\, dy_1 \wedge dy_2 \wedge dy_3}{x} - \frac{g(x, y)\, dx \wedge dy_2 \wedge dy_3}{x}$$

we will have $d\varphi = \psi$ provided that $\partial f/\partial x = \partial g/\partial y_1$, and this implies the surjectivity of d in (2.14). The Poincaré residues are defined by showing that a section η of $\hat{\Omega}_M{}^p(V)$ has the local form

$$\eta = \sigma \wedge \frac{dx}{x} + \tau$$

where σ and τ are holomorphic forms, and then setting Res $\eta = \sigma | V$. It is well-known that this procedure is well-defined and takes closed forms into closed forms.[5]

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[4] When dim $V = 2n - 1$ there are n such sequences.

[5] What is actually going on here is that $\Omega_M{}^k(V)$ are the closed forms in the log complex $\Omega_M{}^p(\log V)$, and the Poincare residue Res: $\Omega_M{}^p(\log V) \to \Omega_V{}^{p-1}$ is defined and commutes with d.

Using 2.2 and (2.11) the exact cohomology sequences of (2.13) and (2.14) give commutative diagrams

$$\begin{cases} 0 \to H^{4,0}(M) \to H^0(\Omega_M{}^4(V)) \xrightarrow{\text{Res}} H^{3,0}(V) \to H^{4,1}(M) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow \qquad\quad \nearrow \eta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad H^{3,0}(M) \end{cases} \tag{2.13a}$$

$$\begin{cases} \qquad H^0(\Omega_M{}^3(V)) \to H^0(\Omega_M{}^4(2V)) \to H^1(\hat{\Omega}_M{}^3(V)) \to 0 \\ H^0(\hat{\Omega}_V{}^2) \to H^1(\hat{\Omega}_M{}^3) \to H^1(\hat{\Omega}_M{}^3(V)) \xrightarrow{\text{Res}} H^1(\hat{\Omega}_V{}^2) \to H^2(\hat{\Omega}_M{}^3) \\ \quad \uparrow \quad \nearrow \eta \qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \quad \nearrow \eta \\ H^0(\Omega_M{}^2) \qquad\qquad\qquad\qquad\qquad\qquad\quad H^1(\Omega_M{}^2) \end{cases} \tag{2.14a}$$

where $\eta \in H^1(\hat{\Omega}_M{}^1)$ represents $c_1(L)$. By the Hard Lefschetz Theorem the η in (2.13a) and the η on the right in (2.14a) are isomorphisms, so that by (2.2) and (2.12)

$$\begin{cases} \text{Res}\, H^0(\Omega_M{}^4(V)) = F^3H_{\text{var}}{}^3(V) \\ \text{Res}\, H^1(\hat{\Omega}_M{}^3(V)) = F^2H_{\text{var}}{}^3(V). \end{cases}$$

Combining these identifications with the definition of primitive classes we obtain from (2.13a) and (2.14a) the sequences

$$0 \to F^4H_{\text{prim}}{}^4(M) \to H^0(\Omega_M{}^4(L)) \to F^3H_{\text{var}}{}^3(V) \to 0 \tag{2.13b}$$

$$\begin{cases} 0 \to F^3H_{\text{prim}}{}^4(M) \to H^1(\hat{\Omega}_M{}^3(V)) \to F^2H_{\text{var}}{}^3(V) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad H^0(\Omega_M{}^4(2L))/dH^0(\Omega_M{}^3(L)) \end{cases} \tag{2.14b}$$

We now think of V as varying in the pencil $|V_t|$. The Hodge bundle decomposes

$$E = E_f \oplus E_v$$

into a fixed and variable part. By (2.12)

$$E_f = F^2H^3(M),$$

while for $t \neq t_1, \ldots, t_N$

$$\begin{aligned}(E_v)_t &= F^2H_{\text{var}}{}^3(V_t)\\ &\cong F^3H_{\text{prim}}{}^4(M) \setminus (H^0(\Omega_M{}^4(2L))/dH^0(\Omega_M{}^3(L)))\end{aligned} \tag{2.15}$$

by (2.14b). In fact, the isomorphism (2.15) also holds for the critical values $t = t_\alpha$: If $\omega \in H^0(\Omega_M{}^4(2L))$ gives a form with a double pole along V_{t_α}, then the formula

$$\lim_{t \to t_\alpha} \int_{\delta_\alpha(t)} \text{Res}\, \omega = C \stackrel{te}{-} \omega(p_\alpha) \tag{2.16}$$

is valid and defines the point residue of ω (c.f. [4]), where here p_α denotes the double point of V_{t_α}. If $\tilde{V}_{t_\alpha} \to V_{t_\alpha}$ is the canonical desingularization, then this point residue is the obstruction to ω inducing a class $\text{Res}\, \omega \in F^2H^3(\tilde{V}_{t_\alpha})$. In general Res ω gives a differential of the 3rd kind on $\tilde{V}_{t_\alpha}$, and its residues there are just $\pm\, C \stackrel{te}{-} \omega(p_\alpha)$. The situation is in fact completely analogous to curves acquiring double points.

Concerning the growth estimate (2.10), suppose that M has dimension $2n$ and $\omega \in H^0(\Omega_M{}^{2n}(kL))$. Then according to [10]

$$\int_\gamma \text{Res}\, \omega = O(-|t|^{n-k} \log |t|),$$

so that for $k \leq n - 2$ we may write

$$\lim_{t \to t_\alpha} F^kH^{2n-1}(V_t) = F^kH^{2n-1}(\tilde{V}_{t_\alpha}),$$

while for $k = n - 1$ this is only true on that part of $F^kH^{2n-1}(V_t)$ having zero residue in the above sense. In particular,

$$\begin{cases} h^{2n-1-k,k}(\tilde{V}_{t_\alpha}) = h^{2n-1-k,k}(V_t), & k \leq n-2 \\ h^{n,n-1}(\tilde{V}_{t_\alpha}) = h^{n,n-1}(V_t) - 1. \end{cases}$$

In all these cases the canonical extensions of the Hodge bundles is given by taking residues of forms in $H^0(\Omega_M{}^{2n}(kL))$.

Now using the above notation E_ν for the variable part of the Hodge bundle and letting $E_\nu{}^{3,0} \subset E_\nu$ be the part corresponding to $F^3H_{\text{var}}{}^3(V_t)$, the sequences (2.13b) and (2.14b) globalize to

$$0 \to F^4H_{\text{prim}}{}^4(M) \to H^0(\Omega_M{}^4(L)) \otimes \mathcal{O}(2) \to E_\nu{}^{3,0} \to 0 \tag{2.13c}$$

$$\begin{cases} 0 \to F^3H_{\text{prim}}{}^4(M) \to \mathfrak{F} \to E_\nu \to 0 \\ 0 \to H^0(\Omega_M{}^3(L)) \otimes \mathcal{O}(1) \to H^0(\Omega_M{}^4(2L)) \otimes \mathcal{O}(2) \to \mathfrak{F} \to 0 \end{cases} \tag{2.14c}$$

For example, if $\omega \in H^0(\Omega_M{}^4(2L)$ and if s_0 and $s_\infty \in H^0(M, L)$ are sections defining V_0 and V_∞ respectively, then $s_0 + ts_\infty$ defines V_t and $\omega/(s_0 + ts_\infty)^2 \in H^0(\Omega_M{}^4(2V_t))$. Setting $t = 1/t'$

$$\operatorname{Res}\left(\frac{\omega}{(s_0 + ts_\infty)^2}\right) = t'^2 \operatorname{Res}\left(\frac{\omega}{(t's_0 + s_\infty)^2}\right),$$

and this defines the map $H^0(\Omega_M{}^4(2L)) \otimes \mathcal{O}_{\mathbf{P}^1}(2) \to E_\nu$. We shall analyze the above two sequences individually.

For the first we tensor with $\mathcal{O}_{\mathbf{P}^1}(-1)$ and $\mathcal{O}_{\mathbf{P}^1}(-2)$ and take cohomology sequences to obtain

$$\begin{cases} H^0(\Omega_M{}^4(L)) \cong H^0(\mathbf{P}^1, E_\nu \otimes \mathcal{O}(-1)) \\ H^0(\Omega_M{}^4) \cong H^0(\mathbf{P}^1, E_\nu \otimes \mathcal{O}(-2)). \end{cases}$$

The second isomorphism means that the top degree holomorphic differentials on M are uniquely of the form

$$\varphi = \omega(t) \wedge dt$$

where $\omega(t) \in H^{3,0}(V_t)$ vanishes to 2nd order at $t = \infty$, generalizing a phenomenon we noted previously for curves on a surface.

Coming to (2.14c) we again tensor with $\mathcal{O}_{\mathbf{P}^1}(-2)$ and take cohomology to obtain

$$\begin{cases} 0 \to H^0(\mathfrak{F}(-2)) \to H^0(\mathbf{P}^1, E \otimes \mathcal{O}(-2)) \to F^3H_{\text{prim}}{}^4(M) \to 0 \\ H^0(\mathfrak{F}(-2)) \cong H^0(\Omega_M{}^4(2L)). \end{cases}$$

The forms in $H^0(\Omega_M{}^4(2L))$ give differentials in $H^0(\Omega_M{}^4(V_0 + V_\infty))$ by the map $\omega \to \omega/s_0 s_\infty$. Taking double residues we obtain a holomorphic 2-form on the base locus B. Consequently the above sequence may be rewritten as

$$0 \to F^2H_{\text{var}}{}^2(B) \to H^0(\mathbf{P}^1, E \otimes \mathcal{O}(-2)) \to F^3H_{\text{prim}}{}^4(M) \to 0. \quad (2.15)$$

We are now ready to sketch the proof of (2.9). Referring to (2.8) and (2.7) and using that $H^0(\mathbf{P}^1, E \otimes \mathcal{O}(-2))$ is dual to $H^1(\mathbf{P}^1, E^*)$ we obtain from these together with (2.15) the diagram

$$\begin{array}{l} H^0(\mathbf{P}^1, J) \to H_{\text{prim}}{}^4(M, \mathbf{Q}) \oplus H_{\text{prim}}{}^2(B, \mathbf{Q}) \longrightarrow \\ \qquad\qquad\qquad 0 \\ \qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad F^3H_{\text{prim}}{}^4(M)^* \cong H_{\text{prim}}{}^{1,2}(M) \oplus H_{\text{prim}}{}^{0,4}(M) \\ \qquad\qquad\qquad \downarrow \\ \qquad \longrightarrow \quad H^1(\mathbf{P}^1, E^*) \\ \qquad\qquad\qquad \downarrow \\ \to \qquad\qquad F^2H^2(B)^* \cong H^{0,2}(B) \end{array} \quad (2.16)$$

What must be verified now is that the "obvious" maps in (2.16) render the diagram commutative. The obvious maps are just the projections

$$\begin{cases} H_{\text{prim}}{}^4(M, \mathbf{Q}) \to H_{\text{prim}}{}^{1,3}(M) \oplus H_{\text{prim}}{}^{0,4}(M) \\ H_{\text{prim}}{}^2(B, \mathbf{Q}) \to H^{0,2}(B) \end{cases}$$

in the Hodge decompositions of cohomology. The proof of this is the same as the case $n = 1$ given above. Once this is done, ignoring questions of torsion we will have found a section $\nu \in H^0(\mathbf{P}^1, J)$ whose fundamental class is a given element of $H_{\text{prim}}{}^4(M, \mathbf{Q}) \cap H^{2,2}(M)$. The differential

$$\begin{array}{c} D\nu \in H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}{}^1(E^{3,0*})) \\ \| \\ H^0(\mathbf{P}^1, E^{3,0*} \otimes \mathcal{O}(-2)) \\ \| \\ 0 \end{array}$$

by (2.13c). Thus the quasi-horizontality condition is automatically satisfied for any $\nu \in H^0(\mathbf{P}^1, J)$, and this completes the proof of (2.9).

c) The discussion in Section 1d) and e) concerning Picard-Fuchs equations and their relative counterparts now carries directly over. In fact we can be quite explicit. Given $\omega \in H^0(\Omega_M{}^4(L))$ we have a holomorphically varying section

$$\omega(t) = \text{Res}\left(\frac{\omega}{s_0 + ts_\infty}\right) \in H^3(V_t) = \mathbf{H}_t. \tag{2.17}$$

The Gauss-Manin connection is then given by

$$\frac{D\omega(t)}{dt} = \text{Res}\left(\frac{-s_\infty \omega}{(s_0 + ts_\infty)^2}\right).$$

Since the global monodromy action on the variable part of $H^3(V_{t_0})$ is irreducible, we infer that for generic t

$$\text{span}\left\{\omega(t), \frac{D\omega(t)}{dt}, \frac{D^2\omega(t)}{(dt)^2}, \cdots\right\} = H_{\text{var}}{}^3(V_t).$$

Recalling that $\dim H_{\text{var}}{}^3(V_{t_0}) = 2g$ is even, there will be a unique normalized O.D.E.

$$\sum_{k=0}^{2g} r_k(t) \frac{d^k\pi(t)}{(dt)^k} = 0, \qquad r_{2g}(t) = 1, \tag{2.18}$$

satisfied by the periods $\pi(t) = \int_\gamma \omega(t)$ of $\omega(t)$. We denote by $P(t, d/dt)$ the differential operator on the left in (2.18). If ν is any normal function viewed as a multi-valued holomorphic section of E^*, then setting $\nu(t) =$

$\langle \nu, \omega(t) \rangle$

$$P(t, d/dt)\nu(t) = r(t) \tag{2.19}$$

is a *rational* function of t. Indeed, it is single-valued and has no essential singularity at the critical values. As before the kernel of the map

$$\nu \to P(t, d/dt)\nu$$

consists exactly of the torsion elements in $H^0(\mathbf{P}^1, J)/J(M)$, where $J(M)$ is the intermediate Jacobian of M viewed as the fixed part of the family $\{J(V_t)\}$.

We may estimate the degree of the rational function $r(t)$ as follows: The singularities of $P(t, d/dt)$ occur at the critical values $t = t_\alpha$ together possibly with the points where the $D^k\omega(t)/(dt)^k$ fail to span $H_{\mathrm{var}}{}^3(V_t)$. The latter are, however, only apparent singularities of $\nu(t)$, while the former are regular singular points. Since around t_α the function $\nu(t)$ has the form

$$\nu(t) = (\text{holomorphic function}) + (\text{period}),$$

it follows that $r(t)$ has a pole of order $\leq 2g$ at each t_α. Thus we obtain the bound:

$$\deg r(t) \leq h_{\mathrm{var}}{}^1(V_{t_0}) \cdot N \tag{2.20}$$

where N is the number of critical values in the pencil $|V_t|$. Geometrically, N represents the degree of the hypersurface M^* dual to M embedded in projective space by the complete linear system $|L|$.

If now $E \in |kL|$ is such that a generic $E_t = E \cap V_t$ is smooth (thus E may have isolated singularities), and if $\omega(t)$ is given by (2.17), then $\omega(t) \in H^{3,0}(V_t)$ restricts to zero as a form on V_t and so defines a relative class in $H_{DR}{}^3(V_t, E_t)$. Letting D_E denote the Gauss-Manin connection for the relative cohomology bundle $\cup H_{DR}{}^3(V_t, E_t)$ there will be a minimal relation

$$P(t, d/dt, E)\omega(t) = \sum_{k=0}^{l} r_k(t, E) \frac{D_E{}^k\omega(t)}{(dt)^k} = 0 \tag{2.21}$$

in $H_{DR}{}^3(V_t, E_t)$. As usual we normalize by taking $r_l(t, E) = 1$.

THEOREM. *For a given normal function ν and each divisor $E \in |kL|$ we consider the equation*

$$P(t, d/dt, E)\nu(t) = 0. \tag{2.22}$$

Thinking of E as variable, those E_0 for which the operator $P(t, d/dt, E_0)$ is obtained by specialization from a generic E[6] *and for which (2.22) is satisfied are exactly the divisors for which the fundamental class η_ν is in the image of*

$$H_{2n}(E_0, \mathbf{Q}) \to H_{2n}(M, \mathbf{Q}). \tag{2.23}$$

The proof of this result is, at this juncture, pretty much verbatim that for the special case of curves on a surface discussed in Section 1e) above. Rather than belabor the details it seems more interesting to offer some observations on the theorem vis á vis the problem of constructing algebraic cycles.

According to (2.9), given a class $\eta \in H_{\text{prim}}{}^{2n}(M, \mathbf{Q}) \cap H^{n,n}(M)$ there will be a normal function ν whose fundamental class is η. We then consider the mapping

$$E \to P(t, d/dt, E)\nu(t), \tag{2.24}$$

and view it as a rational mapping

$$|kL| \to \mathbf{P}^{m(k)}, \tag{2.25}$$

where $\mathbf{P}^{m(k)}$ is obtained by adding the hyperplane at infinity to the vector space of rational functions of some fixed degree $d(k)$.[7] We then claim that:

The Hodge conjecture is true if, and only if, for k sufficiently large the image of the rational mapping (2.24) passes through the origin in $\mathbf{P}^{m(k)}$. (2.26)

Proof. If Hodge is true, then $\eta = \eta_Z$ is the fundamental class of some algebraic cycle Z and (2.22) will be satisfied for those $E_0 \in |kL|$

[6] We will see below that this is not a serious qualifier.

which pass through Z. (Here, there is some problem about the specialization question and this will be discussed below.)

Conversely, we assume by induction that Hodge is true for smooth varieties of dimensions less that dim M. By the various Lefschetz theorems we then reduce to primitive cohomology in the middle dimension (c.f. Section 2a) above). Given $\eta \in H_{\text{prim}}{}^{2n}(M, \mathbf{Q}) \cap H^{n,n}(M)$ coming from a normal function ν we infer that η is in the image of (2.23) for some divisor E_0. Now E_0 must be singular (c.f. the following discussion), but by applying mixed Hodge theory [2] we may conclude that η ultimately comes from (n, n) classes on smooth varieties of dimension less than dim M, and then by the induction assumption η will be algebraic. Q.E.D.

Now of course an obvious possibility would be to show that, for k sufficiently large, the rational mapping (2.25) is surjective. This could be done by counting dimensions and estimating the rank of the Jacobian matrix at a generic E. In fact the latter is not too difficult; a generic fibre is given by all E for which

$$P(t, d/dt, E)\nu(t) = r(t, E) = r(t)$$

is a fixed rational function. Differentiating this equation with respect to E we obtain a homogeneous equation satisfied by $\nu(t)$, and the number of these may be estimated. It is also possible to estimate $m(k)$ by using the regularity theorem to estimate the degree of the rational function $r(t, E)$ in a manner similar to (2.20). When carried out my crude count showed that the dimensions on both sides of (2.25) grow like

$$C \overset{tc}{=} k^n + (\text{lower order terms}),$$

but the coefficient of k^n appears to be larger for the right-hand side of (2.25). This is probably correct since one does not expect to get off so easily.

It is also quite instructive to think about the divisors E_0 for which (2.22) might be satisfied. The basic observation is

[7] If we write $P(t, d/dt, E)$ in the form $\Sigma_k\ [p_k(t, E)/q_k(t, E)][d^k/(dt)^k]$ where the p_k and q_k are polynomials, then as E specializes to E_0 it may happen that some $q_k(t, E_0) \equiv 0$. In this case under the mapping (2.24) E_0 will go to the hyperplane at infinity.

If $\eta \neq 0$ is in the image of the mapping (2.23), then E_0 must be singular. (2.27)

Indeed, the Lefschetz decomposition into primitive homology implies that the primitive cycles are exactly those which are *not* supported on a smooth divisor in the linear system $|kL|$.

Now, given this it none-the-less seems reasonable to hope that for sufficiently large k we may find an E_0 with only ordinary double points which supports η.[8] This raises the question of just how allowing E_0 to have double points might increase the rank of $H_{2n}(E_0)$, and this happens as follows: Let $\Delta = p_1 + \cdots + p_\delta$ be the zero-cycle of double points on E_0 and denote by $\mathcal{I}_\Delta$ the ideal sheaf of Δ. We picture E_0 as the limit of smooth divisors E_t as $t \to 0$. Then to each p_k there corresponds a vanishing cycle $\delta_k(t) \in H_{2n-1}(E_t)$ which shrinks to the point p_k as $t \to 0$. Suppose now that some linear combination

$$\Sigma \lambda_k \delta_k(t) = 0 \quad \text{in} \quad H_{2n-1}(E_t). \tag{2.28}$$

Writing $\Sigma_k \lambda_k \delta_k(t) = \partial\gamma(t)$ for a $2n$-chain $\gamma(t)$ and letting $t \to 0$ we obtain a cycle $\gamma = \gamma(0)$ in $H_{2n}(E_0)$. Conversely, any new cycle in $H_{2n}(E_0)$ arises in this way.

To put the condition (2.28) in algebro-geometric form we recall from Section 2b) above that, if $\tilde{E} \to E_0$ denotes the canonical desingularization, we have

$$h^{2n-1-k,k}(\tilde{E}) = h^{2n-1-k,k}(E), \qquad k \leq n-2,$$

while the expected value for $h^{n,n-1}(\tilde{E})$ is given by

$$h^{n,n-1}(\tilde{E}) = h^{n,n-1}(E) - \deg \Delta. \tag{2.29}$$

Now, assuming for simplicity of notation that L has been replaced by kL so as to have $E_0 \in |L|$, the postulated equation (2.29) will be correct exactly when the zero-cycle Δ imposes independent conditions on the linear system $|K + nL|$. This follows from our discussion of point

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[8] This may be proved by writing Z as a difference of effective smooth subvarieties in the (rational equivalence ring) $\otimes$ **Q**. Many of the following observations were known to Spencer Block, and also to Herb Clemens in connection with his work on double solids.

residues centered around (2.16). The correct formula for the right-hand side of (2.29) is

$$h^{n,n-1}(\tilde{E}) = h^{n,n-1}(E) - \deg \Delta + h^1(\mathcal{I}_\Delta(K + nL)), \tag{2.30}$$

where the superabundance $h^1(\mathcal{I}_\Delta(K + nL))$ exactly measures the failure of the points of Δ to impose independent conditions on $|K + nL|$.[9]

All of this is very clear for curves on a surface. If M is a surface and $E_0 \in |L|$ is an *irreducible* curve with only double points, then on the one hand the linear series $|\mathcal{I}_\Delta(K + L)|$ cuts out the canonical series on E_0, while on the other hand the genus of E_0 is given by

$$\pi(E_0) = \pi(E_t) - \deg \Delta.$$

Consequently, Δ imposes independent conditions on $|K + L|$, and the only way we can have $h^1(\mathcal{I}_\Delta(K + L)) \neq 0$ is for E_0 to become reducible; in this case $\dim H_2(E_0)$ clearly goes up accordingly.

In general, we may say that the zeros of the equation (2.22) as E varies are detecting by analytic and topological means the solutions to the following algebro-geometric problem:

Among zero cycles Δ on M find those for which

$$\begin{cases} \dim |\mathcal{I}_\Delta{}^2(kL)| \geq 0 \\ \dim |\mathcal{I}_\Delta(K + nkL)| < \dim |K + nkL| - \deg \Delta. \end{cases}$$

Fortunately the Lefschetz (1, 1) theorem for curves on a surface was not formulated in this way or else it might not have been proved.

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