The first message, dated Dec. 22, 2013, is simply a comment on the spectral theory of a differential equation on a half-line, a classical topic.

Dear Julia, Dear Michael,

Our conversation this afternoon was difficult, but on reflection perhaps more productive than one might think. There are a number of features of Eisenstein series for groups of rank one, even for SL(2), on which I have never had occasion to reflect at length. One should do so, as has been made clear to me by Michael's question. Somewhat coarsely expressed, the proofs themselves require less reflection than explanations of their various steps do, especially to someone with little analytic experience. I think no one has ever offered a description of them that is intuitively convincing and makes the essential elements clear. To do so in your book would, if possible, be a big help.

One key-term that I used was spectral theory on a half-line or scattering. This means that we have at hand a half-line, $x > 0$, and an eigenvalue problem for an equation which is close to d^2f/dx^2, thus the eigenvalue problem is roughly $d^2f/dx^2 = \lambda f$. There is a boundary condition at 0. There has to be a continuous spectrum, the eigenfunctions being

(A) $$\phi_\lambda(x) = ae^{i\lambda x} + be^{-i\lambda x} + *, \qquad b = b(a),$$

where $*$ guarantees that the boundary condition at 0 is satisfied and decreases sufficiently rapidly. There is one eigenfunction for each $\lambda \geqslant 0$. This is not fully correct. There is an additional feature. The fundamental domain for a rank-one group is to be thought of as a ball, from which emanate a certain finite number of filaments. The core is compact. The eigenfunctions are as in (A) on each filament, but a varies from filament to filament. In addition there is a discrete set of λ with varying eigenvalues that decay on the filaments, so that in (A) one takes $a = b = 0$. These are the "cusp forms."

During our conversation, I appreciated not only that I had forgotten parts of the argument but that I had never mastered it fully as a coherent whole. I think that it would be a good idea to describe them, if not fully, at least in an intuitively adequate fashion, perhaps including a description of the geometric theory as well, where, as I said, the "stable" and "unstable" elements appear and where the filaments appear as discrete sets parametrized by positive integers. Look, for example, at Grothendieck's theorem that every vector bundle over a curve of genus 0 is trivial—taking, for example, its dimension to be 2 and its Chern class 0. The geometric theory may be easier for modern "number theorists." Reflect as well on the geometric theory over a finite field to understand how it sometimes reflects the arithmetic theory and sometimes the geometric theory over **C**.

I am sending this message to Peter Sarnak as well to see whether he is aware of any good references or has any ideas about possible expositions.

Yours, RL

The second message is dated Dec. 27, 2013 and was written simply to emphasize the importance I attached to the first.

Dear Julia, Dear Michael,

It has been a pleasure talking to you and I hope to do so again after a while, but I have decided that after various efforts over the past few years, I need, for health reasons, an absolute break from mathematics for a substantial period. I am starting now and it is clear that some time will pass before I return to work.

I encourage you to continue your efforts, in particular, to try to understand my notes of the past week. The theory of automorphic forms and their relation to number theory as I formulate it has roots in the theory of representations and, in particular, the theory of Eisenstein series. Michael's last message suggested that he did not yet appreciate how important this is, or even the message in my notes. So I urge you both, if you wish to continue the project, to converse with the many specialists in the analytic theory of automorphic forms to be found in New York City and to understand why the theory of Eisenstein series as described in my Yale lectures offered for me the clue to all that follows.

Yours, Robert Langlands

Compiled on November 12, 2024.