Two Applications of Algebraic Geometry to Entire Holomorphic Mappings

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In this paper we shall prove two theorems concerning holomorphic mappings of large open sets of $\mathbb{C}^k$ into algebraic varieties. Both are in response to well-known outstanding problems, and we feel that the techniques introduced should in each case have further applications.

To state our first result, we recall that a holomorphic mapping into an algebraic variety is said to be *algebraically degenerate* in case the image lies in a proper algebraic subvariety.

Theorem I. *Let X be an algebraic variety whose irregularity satisfies*

$$q > \dim X.$$

Then any entire holomorphic curve

$$f : \mathbb{C} \to X$$

is algebraically degenerate.

We remark that the irregularity $q = h^{1,0}(X)$ is the dimension of the space of holomorphic 1-forms on any smooth model for the function field of X. Since such desingularizations exist by Hironaka's well-known theorem, and since q is a birational invariant, our definition of the irregularity makes sense.

When X is a curve, this theorem was proved by Picard [26] in a paper closely related to his proof of the usual Picard theorem. Nowadays this case is an obvious consequence of the uniformization theorem, but unfortunately this latter result does not generalize. Some 47 years later Theorem I was formulated by A. Bloch [1], who established several special cases and contributed the essential technical idea of using jets. To a reader trained in modern mathematics Bloch's paper is obscure to put it mildly, and interest in the subject was revived by Ochiai [25], who considerably clarified matters and who formulated a technical result that would yield what he termed Bloch's conjecture.

Our approach is different in that rather than establishing Ochiai's technical result (which is, in fact, true), we use the method of negative curvature. The

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difference is in a sense more apparent than real, in that the essential ingredient in both proofs is the use of higher-order jets to detect geometric consequences of the assumption $q > \dim X$, consequences of a somewhat subtle character that may not be evident from first-order considerations.

In addition to the work of Bloch and Ochiai, our proof was motivated by the recent paper [2] of Bogomolov, who used symmetric differentials to show the existence of finitely many rational and elliptic curves on a surface of general type with $c_1^2 > c_2$. In this paper we shall use jet differentials to construct a negatively curved jet psuedometric that leads to the proof of Theorem I. For any surface of general type these exist in abundance—even when $H^0(\mathrm{Sym}^m\,\Omega_X^1) = 0$ for all $m \geqslant 1$—and it is our feeling that the systematic use of higher-order differentials presents an algebrogeometric technique that may be useful in other contexts. For this reason we have, in Section 1, attempted to clearly explain the basic concepts. We have also shown that, for any smooth n-dimensional variety X for which $c_1(\Omega_X^1)^n > 0$, the Euler characteristic of the sheaf of jet differentials grows at the maximum rate, and for general type surfaces these jet differentials give a birational embedding of a suitably prolonged projectivized jet bundle.

Our second main result is in response to the following well-known

Conjecture. *An n-dimensional algebraic variety X is measure-hyperbolic if, and only if, X is of general type.*

(Cf. Kobayashi [20]—the relevant definitions together with additional references are given in Section 4 below.) The implication

$$X \text{ general type} \quad \Rightarrow \quad X \text{ measure-hyperbolic}$$

was established some time ago, and so the conjecture pertains to the converse. When $n = 1$ the result is a simple consequence of the uniformization theorem and classification of curves according to their Kodaira dimension. Turning to surfaces, the conjecture would follow from showing that, for any surface X not of general type, there is a holomorphic mapping

$$f : \Delta \times C \to X$$

that takes the origin to a given general point on X and whose Jacobian is not identically zero. Using the classification of surfaces, one is easily reduced to constructing f when X is an algebraic $K3$ surface. These fall into an infinite number of 19-dimensional algebraic families $\mathfrak{F}_n$. Our result is

Theorem II. *An algebraic $K3$ surface $X \in \mathfrak{F}_n$ is not measure-hyperbolic when $n = 1, 2,$ or 3.*

The proof is by showing that on any such X there is a family of ∞^1 elliptic curves (all singular), which then leads to the desired mapping f. These curves are constructed by projective methods. In fact, the construction is valid for all n, but the proof of Theorem II does not go through due to a certain technical point (involving singularities) that we are unable to resolve. This point is one of those issues that are in some sense "geometrically obvious" but whose proof will require a deeper understanding of possible degeneracies than we are able to muster. Nailing it down seems to us a very worthwhile project, as it would have the following geometric consequence:

(*) On any algebraic K3 surface $X \in \mathcal{F}_n$ there are a positive finite number of rational curves that have $(n + 1)$ distinct nodes.

Our proposed proof of this assertion—which has the existence of ∞^1 elliptic curves as an easy consequence—is by induction on the degree $2n$ of $X \subset \mathbb{P}^{n+1}$, and furnishes a technique that may be useful in other contexts.

It is a pleasure to thank Joe Harris for several conversations pertaining to Section 4, and for helping us with several incomplete but enjoyable "proofs" of (*).

Part A. Jet Differentials and Bloch's Conjecture

1. Jets and Jet Differentials

(a) Definition and basic properties of jet spaces. We shall first explain jets for holomorphic mappings into a smooth complex manifold X.

Given $x \in X$, we denote by Δ a disc of any positive radius and consider germs of holomorphic mappings

$$f : \Delta \to X$$

that satisfy $f(0) = x$. In a local holomorphic coordinate system any such f is given by its convergent series

$$f(z) = f^{(0)} + f^{(1)}z + f^{(2)}\frac{z^2}{2!} + f^{(3)}\frac{z^3}{3!} + \cdots, \tag{1.1}$$

where $f^{(k)} \in \mathbb{C}^n$ and $f^{(0)} = x$.

Two germs f and $\tilde{f}$ osculate to order k in case

$$f^{(0)} = \tilde{f}^{(0)}, \quad f^{(1)} = \tilde{f}^{(1)}, \ldots, f^{(k)} = \tilde{f}^{(k)}.$$

The equivalence classes of such germs will be called *jets of order* k at x and denoted $J_k(X)_x$. It is clear that

$$J_k(X) = \bigcup_{x \in X} J_k(X)_x$$

forms a complex manifold of dimension $n + kn$, and if $U \subset X$ is an open set on which we have holomorphic coordinates, then this choice of coordinates induces an isomorphism

$$J_k(U) \cong U \times \mathbb{C}^{kn}.$$

Given a holomorphic arc $f : \Delta \to X$ with $f(z) = x$, we denote by $j_k(f)_x \in J_k(X)_x$ the k-jet defined by the germ of f at x. The notation

$$j_k(f) : \Delta \to J_k(X)$$

will be used to denote the natural lifting of f to k-jets. Intuitively, $J_k(X)_x$

consists of kth-order infinitesimal arcs centered at x, and $j_k(f)$ describes the family of these arcs along the holomorphic curve $f(\Delta)$. In general, a holomorphic mapping

$$h : X \to Y$$

between complex manifolds induces a mapping

$$h_* : J_k(X) \to J_k(Y)$$

on k-jets. For $k = 1$, we have the usual notion of tangent vectors and induced map on tangent spaces.

The jet manifolds $J_k(X)$ are holomorphic fibre bundles over X, but for $k \geqslant 2$ they are not vector bundles. However, there are obvious maps

$$J_{k+1}(X) \to J_k(X) \tag{1.2}$$

whose fibres are affine linear spaces.

Using local coordinates on X so that jets may be expressed in the form (1.1), the fibre of (1.2) amounts to fixing $x = f^{(0)}, f^{(1)}, \dots, f^{(k)}$ and having $f^{(k+1)}$ free to vary over $\mathbb{C}^n$. If, moreover, $f^{(1)} = \cdots = f^{(k)} = 0$, then $f^{(k+1)}$ transforms like a tensor in $T_x(X)$. In other words, the fibres of (1.2) are affine bundles whose associated vector bundle is $T(X)$.

Next, we will define an action of $\mathbb{C}^*$ on jets that amounts to reparametrization by a constant dilation or contraction. Recalling that Δ denotes a disc of unspecified positive radius, given $f : \Delta \to X$ and $t \in \mathbb{C}^*$, we set $f_t(z) = f(tz)$ and define

$$t \cdot j_k(f) = j_k(f_t).$$

In the coordinates (1.1),

$$t \cdot \{ f^{(0)}, f^{(1)}, \dots, f^{(k)} \} = \{ f^{(0)}, tf^{(1)}, \dots, t^k f^{(k)} \}. \tag{1.3}$$

If $J_k^*(X)$ denotes the nonconstant jets—i.e., those with some $f^{(j)} \neq 0$ for $1 \leqslant j \leqslant k$—then this $\mathbb{C}^*$ action preserves $J_k^*(X)$ and we define

$$P_k(X) = J_k^*(X) / \mathbb{C}^*.$$

For $k = 1$ we obtain the projectivized tangent bundle $P_1(X) = \mathbb{P}T(X)$, whose elements will be written (x, ξ), where $x \in X$ and $\xi \in \mathbb{P}T_x(X)$ is a tangent direction. It is clear that $P_1(X)$ is a complex manifold, and is in fact a $\mathbb{P}^{n-1}$-bundle over X.

For $k \geqslant 2$ the objects $P_k(X)$ are perhaps less familiar. The fibre $F_{k,n}$ of $P_k(X) \to X$ is a *weighted projective space* (cf. Dolgacev [8]); it is the quotient of $\mathbb{C}^{kn} - \{0\}$ by the $\mathbb{C}^*$-action

$$t \cdot \{ w^{(1)}, w^{(2)}, \dots, w^{(k)} \} = \{ tw^{(1)}, t^2 w^{(2)}, \dots, t^k w^{(k)} \}, \tag{1.4}$$

where $w^{(j)} \in \mathbb{C}^n$. For $k \geqslant 2$ this action has fixed points, and when also $n \geqslant 2$ the fibre $F_{k,n}$ is a projective algebraic variety having what are usually termed quotient singularities. For example, when $k = n = 2$ the fibre is a quotient of $\mathbb{C}^4 - \{0\}$ under the action

$$t \cdot \{ x, y, u, v \} = \{ tx, ty, t^2 x, t^2 y \}.$$

Taking $t = -1$, the plane $x = y = 0$ is left fixed and $F_{2,2}$ has a singular line. In fact it is biholomorphic to the cone $\{w_1^2 + w_2^2 + w_3^2 = 0\}$ in $\mathbb{C}^4$. The presence of singularities in $P_k(X)$ will not cause any difficulty, since in fact these weighted projective spaces are quite nice varieties that are well understood.

We shall be using jets to study holomorphic curves $f: \Delta \to X$ in general complex analytic varieties X that may have singularities. It is, of course, desirable to define intrinsically the jet spaces associated to X. However, for our purposes this would take us too far afield, and is not necessary, for the following reason: Given any resolution $\tilde{X} \to X$ of the singularities of X, there is a unique lifting $\tilde{f}: \Delta \to \tilde{X}$ of any holomorphic curve whose image does not lie entirely in the singular locus of X. Since we shall be using jets to prove statements like "any entire holomorphic curve $f: C \to X$ is analytically degenerate", this device of using resolution of singularities will suffice for our needs.

(b) Formalism of jet differentials. We will now define the sheaves of jet differentials on a complex manifold X. On the weighted projective space $F_{k,n}$ given by the $\mathbb{C}^*$ action (1.4) we consider polynomials $\phi(w)$ in the kn variables $w_i^{(1)}, \ldots, w_i^{(n)}$. Assigning to $w_i^{(l)}$ the weight l, we consider polynomials that are homogeneous of weight m. Equivalently, the polynomial ϕ should satisfy

$$\phi(t \cdot w) = t^m \phi(w).$$

By taking local coordinates on X and allowing the coefficients of $\phi(w)$ to be holomorphic functions, we may define the *sheaf $\mathcal{J}_{k,m}$ of k-jet differentials on X of weight m.*

For example, when $k = 1$ we have

$$\mathcal{J}_{1,m} = \operatorname{Sym}^m \Omega_X^1.$$

For another example, when $k = 3$, sections of $\mathcal{J}_{3,m}$ are locally

$$\begin{aligned}
\phi &= \textstyle\sum a_i f_i', & m &= 1,\\
\phi &= \textstyle\sum a_{ij} f_i' f_j' + b_i f_i'', & m &= 2,\\
\phi &= \textstyle\sum a_{ijk} f_i' f_j' f_k' + b_{ij} f_i' f_j'' + c_i f_i''', & m &= 3,\\
\phi &= \textstyle\sum a_{ijkl} f_i' f_j' f_k' f_l' + b_{ijk} f_i' f_j' f_k'' & &\\
&\quad + c_{ij} f_i'' f_j'' + d_{ij} f_i' f_j''', & m &= 4,
\end{aligned} \tag{1.5}$$

etc. Here the coefficient functions are holomorphic functions on X, and the obvious symmetry conditions—such as $a_{ij} = a_{ji}$ in the second equation in (1.5)—are assumed satisfied when applicable. In summary:

(1.6) *Sections of $\mathcal{J}_{k,m}$ are locally given by homogeneous polynomials with holomorphic coefficients in the variables $f_i', f_i'', \ldots, f_i^{(k)}$ of total weight m, where $f_i^{(l)}$ is assigned weight l.*

By considering the highest expression in the $f_i^{(k)}$ that occurs, we see that there is a natural filtration on the weighted homogeneous polynomials of total weight m. This gives an intrinsic filtration

$$\mathcal{J}_{k-1,m} = S_0 \subset S_1 \subset \cdots \subset S_{[m/k]} = \mathcal{J}_{k,m},$$

where

$$S_i/S_{i-1} \cong \mathrm{Sym}^i\,\Omega^1_X \otimes \mathcal{J}_{k-1,\,m-ki}.$$

Thus, inductively we have:

$\mathcal{J}_{k,m}$ has a composition series whose factors are

$$\mathrm{Sym}^{i_1}\Omega^1_X \otimes \mathrm{Sym}^{i_2}\Omega^1_X \otimes \cdots \otimes \mathrm{Sym}^{i_k}\Omega^1_X,$$

where each combination of nonnegative indices satisfying

$$i_1 + 2i_2 + \cdots + ki_k = m$$

occurs exactly once.

The simplest example of this is the exact sequence

$$0 \to \mathrm{Sym}^2\,\Omega^1_X \to \mathcal{J}_{2,2} \to \Omega^1_X \to 0.$$

We will now give an alternate definition of $\mathcal{J}_{k,m}$, one that will be useful in our study below. Given any complex analytic variety Y with a $\mathbb{C}^*$ action and analytic quotient space $Z = Y/\mathbb{C}^*$, we denote the projection $Y \to Z$ by $\tilde{\omega}$ and define a sheaf $\mathcal{L}^m$ of $\mathcal{O}_Z$-modules as follows: For an open set $U \subset Z$ we set

$$\mathcal{L}^m(U) = '\{\phi \in \mathcal{O}(\tilde{\omega}^{-1}(U)) : \phi(ty) = t^m\phi(y)\}.$$

The presheaf $U \to \mathcal{L}^m(U)$ then leads to a sheaf $\mathcal{L}^m$ on Z. We note that, under the new $\mathbb{C}^*$ action given by the standard covering $t \to t^m$ of $\mathbb{C}^*$, $\mathcal{L}^m$ becomes the first power of the new sheaf $\mathcal{L}$.

Taking $Y = J_k(X)$ and $Z = P_k(X)$ we have now defined the sheaves $\mathcal{L}^m$ upstairs on $P_k(X)$. In general these sheaves are not invertible (Dolgacev [8]). However, for any multiple $m = l \cdot k!$ of $k!$, the sheaf $\mathcal{L}^m$ is associated to a line bundle. Essentially this is because the action

$$t \times j \to t^{k!}j$$

gives a free action of

$$\mathbb{C}^*/(k\text{th roots of unity}) \cong \mathbb{C}^*,$$

and under a free action the sheaf $\mathcal{L} \to Z$ in the preceding paragraph is the one associated to the line bundle $L = Y \times_{\mathbb{C}^*} \mathbb{C}$ over Z.

The restriction of $\mathcal{L}^m$ to each fibre of $P_k(R) \xrightarrow{\pi} \mathbb{C}$ is Dolgacev's (m) on the weighted projective space $F_{k,n}$. In particular, by the Theorem in §1.4 of [8] we have

$$\begin{aligned} &R^q_\pi(\mathcal{L}^m) = 0 \qquad \text{for } m \geqslant 0,\, q > 0,\\ &R^0_\pi(\mathcal{L}^m) = \mathcal{J}_{k,m} \text{ is locally free on } X. \end{aligned} \tag{1.7}$$

As the second equation suggests, and as is clear from the definitions, the previously defined sheaf $\mathcal{J}_{k,m}$ of m-fold jet differentials is the 0th direct image of $\mathcal{L}^m \to P_k(X)$. In general, by (1.7) and the Leray spectral sequence,

$$H^q(X, \mathcal{J}_{k,m}) \cong H^q(P_k(X), \mathcal{L}^m) \tag{1.8}$$

for all $q \geqslant 0$, $m \geqslant 0$. In particular, for $m = l \cdot k!$ giving the space of global sections $H^0(X, \mathcal{J}_{k,m})$ is equivalent to giving the rational mapping

$$\phi_m : P_k(X) \to P^N$$

$(N + 1 = h^0(X, \mathcal{J}_{k,m}))$ in which each fibre is developed onto a rational image of $F_{k,n}$.

It is clear that a holomorphic mapping $f: X \to Y$ between complex manifolds induces a pullback

$$f^* : H^0(Y, \mathcal{J}_{k,m}) \to H^0(X, \mathcal{J}_{k,m}) \tag{1.9}$$

on jet differentials. Somewhat more interestingly, if f is only assumed to be meromorphic and therefore perhaps not an actual map in codimension two, the usual argument invoking Hartogs' extension theorem shows that the transformation (1.9) is still defined. In particular,

The spaces $H^0(X, \mathcal{J}_{k,n})$ are bimeromorphic invariants of complex manifolds.

As a consequence, the space $H^0(X, \mathcal{J}_{k,m})$ of global jet differentials on any analytic variety may be defined to be $H^0(\tilde{X}, \mathcal{J}_{k,m})$ for any resolution $\tilde{X}$ of X.

To conclude this section we will introduce two formal operations on jet differentials. These will not be used explicitly in our work, but help to clarify the nature of these objects. The first is simply multiplication. More precisely, the projection (1.2) induces inclusions

$$\mathcal{J}_{k,m} \subset \mathcal{J}_{k+1,m},$$

and we shall denote the limit $\bigcup_k \mathcal{J}_{k,m}$ by $\mathcal{J}_{\cdot,m}$. Then multiplication of weighted homogeneous polynomials gives a product

$$\mathcal{J}_{\cdot,m} \otimes \mathcal{J}_{\cdot,m'} \to \mathcal{J}_{\cdot,m+m'}$$

that satisfies obvious algebraic rules.

The second operation is that of differentiation, to be denoted by

$$' : \mathcal{J}_{k,m} \to \mathcal{J}_{k+1,m+1}.$$

It is defined as follows: Given a section ϕ of $\mathcal{J}_{k,m}$ and holomorphic arc $f: \Delta \to X$, we set

$$\phi'(j_{k+1}(f))(z) = \frac{d}{dz}(\phi(j_k(f))(z)).$$

For example, in the case $k = m = 2$ of (1.5),

$$\phi = \sum a_{ij} f_i' f_j' + b_i f_i'',$$

$$\phi' = \sum \frac{\partial a_{ij}}{\partial z_k} f_i' f_j' f_k' + \left(2a_{ij} + \frac{\partial b_j}{\partial z_i}\right) f_i' f_j'' + b_i f_i'''.$$

As usual the Leibniz rule

$$(\phi\psi)' = \phi'\psi + \phi\psi'$$

is valid.

A simple but fundamental observation, one that will be discussed in detail in the next section, is this: We ask whether there may be global sections in

$H^0(X, \mathcal{J}_{k,m})$ that do not ultimately come from ordinary symmetric differentials (i.e., sections in $H^0(X, \mathrm{Sym}^m \Omega^1_X)$). In the following section, the answer to this will turn out to be yes, and as an indication that this should be so we consider the example of a section

$$\phi = \sum a_{ij} f_i' f_j', \qquad a_{ij} = a_{ji},$$

of $\mathrm{Sym}^2 \Omega^1_X$. The derivative

$$\phi' = \sum \frac{1}{3}\left(\frac{\partial a_{ij}}{\partial z_k} + \frac{\partial a_{ki}}{\partial z_j} + \frac{\partial a_{jk}}{\partial z_i} \right) f_i' f_j' f_k' + 2 a_{ij} f_i' f_j''$$

has a coefficient of $f_i' f_j''$ that is symmetric in its indices, but by (1.5) this is not necessarily the case for a general section of $\mathcal{J}_{2,3}$. This led us to suspect that the algebra $\bigoplus_{k,m} H^0(\mathcal{J}_{k,m})$ may be larger than that generated by symmetric differentials and their derivatives.

(c) Existence of jet differentials on a surface of general type. In this section we shall prove the following two results:

(1.10) Proposition. *For any smooth projective variety X,*

$$\begin{aligned} \chi(X, \mathcal{J}_{k,m}) = {} & \frac{m^{(k+1)n-1}}{(k!)^n((k+1)n-1)!} \\ & \times \left(\frac{(-1)^n}{n!} c_1(X)^n (\log k)^n + O\big((\log k)^{n-1}\big) \right) \\ & + O(m^{(k+1)n-2}). \end{aligned}$$

We note that

$$\dim P_k(X) = (k+1)n - 1,$$

so that if $c_1(\Omega^1_X)^n > 0$, then by (1.8) the Euler characteristic of $\mathcal{J}_{k,m}$ grows at the maximum rate.

(1.11) Proposition. *Let X be a surface of general type. Then for k, m sufficiently large, the rational mapping*

$$\phi_m : P_k(X) \to \mathbb{P}^N$$

is birational onto its image.

As we shall see in the next section, there are simple surfaces of general type for which $H^0(X, \mathrm{Sym}^m \Omega^1_X) = 0$ for all $m \geqslant 0$, so that jet differentials definitely give more information than ordinary symmetric ones.

We need to calculate the leading term of $\chi(\mathcal{J}_{k,m})$ for a variety of dimension n. First,

$$\mathrm{ch}(\mathcal{J}_{k,m}) = \sum_{i_1 + 2i_2 + \cdots + k i_k = m} \mathrm{ch}\big(\mathrm{Sym}^{i_1}\Omega^1_X \otimes \mathrm{Sym}^{i_2}\Omega^1_X \otimes \cdots \otimes \mathrm{Sym}^{i_k}\Omega^1_X\big)$$

(with all i's integral and $\geqslant 0$), as $\mathcal{J}_{k,m}$ has a composition series involving exactly these sheaves. If

$$c(\Omega^1_X) = (1+\lambda_1)(1+\lambda_2)\cdots(1+\lambda_n)$$

formally, then

$$\operatorname{ch}(\operatorname{Sym}^i\Omega_X^1) = \sum_{x_1+\cdots+x_n=i} e^{x_1\lambda_1+\cdots+x_n\lambda_n}$$

(with all x's integral and $\geqslant 0$). So

$$\operatorname{ch}(\mathcal{J}_{k,m}) = \sum_{\substack{x_{11}+\cdots+x_{1n}+2(x_{21}+\cdots+x_{2n})\\ +\cdots+k(x_{k1}+\cdots+x_{kn})=m}} \exp\{(x_{11}+\cdots+x_{k1})\lambda_1+\cdots$$

$$+(x_{1n}+\cdots+x_{kn})\lambda_n\}$$

$$= \sum_{\substack{\frac{x_{11}}{m}+\cdots+\frac{x_{1n}}{m}\\ +2(\frac{x_{21}}{m}+\cdots+\frac{x_{2n}}{m})\\ +\cdots+k(\frac{x_{k1}}{m}+\cdots+\frac{x_{kn}}{m})\\ =1}} \exp\left\{m\left(\left(\frac{x_{11}}{m}+\cdots+\frac{x_{k1}}{m}\right)\lambda_1+\cdots + \left(\frac{x_{1n}}{m}+\cdots+\frac{x_{2n}}{m}\right)\lambda_n\right)\right\}$$

(with all x's integral and $\geqslant 0$). This can be approximated by an integral modulo lower-order terms, so

$$\operatorname{ch}(\mathcal{J}_{k,m}) = m^{kn-1} \int\cdots\int_{\substack{y_{11}+\cdots+y_{1n}+2(y_{21}+\cdots+y_{2n})\\ +\cdots+k(y_{k1}+\cdots+y_{kn})=1}} e^{m((y_{11}+\cdots+y_{k1})\lambda_1+\cdots+(y_{1n}+\cdots+y_{kn})\lambda_n)}\,d\omega$$

$$+O(m^{(k+1)n-2})$$

(with all y's $\geqslant 0$), where $d\omega$ is the element of area $dy_{12}\,dy_{13}\cdots dy_{kn}$ pulled back to the hyperplane $y_{11}+\cdots+y_{1n}+\cdots+k(y_{k1}+\cdots+y_{kn})=1$ by the projection map. Recalling that our exponential is purely formal and represents a polynomial of degree n, we have

$$\operatorname{ch}(\mathcal{J}_{k,m}) = m^{kn-1} \int\cdots\int_{\substack{y_{11}+\cdots+y_{1n}+2(y_{21}+\cdots+y_{2n})\\ +\cdots+k(y_{k1}+\cdots+y_{kn})=1}} \frac{m^n}{n!}((y_{11}+\cdots+y_{k1})\lambda_1$$

$$+\cdots+(y_{1n}+\cdots+y_{kn})\lambda_n)^n\,d\omega + O(m^{(k+1)n-2})$$

(with all y's $\geqslant 0$). By a substitution,

$$\operatorname{ch}(\mathcal{J}_{k,m}) = \frac{m^{(k+1)n-1}}{n!(k!)^n} \times \int\cdots\int_{y_{11}+\cdots+y_{kn}=1} \left(\left(y_{11}+\frac{y_{21}}{2}+\cdots+\frac{y_{k1}}{k}\right)\lambda_1\right.$$

$$\left.+\cdots+\left(y_{1n}+\frac{y_{2n}}{2}+\cdots+\frac{y_{kn}}{k}\right)\lambda_n\right)^n d\mu$$

$$+O(m^{(k+1)n-2})$$

(with all y's $\geqslant 0$), where $d\mu$ is $dy_{12} \cdots dy_{kn}$ pulled up by projection. Thus

$$\begin{aligned}\operatorname{ch}(\mathcal{J}_{k,m}) = {}& \frac{m^{(k+1)n-1}}{(k!)^n} \times \int_{y_{11}+\cdots+y_{kn}=1} \cdots \int \sum_{q_1+\cdots+q_n=n} \frac{1}{q_1!\cdots q_n!} \\ & \times \left(y_{11} + \frac{y_{21}}{2} + \cdots + \frac{y_{k1}}{k}\right)^{q_1} \cdots \\ & \times \left(y_{1n} + \frac{y_{2n}}{2} + \cdots + \frac{y_{kn}}{k}\right)^{q_n} \lambda_1^{q_1} \cdots \lambda_n^{q_n}\, d\mu \\ & + O(m^{(k+1)n-2})\end{aligned}$$

(with all y's $\geqslant 0$, all q's integral and $\geqslant 0$). Setting

$$\begin{aligned}F_k(q_1, \ldots, q_n) = {}& \int_{y_{11}+\cdots+y_{kn}=1} \cdots \int \left(y_{11} + \frac{y_{21}}{2} + \cdots + \frac{y_{k1}}{k}\right)^{q_1} \cdots \\ & \times \left(y_{1n} + \frac{y_{2n}}{2} + \cdots + \frac{y_{kn}}{k}\right)^{q_n} d\mu \end{aligned} \tag{1.12}$$

(with all y's $\geqslant 0$), we have

$$\begin{aligned}\operatorname{ch}(\mathcal{J}_{k,m}) = {}& \frac{m^{(k+1)n-1}}{(k!)^n} \sum_{q_1+\cdots+q_n=n} \frac{F_k(q_1 \cdots q_n)}{q_1!\cdots q_n!} \lambda_1^{q_1} \cdots \lambda_n^{q_n} \\ & + O(m^{(k+1)n-2})\end{aligned} \tag{1.13}$$

(with all q's integral and $\geqslant 0$).

To evaluate $F_k(q_1, \ldots, q_n)$, we introduce two notations. Let

$$I_{j_1,\ldots,j_r}(p) = \int_{y_1+\cdots+y_p=1} \cdots \int y_1^{j_1} \cdots y_r^{j_r}\, dy_1 \cdots dy_p$$

(with all y's $\geqslant 0$, $r, j_1, \cdots, j_r$ integers $\geqslant 0$, $r \leqslant p$). By calculus,

$$I_{j_1,\ldots,j_r}(p) = \frac{j_1! \cdots j_r!}{(j_1 + \cdots + j_r + p)!}\,.$$

The sums

$$S_{j_1,\ldots,j_r}(k) = \sum \frac{1}{i_1^{j_1} i_2^{j_2} \ldots i_r^{j_r}}\,, \qquad 0 \leqslant j_1 \leqslant j_2 \cdots \leqslant j_r \text{ integers} \tag{1.14}$$

(where the summation is over $i_s < i_t$ if $j_s = j_t$ and $s < t$, with all i's integers $\geqslant 1$ and $\leqslant k$) grow asymptotically like a constant times $(\log k)^\nu$, where ν is the number of j_i's equal to 1. In particular,

$$S_{11\ldots1}(k) \sim \frac{1}{r!}(\log k)^r.$$

Returning to the expression (1.12) for $F_k(q_1, \ldots, q_n)$, we have

$$F_k(q_1, \ldots, q_n) = \sum \frac{q_1!\, q_2! \cdots q_n!}{j_{1,1}! \cdots j_{n,q_n}!} S_{j_{1,1}, \ldots, j_{1,q_1}}(k) \cdots$$

$$\times S_{j_{n,1}, \ldots, j_{n,q_n}}(k) I_{j_{1,1}, \ldots, j_{n,q_n}}(kn-1) \tag{1.15}$$

(where in the summation $j_{\nu,1} + \cdots + j_{\nu,q_\nu} = q_\nu$, $0 \leqslant j_{\nu,1} \leqslant j_{\nu,2} \leqslant \cdots \leqslant j_{\nu,q_\nu}$ for $\nu = 1, \ldots, n$, all j's integers). So

$$F_k(q_1, \ldots, q_n) = \sum \frac{q_1! \cdots q_n!}{((k+1)n-1)!} S_{j_{1,1}, \ldots, j_{1,q_1}}(k) \cdots S_{j_{n,1}, \ldots, j_{n,q_n}}(k) \tag{1.16}$$

with the same conditions on the summation indices. Thus

$$F_k(q_1, \ldots, q_n) = \frac{1}{((k+1)n-1)!} (\log k)^n + O\big((\log k)^{n-1}\big)$$

since the only way $(\log k)^n$ can occur is when all of the j's are 1. So

$$\begin{aligned}
\operatorname{ch}(\mathcal{J}_{k,m}) &= \frac{m^{(k+1)n-1}}{(k!)^n((k+1)n-1)!} (\log k)^n \sum_{\substack{q_1 + \cdots + q_n = n \\ q\text{'s integers} > 0}} \frac{\lambda_1^{q_1} \cdots \lambda^{q_n}}{q_1! \cdots q_n!} \\
&\quad + O\big((\log k)^{n-1}\big) + O(m^{(k+1)n-2}) \\
&= \frac{m^{(k+1)n-1}}{(k!)^n((k+1)n-1)!} \\
&\quad \times \left(\frac{(-1)^n}{n!} c_1(X)^n (\log k)^n + O\big((\log k)^{n-1}\big) \right) \\
&\quad + O(m^{(k+1)n-2}).
\end{aligned}$$

By the Hirzebruch Riemann–Roch theorem,

$$\begin{aligned}
\chi(\mathcal{J}_{k,m}) &= \frac{m^{(k+1)n-1}}{(k!)^n((k+1)n-1)!} \\
&\quad \times \left(\frac{(-1)^n}{n!} c_1(x)^n (\log k)^n + O\big((\log k)^{n-1}\big) \right) \\
&\quad + O(m^{(k+1)n-2}),
\end{aligned} \tag{1.17}$$

which proves the desired result on the leading term of $\chi(\mathcal{J}_{k,m})$.

Returning to the explicit formula (1.16) for $F_k(q_1, \ldots, q_n)$, we can calculate the leading term explicitly for low dimensions. The leading terms are the following combinations of Chern numbers, where c_i denotes $c_i(\Omega^1_X)$:

$$n = 1, \qquad \frac{1}{(k!)^2} c_1,$$

$$n = 2, \qquad \frac{1}{(k!)^2(2k+1)!}\big((S_{1,1}(k) + S_2(k))c_1^2 - S_2(k)c_2\big),$$

$$n = 3, \qquad \frac{1}{(k!)^3(3k+2)!}\big((S_{1,1,1}(k) + S_{2,1}(k) + S_3(k))c_1^3$$

$$+ (S_{2,1}(k) - 2S_3(k))c_1c_2 + (S_3(k) - S_{2,1}(k))c_3\big).$$

In particular, for surfaces the leading term is

$$k = 1, \qquad \frac{1}{3!}(c_1^2 - c_2),$$

$$k = 2, \qquad \frac{1}{4^3 \cdot 3!}(7c_1^2 - 5c_2),$$

$$k = 3, \qquad \frac{1}{6^5}(85c_1^2 - 49c_2).$$

For surfaces of general type, we have the result of Bogomolov [2]:

(1.18) *If a section of*

$$H^0\big(X, \mathrm{Sym}^{i_1}\Theta_X \otimes \mathrm{Sym}^{i_2}\Theta_X \otimes \cdots \otimes \mathrm{Sym}^{i_k}\Theta_X \otimes K^{(i_1 + \cdots + i_k)/2}\big),$$
$$i_1 + \cdots + i_k \textit{ even},$$

vanishes at a point of X, *it vanishes identically*.

Thus if $i_1 + \cdots + i_k$ is even and $q < (i_1 + \cdots i_k)/2$,

$$H^0(X, \mathrm{Sym}^{i_1}\Theta_X \otimes \cdots \otimes \mathrm{Sym}^{i_k}\Theta_X \otimes K_X^q) = 0.$$

By squaring, we see we may drop the hypothesis that $i_1 + \cdots + i_k$ is even. Then using Serre duality,

$$H^2(X, \mathrm{Sym}^{i_1}\Omega^1_X \otimes \cdots \otimes \mathrm{Sym}^{i_k}\Omega^1_X) = 0 \quad \text{for } i_1 + \cdots + i_k > 2. \tag{1.19}$$

As $\mathcal{J}_{k,m}$ has a composition series involving $\mathrm{Sym}^{i_1}\Omega^1_X \otimes \cdots \otimes \mathrm{Sym}^{i_k}\Omega^1_X$ with $i_1 + 2i_2 + \cdots + ki_k = m$, we infer that

(1.20) $H^2(X, \mathcal{J}_{k,m}) = 0$ *for* $m > 2k$, $k \geqslant 1$, *and for* X *a surface of general type*.

Since

$$X(\mathcal{J}_{k,m}) = h^0(X, \mathcal{J}_{k,m}) - h^1(X, \mathcal{J}_{k,m}) + h^2(X, \mathcal{J}_{k,m}),$$

we conclude that for a minimal surface of general type

$$h^0(X, \mathcal{J}_{k,m}) \geqslant Am^{n(k+1)-1} + O(m^{n(k+1)-2}), \qquad A > 0,$$

for k sufficiently large.

From a result of Iitaka [18], it follows that

(1.21) *For X a surface of general type, if*

$$\left(\sum_{1 \leqslant i < j \leqslant k} \frac{1}{ij} + \sum_{1 \leqslant i \leqslant k} \frac{1}{i^2} \right) c_1^2 - \left(\sum_{1 \leqslant i \leqslant k} \frac{1}{i^2} \right) c_2 > 0,$$

then

$$\phi_m : P_k(X) \to \mathbb{P}_N$$

is birational to its image for m sufficiently large.

Less specifically, the hypothesis on the Chern classes always holds for k sufficiently large. This follows because $c_1^2(\Omega_X^1) > 0$ for a minimal surface of general type.

(d) Examples

(1) *Smooth hypersurfaces in $\mathbb{P}_n$.* Let X be a smooth hypersurface in $\mathbb{P}_n$ of degree d. The main facts are:

(1.22) $H^0(X, \mathrm{Sym}^k\, \Omega^1) = 0$ *for all $k \geqslant 1$ if $n \geqslant 3$.*

(1.23) $\phi_{\mathcal{J}_{2,m}}$ *is a birational embedding for m sufficiently large, for X a surface and* $d \geqslant 16$.

To see (1.22), which is due to F. Sakai [28], begin with the exact sequence

$$0 \to \mathcal{O} \to \bigoplus_{n+1} \mathcal{O}(1) \to \Theta_{P_n} \to 0$$

and its analogue

$$0 \to \bigoplus_{\binom{n+k-1}{k-1}} \mathcal{O}(k-1) \to \bigoplus_{\binom{n+k}{k}} \mathcal{O}(k) \to \mathrm{Sym}^k\, \Theta_{P_n} \to 0.$$

Dualizing,

$$0 \to \mathrm{Sym}^k \Omega_{P_n}^1 \to \bigoplus_{\binom{n+k}{k}} \mathcal{O}(-k) \to \bigoplus_{\binom{n+k-1}{k-1}} \mathcal{O}(1-k) \to 0.$$

Thus

$$H^i\big(X, \mathrm{Sym}^k\, \Omega_{P_n}^1\big|_X \otimes \mathcal{O}(l)\big) = 0$$

unless either

(1) $i = 0$, $l - \mathrm{k} \geqslant 0$, or
(2) $i = \mathrm{n} - 1$, $d - (n+1) + k - l \geqslant 0$.

From this and the sequence

$$0 \to \mathrm{Sym}^{k-1}\, \Omega_{P_n}^1|_X \otimes \mathcal{O}(-d) \to \mathrm{Sym}^k\, \Omega_{P_n}^1|_X \to \mathrm{Sym}^k\, \Omega_X^1 \to 0$$

we conclude

$$H^i\big(X,\ \mathrm{Sym}^k\, \Omega_X^1 \otimes \mathcal{O}(l)\big) = 0$$

unless either

(1) $i = 0$, $l - k \geqslant 0$,
(2) $i = n - 2$, $k - (n + 2) - l \geqslant 0$, or
(3) $i = n - 1$, $d - (n + 1) + k - l \geqslant 0$.

For $l = 0$ we conclude

$$H^0(X, \operatorname{Sym}^k \Omega_X^1) = 0.$$

To see (1.23), the exact sequence

$$0 \to \Theta_X \to \Theta_{P_n}|_X \to \mathcal{O}(d) \to 0$$

implies

$$(1 + c_1(\Theta_X) + c_2(\Theta_X))(1 + dH) = (1 + H)^4,$$

where H is the hyperplane class. Thus

$$c_1^2(\Theta_X) = (d - 4)^2 d,$$
$$c_2(\Theta_X) = (d^2 - 4d + 6)d.$$

Therefore

$$c_1^2(\Omega_X^1) - c_2(\Omega_y^1) = (10 - 4d)d,$$
$$7c_1^2(\Omega_X^1) - 5c_2(\Omega_X^1) = 2(d^2 - 18d + 41)d.$$

Thus $\phi_{\mathcal{J}_{1,m}}$ cannot be a birational embedding—indeed, we have seen there are no symmetric differentials—while $\phi_{\mathcal{J}_{2,m}}$ is a birational embedding for large m when $d \geqslant 16$.

(2) *Subvarieties of Abelian Varieties*. These will be discussed at length in Section 3 when we give the proof of Bloch's conjecture. Here we will merely assert without proof that:

(1.24) *For $X_n \subset A_N$, if $k \geqslant n/(N - n)$ and X is not ruled by subtori, then $P_k(X) \xrightarrow{\phi_{L^m}} \mathbb{P}_M$ is a birational embedding for m sufficiently large.*

For X a smooth surface, from the formula

$$(1 + c_1(X) + c_2(X))(1 + c_1(N_X) + c_2(N_X)) = 1$$

we conclude that

$$c_1^2(X) - c_2(X) = 0, \qquad N = 3,$$

and with a little geometry that

$$c_1^2(X) - c_2(X) > 0, \qquad N > 3.$$

Thus, for $X_2 \subset A_3$, 1-jets are not enough, while 2-jets are. For $X_2 \subset A_N$, $N > 3$, 1-jets are enough.

2. Metrics of Negative Curvature from Jet Differentials

(a) The Ahlfors lemma. The Ahlfors lemma is central to differential-geometric methods of studying holomorphic mappings, recurrently surviving all changes in viewpoint. We will use a variant of it here.

Definition. Let X be a complex space. A *jet pseudometric* is given by a function

$$| \ | : J_k(X) \to \mathbb{R}^+$$

that is continuous and smooth except when it is zero, and satisfies

$$|tj| = |t| \cdot |j| \qquad (j \in J_k(X),\ t \in \mathbb{C}^*).$$

Here the action of $\mathbb{C}^*$ on jets is by reparametrization as discussed in Section 1 (a). Intuitively, $| \ |$ assigns a length to kth-order infinitesimal arcs in X. In local coordinates in a neighborhood consisting of smooth points on X a jet pseudometric will be given by

$$|j_k(f)(z)| = F\big(f_1'(z), \ldots, f_n'(z), \ldots, f_1^{(k)}(z), \ldots, f_N^{(k)}(z)\big),$$

where $F(f_1', \ldots, f_n', \ldots, f_1^{(k)}, \ldots, f_N^{(k)})$ is a nonnegative function, smooth except when zero, that satisfies a suitable weighted homogeneity condition.

Definition. The jet pseudometric $| \ |$ *has holomorphic sectional curvatures* $\leqslant -A$ $(A > 0)$ *on discs* if for any holomorphic mapping $f : \Delta \to X$ and point $x \in f(\Delta)$ we have at x either $|j_k(f)| = 0$ or

$$\sqrt{-1}\, \partial\bar{\partial} \log|j_k(f)|^2 \geqslant A|j_k(f)|^2. \tag{2.1}$$

(Compare this definition with Wu [29].)

We remark that, multiplying $| \ |$ by A^{-1}, we may always make the constant in (2.1) to be -1.

Examples

(i) The standard example is the Poincaré metric

$$\rho(z)\,|dz| = \frac{|dz|}{1-|z|^2}\ ;$$

it has constant holomorphic section curvature -1.

(ii) In [10] Grauert and Reckziegel introduced negatively curved Finsler metrics (cf. also Cowen [5]), given by a nonnegative function $F(x, \xi)$ on the tangent bundle satisfying

$$F(x, t\xi) = |t| F(x, \xi) \qquad (\xi \in T_x(X),\ t \in C).$$

A useful remark they made is that the sum of two negatively curved Finsler metrics is again negatively curved; the same is true for jet pseudometrics having holomorphic sectional curvatures $\leqslant -A$.

(iii) The Kobayashi metric $| \ |_\kappa$ is the pseudometric on $J_1(X)$ defined by

$$|\xi|_\kappa = \inf_f \left|\xi / f_*\!\left(\frac{\partial}{\partial z}\right)\right|,$$

where the inf is taken over all holomorphic mappings $f : \Delta \to X$ that satisfy $f(0) = x$ and $f_*(\partial/\partial z)_0 = a\xi$, $a \in \mathbb{C}^*$ (cf. Kobayashi [20]).

If we have $f : \Delta_r \to X$ with $f(0) = x$ and $f_*(\partial/\partial z)_0 = \xi$, then setting $f_r(z) = f(z/r)$ gives $f_r : \Delta \to X$ with $f_r(0) = x$ and $(f_r)_*(\partial/\partial z_0) = r\xi$. It follows that

$|\xi|_\kappa = 0$ in case there is an entire holomorphic curve passing through x in the direction $\xi \in T_x(X)$. For X compact the converse is due to Brody [3].

(iv) Let $H_0, \ldots, H_{n+1}$ be a collection of $n+2$ hyperplanes in general position in $\mathbb{P}^n$. There has been an extensive study of the position of a nondegenerate holomorphic curve in $\mathbb{P}^n$ relative to these hyperplanes (cf. the introductions to Cowen and Griffiths [6] and Green [11]). In particular, for $X = \mathbb{P}^n - \bigcup_{\mu=0}^{n+1} H_\mu$ a classical theorem of E. Borel says that an entire holomorphic curve $f: \mathbb{C} \to X$ must lie in a $\mathbb{P}^{n-1}$. The corresponding defect relations were established by H. Cartan and Ahlfors.

In Cowen and Griffiths [6] there is a proof of these defect relations and Borel's theorem using what amounts to a negatively curved jet pseudometric on X; cf. (6.3) and (6.4) on p. 132 of that paper. Comparing (6.3) and (5.13) one sees that for $n \geqslant 2$ higher derivatives enter in an essential way in this metric. In fact, the pseudometric vanishes at a point in case the curve osculates to high order to a hyperplane at that point, and it vanishes identically exactly when the image $f(\Delta)$ lies in a $\mathbb{P}^{n-1}$.

As with ordinary metrics, the basic fact concerning jet pseudometrics is the

Ahlfors lemma for jet pseudometrics. *On a complex space X we let $|\ |$ be a jet pseudometric that has holomorphic sectional curvatures $\leqslant -1$ on discs. Then any holomorphic mapping $f: \Delta \to X$ is distance decreasing relative to the Poincaré metric; i.e.,*

$$|j_k(f)(z)| \leqslant \rho(z) \tag{2.2}$$

for all $z \in \Delta$.

Proof. If not identically zero, the pseudometric $|j_k(f)(z)|^2 |dz|^2$ has Gaussian curvature $\leqslant -1$ at the points where it does not vanish. The result now follows from the usual Ahlfors lemma (Kobayashi [20]). $\square$

To state a corollary we let $(x, \xi) \in T(X)$ and denote by $J_k(X)_{(x,\xi)}$ the set of all jets $j \in J_k(X)_x$ that project onto ξ; i.e., the linear part of j is ξ.

(2.3) Corollary. *Let $|\ |$ be a jet pseudometric whose holomorphic sectional curvatures on discs are $\leqslant -1$. Then the Kobayashi length satisfies*

$$|\xi|_\kappa \geqslant \inf_{j \in J_k(X)_{(x,\xi)}} |j|.$$

The proof is immediate from the Ahlfors lemma and the definition of $|\ |_\kappa$.

(2.4) Corollary. *Let $|\ |$ be a jet pseudometric on $J_k(X)$ having holomorphic sectional curvatures $\leqslant -1$ on discs. Then if $f: \mathbb{C} \to X$ is an entire holomorphic curve, then*

$$|j_k(x)(z)| \equiv 0.$$

(b) Construction of negatively curved pseudometrics from jet differentials. We will now show how negatively curved jet pseudometrics may be constructed by having enough holomorphic sections of a suitable line bundle.

(2.5) Proposition. *Let X be a projective algebraic variety and $E \to P_k(X)$ a very ample line bundle. If $t_0, \ldots, t_M$ is any basis for $H^0(P_k(X), E)$ and $s_0, \ldots, s_N$ any basis for $H^0(P_k(X), L^m \otimes E^{-1})$, then for a suitable constant $A > 0$ the jet pseudometric*

$$|j|^2 = A\Big(\sum_{i,\alpha} |s_i t_\alpha(j)|^2\Big)^{1/m}$$

has holomorphic sectional curvatures $\leqslant -1$ on discs.

Proof. Let $U \subset P_k(X)$ be an open set over which E and L are trivial, and suppose that $f: \Delta \to X$ is a holomorphic mapping such that $j_k(f)(z) \in U$ for all $z \in \Delta$. Then using these trivializations,

$$t_\alpha(j_k(f))(z) = u_\alpha(z),$$
$$s_i(j_k(f))(z) = v_i(z)$$

are holomorphic functions of z and

$$|j_k(f)(z)|^2 = A\Big(\sum_{i,\alpha} |v_i(z) u_\alpha(z)|^2\Big)^{1/m}.$$

Assuming that $|j_k(f)|^2$ is not identically zero, the (1, 1) form

$$\sqrt{-1}\ \partial\bar\partial \log|j_k(f)(z)|^2 = \frac{\sqrt{-1}}{m}\ \partial\bar\partial \log\big(\sum|v_i(z)|^2\big) + \frac{\sqrt{-1}}{m}\ \partial\bar\partial \log\big(\sum|u_\alpha(z)|^2\big)$$
$$= \frac{\alpha}{m} + \frac{\beta}{m}$$

is intrinsically defined—i.e., does not depend on the trivializations used. Each of the forms α and β is nonnegative; and β has the following geometric interpretation: Let $\phi_E : P_k(X) \to \mathbb{P}^M$ be the projective embedding induced by the sections $t_0, \ldots, t_M$ and

$$\omega = \phi_E^* \qquad \text{(Fubini–Study metric on } \mathbb{P}^M).$$

If we denote by

$$f_k : \Delta \to P_k(X)$$

the canonical lifting of $f: \Delta \to X$ given by $f_k(z) = j_k(f)(z)$, then

$$\beta = f_k^*(\omega). \quad \square$$

Next we need to know that:

(2.6) $\beta(z) = 0 \Leftrightarrow j_{k+1}(f)(z)$ *is a constant jet.*

Proof. We shall prove that the right-hand side is equivalent to the differential of f_k vanishing at z. Taking $z = (0)$ and a local embedding of a neighborhood of

$f(0)$ in X as a subvariety in an open set in C^N, we write

$$f(z) = f^{(j)}(0)\,\frac{z^j}{j!} + f^{(j+1)}(0)\,\frac{z^{j+1}}{(j+1)!} + \cdots$$

where $f^{(j)}(0) \neq 0$. If $j \leqslant k+1$ and if the differential of $j_k(f) \in P_k(X)$ is zero at $z = 0$, then this means that

$$\frac{d}{dz}\,(j_k(f)(z))_{z=0} = t(j_k(f)(0)).$$

The right-hand side is the reparametrization of the jet $j_k(f)(0)$, and all terms of order $\leqslant j$ are zero. But the left-hand side has a nonzero term of order $j-1$, which contradicts our assumption $j \leqslant k+1$.

Now both of the mappings

$$z \to \beta(z),$$

$$z \to |j_k(f)(z)|^2$$

are quadratic with respect to a reparametrization, and consequently the ratio

$$\frac{|j_k(f)(z)|^2}{\rho(z)}$$

is locally bounded from above on the projectivized tangent bundle of $P_k(X)$. Since it is intrinsic and X is compact, this ratio will everywhere be $\leqslant B$ for some constant B. This implies that

$$\sqrt{-1}\,\partial\bar{\partial}\log|j_k(f)|^2 \geqslant \frac{\beta}{m} \geqslant \frac{1}{Bm}\,|j_k(f)|^2$$

for any holomorphic mapping $f: \Delta \to X$. Adjusting constants yields the proposition. □

(2.7) Corollary. *Consider the map*

$$\phi_{L^m} : P_k(X) \to \mathbb{P}^N$$

defined by the linear system $|L^m|$ *on* $P_k(X)$. *Let* $B_{k,m}$ *be the union of the base locus of* ϕ_{L^m} *and the points* $j \in P_k(X)$ *such that* $\dim(\phi_{L^m}^{-1}(\phi_{L^m}(j))) \geqslant 1$. *Then there exists a jet pseudometric on* $P_k(X)$ *with holomorphic sectional curvatures* $\leqslant -1$ *on discs and vanishing at most on* $B_{k,m}$.

Remark. Noguchi [24] has a similar observation in the case of symmetric differentials.

Proof. Let $E \to P_k(X)$ be a very ample line bundle. It will suffice to show that for sufficiently large l, the base of the linear system $|L^{lm} \otimes E^{-1}|$ is contained in $B_{k,m}$.

By blowing up we may assume that the base of $|L^m|$ is a divisor F on the blown-up variety $P_k(X)$, and we set $\tilde{L} = L^m \otimes F^{-1}$. Then

$$\phi_{\tilde{L}} : P_k(X) \to \mathbb{P}^N$$

is a holomorphic mapping that is finite-to-one outside the total transform $\tilde{B}_{k,m}$ of $B_{k,m}$.

Given $j \in P_k(X) - B_{k,m}$, we may choose a divisor $D \in |E|$ such that

$$\phi_{\tilde{L}}^{-1}(\phi_{\tilde{L}}(j)) \notin \tilde{D},$$

where $\tilde{D}$ is the total transform of D. If we choose a hypersurface of sufficiently high degree l in $\mathbb{P}^N$ that passes through $\phi_{\tilde{L}}(\tilde{D})$ but does not contain $\phi_{\tilde{L}}(j)$, then we obtain a section of $\tilde{L}^l - \tilde{D}$ on $P_k(X)$ that does not pass through j. Projecting its divisor down to $P_k(X)$ gives a divisor in $|L^{lm} - D|$ that does not pass through j. $\square$

Combining Corollary 2.4 with Corollary 2.7 gives the

(2.8) Corollary. *Let $f : \mathbb{C} \to X$ be an entire holomorphic curve with canonical lifting $f_k : \mathbb{C} \to P_k(X)$. Then, with the above notation,*

$$f_k(\mathbb{C}) \subset \bigcup_m B_{k,m}.$$

Remark. Observe that the right-hand side of this inclusion is defined purely in terms of the geometry of the linear systems $|L^m|$ on $P_k(X)$. These are in turn described by the jet differentials on X.

3. Proof of Bloch's Conjecture (Theorem I)

(a) Proof of Theorems I and I′. We begin by establishing

Theorem I′. *Let X be an analytic subvariety of a complex torus A. If X is not the translate of a subtorus of A, then any entire holomorphic curve $f : \mathbb{C} \to X$ lies in a proper analytic subvariety of X.*

Remark. By induction, then, the image curve $f(\mathbb{C})$ will lie in a proper subtorus. In this connection, when A is a simple abelian variety an elementary proof of Bloch's conjecture has been given by one of us (cf. Green [12]).

Proof. Writing $A = \mathbb{C}^N/\Lambda$ where Λ is a lattic in $\mathbb{C}^N$ and using the monodromy theorem, we may assume that any holomorphic mapping $f : \Delta_r \to X$ has been lifted to $\mathbb{C}^N$. We shall continue to denote this lifting by f, and remark that it is unique up to translation by a constant vector in Λ. Thus $f(z) = (f_1(z), \ldots, f_N(z))$ where the $f_i(z)$ are holomorphic functions.

We shall also use the notation

$$\begin{aligned} u(j_k(f)) &= (f_1', \ldots, f_N', \ldots, f_1^{(k)}, \ldots, f_N^{(k)}) \\ &= (f_i', \ldots, f_i^{(k)}) \end{aligned}$$

(here, the index i is thought of as running from 1 to n) for the indicated global coordinates on the jet spaces $J_k(X)$. Equivalently, u is the composite map in the diagram

$$\begin{tikzcd} J_k(X) \arrow[r] \arrow[rrr, bend left, "u"] & J_k(A) \cong A \times \mathbb{C}^{kN} \arrow[r] & \mathbb{C}^{kN}. \end{tikzcd}$$

If we take any basis $P_0, \ldots, P_M$ for all polynomials with constant coefficients in the indeterminates $f_1', \ldots, f_N'; \ldots; f_1^{(k)}, \ldots, f_N^{(k)}$ that are homogeneous of total weight m when $f_i^{(l)}$ is assigned weight l, then the P_α form part of a basis for

$$H^0(P_k(X), L^m) = H^0(X, \mathcal{J}_{k,m}).$$

Using these to define a mapping U_m to projective space, we will have a diagram

$$\begin{tikzcd} & \mathbb{P}^{M'} \arrow[d] \\ P_k(X) \arrow[ur, "\phi_{L^m}"] \arrow[r, "U_m"'] & \mathbb{P}^M \end{tikzcd}$$

where ϕ_{L^m} is the mapping defined by the complete linear system $|L^m|$ and the vertical arrow is a linear projection. For $m = k!$ the mapping U_m involves $(f_i^{(l)})^{m/l}$ for all i and $l \leqslant k$, and consequently U_m has no base locus. Moreover, for this same m

$$\begin{aligned} U_m(j_1) = U_m(j_2) \quad &\Leftrightarrow \quad u(j_1) = u(j_2) \\ &\Leftrightarrow \quad j_1 = j_2 + a, \end{aligned}$$

where $j_1, j_2 \in J_k(X)$ and $j_2 + a$ denotes the translation of j_2 by $a \in \mathbb{C}^N$. Summarizing: *If the jet j lies in the subvariety $B_{k,k!}$ (cf. Section 2), then*

$$\dim\{a \in A : j \in J_k(X) \cap J_k(X + a)\} \geqslant 1. \tag{*}$$

Now let $f: \mathbb{C} \to X$ be an entire holomorphic curve. By Corollaries 2.7 and 2.8 to Proposition 2.5, $j_k(f) \in B_{k,l}$ for all k and l. We define the sequence of complex-analytic varieties

$$V_k(f) = \{a \in A : j_k(f)(0) \in J_k(X) \cap J_k(X + a)\}.$$

These form a nested sequence

$$V_1(f) \subseteq V_2(f) \subseteq V_3(f) \subseteq \cdots$$

that eventually stabilizes at a variety V. By power series,

$$a \in V \quad \Leftrightarrow \quad f(\mathbb{C}) \subseteq X \cap (X + a).$$

On the other hand, by (*) above

$$\dim V = \dim\{a \in A : f(\mathbb{C}) \subseteq X \cap (X + a)\} \geqslant 1.$$

Now, either $X \cap (X + a)$ is a proper analytic subvariety of X for some $a \in A - \{0\}$—in which case we are done—or else

$$X = X + a \quad \text{for all } a \in V.$$

Assuming this alternative holds, we note that

$$\{a \in A : X = X + a\} = B$$

is a subgroup of A that must have positive dimension; in this case we shall say that X *is ruled by subtori*.

Letting $B^0 \subset A$ be the identity component of the group B and setting $\bar{A} =$

A/B^0, we have a diagram of entire holomorphic mappings

$$\begin{tikzcd} & X \subset A \arrow[d, "\pi"] \\ \mathbb{C} \arrow[ur, "f"] \arrow[r, "\bar{f}"'] & \bar{X} \subset \bar{A} \end{tikzcd}$$

where $\bar{X}$ is not ruled by subtori. Applying the argument thus far, if $\bar{X}$ is not a point, then $\bar{f}(\mathbb{C})$ lies in a proper analytic subvariety $\bar{Z}$ of $\bar{X}$, $f(\mathbb{C})$ lies in $Z = \pi^{-1}(\bar{Z})$, and we are done. If $\bar{X}$ is a point, then X is a subtorus, and this contradicts our initial assumption. □

Theorem I is an easy consequence of Theorem I′. If X is any algebraic variety with irregularity $q > \dim X$, then we denote by A the Albanese variety of X and by

$$\alpha : X \to A$$

the standard map. Setting $\alpha(X) = Y$, Y is not a subtorus of A, and consequently the image of

$$\alpha \circ f : \mathbb{C} \to Y$$

lies in a proper subvariety of Y. The same must be true of the image $f(\mathbb{C}) \subset X$.

Remark. The same argument applies whenever X is a compact Kähler manifold or a Moisezon space.

(b) Some remarks on analytic subvarieties of complex tori. We want to make some general observations about subvarieties of abelian varieties, and then in the following section shall give some related remarks on how our proof compares with the argument of Bloch and Ochiai.

Given an analytic subvariety $X \subset A$ of an abelian variety, the most obvious way to study its geometry is via its *Gauss mapping*

$$\gamma : X \to G(n, N). \tag{3.1}$$

Here, γ is defined at the smooth points of X by

$$\begin{aligned} \gamma(x) &= T_x(X) \\ &= \text{translate to the origin of the tangent space at } x \in X, \end{aligned}$$

and is then extended to all of X as a natural mapping (see Griffiths [14] for further discussion). Actually, for the purposes of this discussion the singularities of X are not that essential, so the reader may either assume X is smooth or replace X by its Nash blowup on which γ is everywhere defined [14]. By definition

$$T^*(X) = \gamma^* E,$$

where $E \to G(n, N)$ is the *dual* of the universal subbundle.

The first step is to analyze the case when γ is degenerate in the sense that $\dim \gamma(X) < \dim X$. In this regard there is a classical structure theorem which may be found, e.g., in Section 4 of [15]:

(3.2) *Given $X \subset A$, we may pass to a finite unramified covering of A and make a translation to have*

$$A = A' \times A'',$$
$$X = A' \times X'',$$

where A', A'' are abelian subvarieties of A, $X'' \subset A''$ is an analytic subvariety whose Gauss mapping is nondegenerate, and the Gauss mapping of X has fibres $A' \times \{x''\}$.

Briefly, the fibres of γ are translates of abelian subvarieties that give a ruling of X.

We should like to make two further observations, of which the first is this:

(3.3) *For an n-dimensional subvariety $X \subset A$ of an abelian variety, the Kodaira number*

$$\kappa(X) = n$$

if, and only if, the Gauss mapping of X is nondegenerate.

In (3.3) X is assumed to be irreducible, but it may have singularities. We observe that we have a diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[dr, "\phi_K"'] & G(n,N) \arrow[r, "p"] & \mathbb{P}^{\binom{N}{n}-1} \\ & \mathbb{P}^{p_g-1} \arrow[ur, "\pi"'] & \end{tikzcd}$$

where p is the Plücker embedding, ϕ_K is the canonical map of X ($p_g = h^{n,0}(X)$), and π is a linear projection. From this it follows that

$$\dim \gamma(X) = n \quad \Rightarrow \quad \kappa(X) = n,$$

and the converse is provided by the structure theorem (3.2).

Our second remark is that we have always taken A to be an abelian variety as opposed to just a complex torus. There is no particular reason for this, and there is also no essential loss of generality, because of the following:

(3.4) *Suppose $X \subset A$ is an analytic subvariety of a complex torus that is not contained in a subtorus, and assume that X is not ruled by subtori. Then A is an abelian variety.*

Proof. If X is not ruled by subtori, then the Gauss map of X is nondegenerate. Since there is an equidimensional mapping of X to the projective algebraic variety $\gamma(X)$, X is a Moisezon space. The Albanese variety $\mathrm{Alb}(X)$ is then an abelian variety, and there is a diagram of holomorphic mappings

$$\begin{tikzcd} & \mathrm{Alb}(X) \arrow[d] \\ X \arrow[ur] \arrow[r] & A \end{tikzcd}$$

where the vertical arrow is surjective, since X is not contained in an abelian subvariety. Being a quotient of $\mathrm{Alb}(X)$, A must be an abelian variety. □

Returning to our general discussion, we assume that the Gauss mapping (3.1) is equidimensional. Then one might think that $T^*(X)$, being generated by its global sections and pulled back from the universal bundle by a nondegenerate mapping, might be close to being ample. For example, we might hope that Sakai's λ-invariant (cf. [28])

$$\lambda = \operatorname{Tr}\deg\Big\{\bigoplus_{m>0} H^0(X, \mathrm{Sym}^m \Omega_X^1)\Big\} - n$$

would achieve its maximum possible value $\max(n, N-n)$. In this regard we first observe the

(3.5) Lemma. *The universal bundle $E \to G(n,N)$ is not ample if $n \geqslant 2$. The transcendence degree of $\bigoplus_{m\geqslant 0} H^0(G(n,N); \mathrm{Sym}^m E)$ is N.*

Proof. We denote by $P = P(E^*)$ the projective bundle of hyperplanes in E, by $\mathcal{O}_P(1)$ the tautological line bundle over P, and recall that

$$\begin{aligned} H^0(P, \mathcal{O}_P(m)) &\cong H^0(G(n,N), \mathrm{Sym}^m E) \\ &\cong \mathrm{Sym}^m(\mathbb{C}^{N*}). \end{aligned} \tag{3.6}$$

Moreover, by definition E is ample if, and only if, $\mathcal{O}_P(1)$ is ample on P.

We may realize P as the subvariety of $G(n,N) \times \mathbb{P}^{N-1}$ defined by incidence:

$$P = \{(\Lambda, p) : p \in \Lambda\} \subset G(n,N) \times \mathbb{P}^{N-1}. \tag{3.7}$$

Since $H^0(\mathcal{O}_P(1)) = \mathbb{C}^{N*}$, the mapping given by the complete linear system $|\mathcal{O}_P(1)|$ is projection on the second factor in

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & \mathbb{P}^{N-1} \\ {\scriptstyle \pi_1}\downarrow & & \\ G(n,N) & & \end{array}$$

This mapping is everywhere defined and has fibres

$$\begin{aligned} \pi_2^{-1}(p) &= \{\Lambda : p \in \Lambda\} \\ &\cong G(n-1, N-1). \end{aligned}$$

It follows that

$$\mathcal{O}_P(1) = \pi_2^* \mathcal{O}_{\mathbb{P}^{N-1}}(1)$$

cannot be ample if $n \geqslant 2$. By (3.6),

$$\begin{aligned} \bigoplus_{m>0} H^0(\mathcal{O}_P(m)) &\cong \bigoplus_{m>0} \mathrm{Sym}^m(\mathbb{C}^N)^* \\ &= C[z_1, \ldots, z_N] \end{aligned}$$

has transcendence degree N, and for $n \geqslant 2$

$$N < n(N-n) + n = \dim P + 1. \quad \square$$

Because of the lemma we have the possibility that $T^*(X)$ may not be ample, even if γ is an embedding. To determine what λ is we consider the tangential variety

$$\tau(X) \subset \mathbb{P}^{N-1},$$

defined to be the union of the projectivized tangent spaces $\mathbb{P}T_x(X)$ as x varies over X. Alternatively, in the diagram

$$\begin{tikzcd} P_1(X) \arrow[r, "\gamma^*"] \arrow[d] \arrow[rr, bend left, "\tau"] & P \arrow[r, "\pi_2"] \arrow[d] & \mathbb{P}^{N-1} \\ X \arrow[r, "\gamma"] & G(n,N) & \end{tikzcd}$$

the tangential variety is the image of $\tau = \pi_2 \circ \gamma^*$. It follows that

$$\lambda = \dim \tau(X) - n + 1,$$

so all we can easily say is that $\lambda \geqslant 1$. In general the behavior of $\tau(X)$ is not well understood, especially when $\dim X \geqslant 4$ (cf. Section 5 of Griffiths and Harris [15]), but in any case $T^*(X)$ cannot be ample when $\operatorname{codim} X < \dim X$.

(c) Some observations about jet differentials associated to subvarieties of abelian varieties. Our approach to Theorem I′ differs from Bloch and Ochiai's in two respects: we substitute negative-curvature arguments for Nevanlinna theory, and we make a different geometric computation.

In fact these are related. The most naive way to use negative curvature is via the observation that holomorphic sectional curvatures decrease on submanifolds. The flat Euclidean metric on an abelian variety $A = \mathbb{C}^N/\Lambda$ induces on any subvariety $X \subset A$ a metric whose holomorphic sectional curvatures $K(\xi)$ are $\leqslant 0$ (here $\xi \in T_x(X)$ is a tangent vector). The condition $K(\xi) < 0$ is closely related to the tangential variety having dimension $2n-1$. More precisely, from Section 4 of Griffiths and Harris [15] we have:

(3.8) *If* $\mathrm{II}(\xi, \eta)$ *denotes the 2nd fundamental form of* $X \subset A$, *then for* $\xi \in T(X)$

$$\mathrm{II}(\xi,\xi) = 0 \quad \Leftrightarrow \quad K(\xi) = 0.$$

On the other hand, if $\mathrm{II}(\xi,\xi) \neq 0$ for every nonzero tangent vector, then the linear system $|\mathrm{II}|$ has no base points, and it follows from Section 5 of [15] that the tangential mapping

$$\tau : P_1(X) \to \mathbb{P}^{N-1} \tag{3.9}$$

is equidimensional.

However, even if τ is equidimensional, it may have a branch locus or, worse still, blow down a subvariety of $P_1(X)$. For example, if C_1, C_2 are curves in A and we consider the translation-type surface

$$X = C_1 + C_2 = \{ p_1 + p_2 \mid p_1 \in C_1, p_2 \in C_2 \},$$

then for each $(p,\xi) \in T(C_1)$, the curve $E = \{(p,\xi) \times (q,0) \mid q \in C_2\}$ in $P(X)$ collapses to a point under τ. In fact, the union of such E's projects down onto all of X. The main hitch in completing Bloch's argument was to get these blown-down varieties under control.

Moreover, in general the tangential mapping (3.9) will not be equidimensional (e.g., if $\operatorname{codim} X < \dim X$), and apparently the conclusion to be drawn is that it is necessary to go to higher-order jets to detect the geometry necessary to force negative curvature.

As mentioned in the introduction, jets already appeared in the original paper of Bloch in 1926, as well as in the work of Ochiai [25]. Their main computation centered on determining the branch locus of the mappings

$$u_k : J_k(X) \to \mathbb{C}^{kN} \tag{3.10}$$

that we encountered in the proof of Theorem I′. (We remark that we only needed the exceptional locus of u_k, and not the full branch locus.) We have found a geometric interpretation of their computation that may illuminate what is going on. Given a line L through the origin in $\mathbb{C}^N$, we define the Schubert cycle

$$\Sigma_L = \{\Lambda \in G(n,N) : L \subset \Lambda\}.$$

If we consider the Gauss mapping

$$\gamma : X \to G(n,N),$$

then we note that

$\gamma^{-1}(\Sigma_L) =$ projection to X of the fibre $\tau^{-1}(L)$ of the tangent mapping (3.9).

Now, rather than stop with the Gauss mapping alone, we extend to k-jets to obtain

$$\gamma_k : J_k(X) \to J_k(G(n,N)). \tag{3.11}$$

The main computation in Ochiai [25] may be expressed by saying that the branch locus B_k of the mapping (3.11) satisfies

$$B_k \subseteq \bigcup_{L \in \mathbb{P}^{N-1}} \gamma_k^{-1}(J_k(\Sigma_L)) \cup \pi_k^{-1}(X_{\text{sing}}),$$

where $\pi_k : J_k(X) \to X$ is the projection. As a consequence we have:

(3.12) *For a holomorphic mapping $f : \Delta \to X$ that satisfies $j_k(f)(z) \in B_k$ for all k and $z \in \Delta$, one of the following alternatives must hold*:

$$f(\Delta) \subset X_{\text{sing}},$$

$$f(\Delta) \subset \gamma^{-1}(\Sigma_L) \quad \textit{for some } L \in \mathbb{P}^{N-1}.$$

Using the interpretation (3.12), we may complete Ochiai's argument. Alternatively, we may use the jet forms to construct a negatively curved jet pseudometric on $J_k(X) - B_k$, which is the approach we have followed in this paper.

Part B. Measure Hyperbolic Algebraic Surfaces

4. Proof of Theorem II

(a) Reduction to the K3 case. We first recall the definition of the Kobayashi–Eisenman intrinsic volume form Ψ defined on any n-dimensional complex analytic variety X (cf. Kobayashi [20]). Let

$$\mu = \prod_{j=1}^{n} \frac{4\sqrt{-1}\, dz_j\, d\bar{z}_j}{\left(1 - |z_j|^2\right)^2}$$

denote the Poincaré volume form on the unit polycylinder Δ^n in $\mathbb{C}^n$. Given a smooth point $x \in X$, we consider all holomorphic mappings

$$f : \Delta^n \to X$$

that satisfy $f(0) = x$ and $J_f(0) \neq 0$, where $J_f = \Lambda^n f_*$ is the Jacobian determinant of f. Then by definition

$$\Psi(x) = \inf_f (f^{-1})^* \mu.$$

To better understand this definition, if $\Psi(x) = 0$, then we must have a sequence

$$f_k : \Delta^n \to X$$

of holomorphic mappings that satisfy

$$f_k(0) = x, \qquad |J_{f_k}(0)| \geqslant k.$$

If we let

$$\Delta^n(k, 1) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| < k, |z_2| < 1, \ldots, |z_n| < 1\}$$

and replace z_1 by z_1/k, then we obtain a sequence of holomorphic mappings

$$g_k : \Delta^n(k, 1) \to X$$

satisfying

$$g_k(0) = x, \qquad |J_{g_k}(0)| \geqslant 1.$$

The analytic variety X is said to be *measure hyperbolic* in case Ψ is positive outside a proper subvariety E of X. In this case, for any point $x \in X - E$ there is an upper bound on the size of polydiscs $\Delta^n(k, 1)$ that can be mapped into X sending the origin to x and having Jacobian $\geqslant 1$ there. In particular:

(4.1) *If there is a holomorphic mapping*

$$f : \Delta^n(\infty, 1) \to X$$

whose image contains a Zariski open subset (*actually, any open set will do*), *then X fails to be measure hyperbolic.*

Turning to the conjecture of the introduction, we may assume that X is a smooth projective variety and recall that X is said to be of *general type* in case, for some $m > 0$, the rational map

$$\phi_{mK} : X \to \mathbb{P}^N$$

defined by the pluricanonical system $|mK_X|$ is equidimensional—i.e, the image $\phi_{mK}(X)$ is an n-dimensional algebraic subvariety of $\mathbb{P}^N$. Equivalently, the canonical ring

$$\bigoplus_{m \geqslant 0} H^0(mK_X)$$

should have maximal transcendence degree $n + 1$. It is known that

$$X \text{ general type} \quad \Rightarrow \quad X \text{ measure hyperbolic}$$

(see Griffiths [13] for the case $m = 1$, and Kobayashi and Ochiai [21] for the extension of this idea to the general situation).

For any variety X the *Kodaira number* $\kappa = \kappa(X)$ is defined to be the maximal dimension of the pluricanonical images $\phi_{mK}(X)$ $(m > 0)$. For algebraic surfaces that contain no exceptional curve of the 1st kind the classification theorem (Griffiths and Harris [17, p. 590]) gives the following list:

(a) $\kappa = -1 \Rightarrow X$ is $\mathbb{P}^2$ or is ruled by $\mathbb{P}^1$'s.
(b) $\kappa = 0 \Rightarrow$ (i) X is a $K3$ surface if $q = 0$ and $p_g = 1$,
(ii) X is an Enriques surface if $q = p_g = 0$,
(iii) X is a hyperelliptic surface if $q = 1$,
(iv) X is an abelian surface if $q = 2$.
(c) $\kappa = 1 \Rightarrow X$ is an elliptic surface.
(d) $\kappa = 2 \Rightarrow X$ is of general type.

The surfaces of class (a) and (b)(iv) clearly fail to be measure hyperbolic (cf. (4.1)—in these cases there is a nondegenerate mapping of all of $\mathbb{C}^2$ to X). To treat the remaining ones we shall utilize the

(4.2) Lemma. *If on a surface X there is an algebraic family consisting of ∞^1 algebraic curves whose general member is either rational or elliptic, then X fails to be measure hyperbolic.*

Remark. We do *not* require that the general curve E in our family should be smooth—to say that E is rational or elliptic means that the genus of its normalization should be zero or one.

Proof. We may describe these curves as being a family $\{E_t\}_{t\in B}$ where B is an algebraic parameter curve. If a generic E_t is rational, then there is a finite covering $\tilde{B}$ of B and a surjective rational mapping

$$\tilde{f} : \tilde{B} \times \mathbb{P}^1 \to X. \tag{4.3}$$

By deleting the finite set of points Z in $\tilde{B}$ over which $\tilde{f}$ may not be defined as a holomorphic mapping and setting

$$B^* = \tilde{B} - Z, \qquad f^* = \tilde{f}|_{B^*},$$

we arrive at a holomorphic mapping

$$f^* : B^* \times \mathbb{P}^1 \to X$$

whose image is Zariski open in X. Finally, passing to the universal covering of B^* gives a mapping of $C \times \mathbb{P}^1$ $(C = \Delta, C,$ or $\mathbb{P}^1)$ to X with Zariski dense image, and we may apply (4.1).

If a generic E_t is elliptic, then removing a finite set Z from B we may assume that for each $t \in B^* = B - Z$ the curve E_t is irreducible and the normalization $\tilde{E}_t$ is a compact Riemann surface of genus one. We also assume that the universal covering of B^* is the disc Δ and the E_t have nonconstant j-invariant—otherwise the argument is similar but easier. The covering $\Delta \to B^*$ will be denoted by $z \to t(z)$, and we may then write

$$\tilde{E}_{t(z)} = \mathbb{C}/\Lambda_z$$

where Λ_z is a holomorphically varying lattice in C. More precisely, enlarging Z

to include all t where E_t has automorphisms, we may lift the mapping

$$B^* \to \{\text{moduli of elliptic curves}\}$$

to a holomorphic mapping

$$\Delta \xrightarrow{\tau} \{\text{upper half plane}\}$$

such that

$$\Lambda_z = \{m + n\tau(z)\}_{m,\, n \in Z}.$$

We then obtain a holomorphic mapping

$$f : \Delta \times \mathbb{C} \to X$$

defined by

$$f(z, w) = w \in \mathbb{C}/\Lambda_z;$$

by construction the image of f is Zariski dense in X and we may apply (4.1). $\square$

We again emphasize it is not required that the general E_t be smooth.

Also, if we observe that on $\Delta \times \mathbb{P}^1$ any holomorphic section of $\mathrm{Sym}^m(\Omega^2_{\Delta \times \mathbb{P}^2})$ must be identically zero, then from (4.3) we have the corollary:

(4.4) *If $H^0(mK_X) \neq 0$ for some $m > 0$, then any rational curve on X must be isolated.*

More precisely, for any holomorphic mapping

$$f : \Delta \times \mathbb{P}^1 \to X$$

the Jacobian J_f must be identically zero.

Using the lemma, we see that elliptic surfaces, hyperelliptic surfaces, and Enriques surfaces—these all have elliptic pencils—fail to be measure hyperbolic. To establish the conjecture for algebraic surfaces it will suffice to show that any algebraic $K3$ surface fails to be measure hyperbolic, and again using Lemma 4.2, this would follow from the assertion:

(4.5) *On any smooth algebraic $K3$ surface X there are ∞^1 elliptic curves.*

As mentioned in the introduction we shall give a construction of ∞^1 curves on any X that we can show to be elliptic for the first three families of $K3$ surfaces, and in general serve to reduce the conjecture to establishing a certain technical algebro-geometric point to be explained below.

(b) Informal discussion of the proof. In this subsection we shall discuss the idea behind the proposed construction of the ∞^1 elliptic curves on any algebraic $K3$ surface X. We recall (Mayer [22] and Saint-Donat [27]) that these surfaces fall into a sequence of irreducible families $\mathfrak{F}_n$ $(n \geqslant 1)$ that may be described as follows:

(4.6)

(i) *The surfaces $X \in \mathfrak{F}_1$ are 2-sheeted coverings $X \to \mathbb{P}^2$ branched over a smooth curve of degree six.*

(ii) *The surfaces* $X \in \mathfrak{F}_n$ $(n \geq 2)$ *are nondegenerate smooth surfaces* $X \subset \mathbb{P}^{n+1}$ *that have degree* $2n$ *and* $p_g \neq 0$.

(iii) *If* $X \in \mathfrak{F}_n$, *then the general hyperplane section* $C \in |\mathcal{O}_X(1)|$ *is a smooth canonical curve of genus* $n+1$ (*if we take* $\mathcal{O}_X(1)$ *to be* $f^*\mathcal{O}_{\mathbb{P}^2}(1)$, *then this is also valid for* $n = 1$).

We recall that for generic $X \in \mathfrak{F}_n$ the Neron–Severi group $\mathrm{Pic}(X) \otimes Q$ is generated by the class of a hyperplane section. Since a smooth rational or elliptic curve E has respectively

$$E^2 = -2, \qquad E^2 = 0,$$

it follows that on a general algebraic $K3$ surface there are no smooth curves of genus 0 or 1 (actually, for generic $X \in \mathfrak{F}_n$ there are not smooth curves of genus $\leq n$), so the curves we are seeking in order to establish (4.5) must be singular.

It is also the case that for generic $X \in \mathfrak{F}_n$ there are no curves other than those cut out by hypersurfaces in $\mathbb{P}^{n+1}$ (cf. Saint-Donat [27]). If $V \subset \mathbb{P}^{n+1}$ is a hypersurface of degree d such that

$$C = V \cap X$$

is smooth, then the genus $g(C) = d^2 n$. It follows that if on any algebraic $K3$ surface $X \in \mathfrak{F}_n$ we are to find a curve of genus $\leq n$, then we should look for those of the form

$$C = H \cap X$$

where $H \in \mathbb{P}^{n+1*}$ is a hyperplane that fails to meet X transversely—i.e., H should be a tangent hyperplane to X

Now the tangent hyperplanes constitute the *dual variety* $X^* \subset \mathbb{P}^{n+1*}$. At a smooth point of X^* the corresponding hyperplane H is simply tangent to X at one point, and consequently the section $C = H \cap X$ has one ordinary double point (= node), and the genus of its normalization is $g(\tilde{C}) = n$. Suppose next that H is simply tangent at two points; then $C = H \cap X$ has two ordinary nodes and $g(\tilde{C}) = n - 1$. In general:

(4.7) *If a hyperplane* H *is tangent to* X *at* k *distinct points, then the corresponding section* $C = H \cap X$ *has normalization* $\tilde{C}$ *with genus* $g(\tilde{C}) \leq n - k + 1$.

Our main result is the following:

(4.8) *Proposition*

(i) *For any algebraic* $K3$ *surface* $X \in \mathfrak{F}_n$, *there are* ∞^{n-k+1} *hyperplanes that are tangent at* k *points.*

(ii) *If* $n = 1$ *or* 2 *or* $n = 3$ *and* X *is generic, then these* k *points may be taken distinct.*

It is clear that Theorem II follows from Proposition (4.8), (4.7), and Lemma (4.2). Moreover, the full conjecture of the introduction would follow if part (ii) of (4.8) were established for all n. As we shall presently discuss, there are compelling reasons that this should be the case, but as will also be seen during the proof of (4.8) in the next section, there is one technical issue dealing with the precise meaning of "k-fold tangent point" that we are unable to overcome.

The idea behind the proof of Proposition (4.8) is simply the following count of constants:

(4.9) *The dimension of X^* is n, and it is "k conditions" that a hyperplane H is tangent to X at k points.*

Obviously, there are several matters that require extreme caution here, the most serious of which is that the dual X^* may not have any k-fold points when $k = n$. This particular difficulty will be overcome in the next section. The next most serious question—and the one on which we are stuck— is just how to control the genus of the section $H \cap X$ where H is a k-fold tangent hyperplane but where the points of tangency may not be distinct. This much, however, can be said:

(4.10) *If there are ∞^1 hyperplanes H such that a general one is tangent to X at n distinct points, then these are points of simple tangency.*

Proof. For the corresponding section $C = H \cap X$ we consider the normalization $\tilde{C}_0$ of any irreducible component C_0 of C. Then the genus $g(\tilde{C}_0) \leqslant 1$, with equality holding if, and only if, $C_0 = C$ and the points of tangency are simple. The result then follows from (4.4). □

The intuitive reason, then, why (ii) in Proposition (4.8) should hold for all n is that in any case by part (i) there are ∞^1 n-fold tangent hyperplanes, and if a general one of these were not simply tangent at n distinct points, then (4.4) would be violated.

(c) Existence of n-fold tangent hyperplanes to a $K3$ surface. In this subsection we shall establish (i) in Proposition (4.8). It is instructive to begin by discussing the pitfalls in trying to directly rigorize the naive dimension count (4.9).

For example, consider the statement: "It is one condition that a hyperplane is tangent to X." What this means is that the dual variety X^* is a hypersurface in $\mathbb{P}^{n+1*}$. Although this is generally true, there are certainly smooth nondegenerate varieties $V \subset \mathbb{P}^{n+1}$ for which V^* fails to be a hypersurface (cf. Section 3 of Griffiths and Harris [15]). However, for any smooth surface or any variety X whose Kodaira number $\kappa(X) \geqslant 0$, the dual X^* is a hypersurface [15, Section 3]. Both reasons are applicable in our present case.

A more serious objection concerns the singular locus of X^*. For example, the hyperplanes tangent at two distinct points occur on the double locus of X^*, and there are varieties for which X^* is a smooth hypersurface (e.g., nonsingular quadrics) or, even worse, X^* may be a hypersurface whose singularities occur in high codimension (e.g., according to Donagi [9] the dual of the Plücker image of the Grassmannian G(3.6) in $\mathbb{P}^{19}$ is a hypersurface whose singularities occur in codimension five). About all that can be easily said is this:

(4.11) *If the dual $X^* \subset \mathbb{P}^{N*}$ of an algebraic variety $X \subset \mathbb{P}^N$ is a hypersurface, and if there is one hyperplane that is tangent at k distinct smooth points of X, then there are at least ∞^{N-k} such k-fold tangent hyperplanes.*

This is because if there is one point in $\mathbb{P}^{N*}$ that lies on k distinct local branches of a hypersurface X^*, then the k-fold locus of X^* has codimension $\leqslant k$.

We remark that in our problem dealing with algebraic $K3$ surfaces it is expected that all such inequalities should in fact be equalities. The reason is that if, e.g., there were ∞^2 n-fold tangent hyperplanes to $X \in \mathfrak{F}_n$, then we would have on X either (i) ∞^1 elliptic curves with the same j-invariant, or (ii) ∞^1 rational curves. Both of these are impossible.

We also remark that, in general, the k-fold locus X_k^* of X^* may be defined by a condition on Fitting ideals (see the beautiful survey paper [19] of Kleiman). Since this is a determinantal condition, it follows that

$$X_k^* \text{ nonempty} \quad \Rightarrow \quad \operatorname{codim} X_k^* \leqslant k.$$

What our proof of (i) in Proposition (4.8) will give us is that

(4.12) $\operatorname{codim} X_k^* \leqslant k$ *for any algebraic $K3$ surface X.*

This is a fairly strong condition, but, as will be discussed in the next section, it does not yield the conjecture of the introduction, since it need not be the case that for any $H \in X_k^*$ and C_0 any irreducible component of $H \cap X$, we have $g(\tilde{C}_0) \leqslant n - k + 1$.

To establish (i) in Proposition (4.8) we shall use induction on n for the families $\mathfrak{F}_n$, together with the following linkage between $\mathfrak{F}_{n-1}$ and $\mathfrak{F}_n$:

Let $X_0 \in \mathbb{P}^{n+1}$ be a $K3$ surface having one ordinary double point p_0. It is well known that such exist for all $n \geqslant 1$,[3] and projecting x_0 from p_0 gives a smooth $K3$ surface $X^1 \subset \mathbb{P}^n$. In fact, X^1 belongs to the family $\mathfrak{F}_{n-1}$ and is biholomorphic to the standard desingularization $\tilde{X}_0$ of X_0.

It will suffice to prove (4.12) in the crucial case $k = n$. Suppose first that $H^1 \cap \mathbb{P}^n$ is a hyperplane that is tangent to X^1 at $n-1$ distinct points. Then the inverse image of H^1 under the projection $\mathbb{P}^{n+1} - \{p_0\} \to \mathbb{P}^n$ gives a hyperplane $H_0 \subset \mathbb{P}^{n+1}$ that passes through the double point and is tangent to X_0 at $n-1$ points corresponding to the tangencies of H^1 and X^1. If $E \subset \tilde{X}_0$ is the exceptional curve appearing in the resolution of p_0, then in general we may expect that none of these tangencies of H^1 and X^1 will occur along E. In this case H_0 is tangent to X_0 at $n-1$ distinct points away from p_0 and passes through this double point.

Assuming that this is the situation, suppose that $X \in \mathfrak{F}_n$ is a smooth $K3$ surface that is close to X_0, and let $U \subset X$ be the inverse image of a neighborhood U_0 of the double point under the collapsing map $X \to X_0$. Now the set of tangent hyperplanes to U forms an open piece $U^* \subset X^*$ of the hypersurface X^*, and the crucial observation is that under the specialization $U \to U_0$ we have

$$U^* \to U_0^* + 2p_0^*.$$

Here, U_0^* is the closure in $\mathbb{P}^{n+1*}$ of the set of tangent hyperplanes to the complex manifold $U_0 - \{p_0\}$, and p_0^* is the $\mathbb{P}^n$ of hyperplanes through p_0^*. In

[3] For example we may consider trigonal $K3$'s. These appear as hypersurfaces in a 3-dimensional scroll W (cf. Mayer [22] and Saint-Donat [27]), and X_0 may be taken to be a singular section of W.

the language of Section 5 of Griffiths [14], $2p_0^*$ is the *Plücker defect* associated to the degeneration $U \to U_0$. In particular, every hyperplane through p_0 is the specialization of two tangent hyperplanes to X.

Now by (4.11) where $k = 2$, there are ∞^2 hyperplanes that are tangent to X at $n-1$ distinct points close to the $n-1$ points where H_0 is tangent to $X_0 - \{p_0\}$, and by the above observation ∞^1 of these must also be tangent to U. This establishes (i) in Proposition (4.8) provided that there is a hyperplane H^1 that is tangent to X^1 at $n-1$ distinct points none of which is on the exceptional curve E. In the general case the same argument goes through provided that we adopt the definition given by Kleiman [19] for the k-fold locus X_k^*. Rather than write all this out in detail, we shall examine the low cases $n = 1, 2, 3$ and discuss what is needed to establish (ii) in (4.8) for all n.

(d) Completion of the proof of (ii) in Proposition (4.8) When $n = 1$ we have that any smooth $K3$ surface $X \in \mathfrak{F}_1$ is a 2-sheeted covering

$$X \xrightarrow{\pi} \mathbb{P}^2$$

branched along a smooth sextic curve B. The "hyperplane sections" are $\pi^{-1}(L)$ where $L \subset \mathbb{P}^2$ is a line, and the section is singular exactly when L is tangent to B. Consequently, the $\pi^{-1}(L)$ for $L \in B^*$ give the desired ∞^1 elliptic curves E_L on X. We note that E_L becomes rational when L is bitangent to B, and that such L always exist.

When $n = 2$ a smooth $K3$ surface $X \in \mathfrak{F}_2$ is a quartic $X \subset \mathbb{P}^3$. This case illustrates the difficulty in the general situation. Namely, a "nice" bitangent plane H will be simply tangent to X at two distinct points, and the corresponding section $E = H \cap X$ will be a plane quartic curve having two ordinary nodes. It is then clear that $g(\tilde{E}) = 1$. However, in exceptional cases we may imagine that E has either one tacnode (= two infinitely near nodes) or one cusp, and both of these contribute to the locus X_2^*. In the first case we still have $g(\tilde{E}) = 1$, but in the second $g(\tilde{E}) = 2$.

Because (ii) of Proposition (4.8) is true when $n = 1$, we may use the induction argument above to infer that a generic $X \in \mathfrak{F}_2$ has ∞^1 planes $H \in X_2^*$ that are tangent at two distinct points. Then, by specialization on any $X \in \mathfrak{F}_2$, there are ∞^1 planes $H \in X_2{}^*$ for which $g(H \cap X) = 1$. Actually, in this case we can say more. For any smooth surface $X \subset \mathbb{P}^3$ we may take a net $\{H_t\}_{t \in \mathbb{P}^2}$ of hyperplane sections and plot the discriminant curve $B \subset \mathbb{P}^2$ where $E_t = H_t \cap X$ is singular. For a generic choice of net this curve B will have δ ordinary double points and κ cusps, and there are classical Plücker-type formulas for the numbers of each (see Castelnuovo and Enriques [4]). In particular, in the case at hand we have $\delta > 0$, and so there exists one—and hence ∞^1—planes that are tangent to X at two distinct points. This in turn yields (ii) of (4.8) when $n = 2$, and then the assertion about $n = 3$ follows as before from the induction argument.

It is pretty clear that for increasing n the possibilities for what an n-fold tangent hyperplane $H \in X_n^*$ may cut out on X quickly get out of hand, so that some more efficient method for dealing with the singularities must be devised in order to establish the second part of Proposition (4.8) in general.

(e) Concluding remarks. We will conclude with an algebro-geometric implication of the above argument, assuming that it can be pushed through in general. Namely, the same method would establish the following result:

(4.13) *On any algebraic $K3$ surface $X \subset \mathbb{P}^{n+1}$ there are a finite number of rational curves of degree $2n$.*

In fact, these will be sections $C = H \cap X$ where H is tangent to X at $n+1$ distinct points. We note that C is a *Castelnuovo canonical curve* in the sense of [16]. Giving such a Castelnuovo canonical curve *in abstracto* is the same as giving $2n+2$ marked points on $\mathbb{P}^1$; consequently there are ∞^{2n-1} such curves and they form a family that has codimension $n+1$ in the Deligne–Mumford compactification [7] of curves of genus $n+1$.

The above result is related to a special case of the recent beautiful theorem of Mori [23]:

Let V be a smooth algebraic variety of dimension n such that $-K_V$ is ample. Then V contains a rational curve C such that $C \cdot (-K_V) \leqslant n+1$.

Mori's proof is in two steps: He first uses a characteristic p argument to produce a rational curve $C_1 \subset V$, and then he employs elementary deformation-theoretic techniques to reduce the degree of C_1 to $n+1$.

When $\dim V = 3$ we may use Kodaira vanishing plus the Riemann–Roch theorem to find a surface $X \in |-K_V|$. In case X is smooth, it is a $K3$ surface and (4.13) yields a rational curve. When X is not smooth it should be even easier to find a rational curve, but we have not tried to do this.

We feel that it would be an instructive project to establish Mori's result by projective methods. In particular, a consequence of Mori's theorem is that X is not measure hyperbolic in case $-K_X$ is ample. According to the conjecture of the introduction, this should be true if we only assume that $-K_X \geqslant 0$, and a different argument for Mori's theorem might shed some light on this question.

References

[1] A. Bloch, Sur les systèmes de fonctions uniformes satisfaisant à l'equation d'une variété algébrique dont l'irrégularité dépasse la dimension. *J. de Math.* V, 19–66 (1926).

[2] F. Bogomolov, Families of curves on a surface of general type. Sov. Math. Dokl. **18**, 1294–1297 (1977).

[3] R. Brody, Intrinsic metrics and measures on compact complex manifolds. Ph.D. Thesis, Harvard Univ., 1975.

[4] G. Castelnuovo and F. Enriques, Grundeigenschaften der Algebraischen Flächen. In *Encyklop. d. Math. Wissensch.*, Vol. 3, 1903, pp. 635–768.

[5] M. Cowen, Families of negatively curved Hermitian manifolds. *Proc. Amer. Math. Soc.* **39** 362–366 (1973).

[6] M. Cowen and P. Griffiths, Holomorphic curves and metrics of negative curvature. *J. Analyse Math.*

[7] P. Deligne and D. Mumford, The irreducibility of the space of curves of a given genus. *Publ. Math. IHES* **36**, 75–109 (1969).

[8] I. Dolgacev, Weighted Projective Varieties, to appear.

[9] R. Donagi, On the geometry of Grassmannians. *Duke Math. J.* **44**, 795–837 (1977).

[10] H. Grauert and H. Reckziegel, Hermitesche Metriken und normale familien holomorpher Abbildungen. *Math. Z.* **89**, 108–125 (1965).

[11] M. Green, Holomorphic maps into complex projective space omitting hyperplanes. *Trans. Amer. Math. Soc.* **169**, 89–103 (1972).

[12] M. Green, Holomorphic maps into complex tori. *Amer. J. Math.* **100**, 615–620 (1978).

[13] P. Griffiths, Complex differential and integral geometry and curvature integrals associated to singularities of complex analytic varieties. *Duke Math. J.* **45**, 427–512 (1978).

[14] P. Griffiths, Holomorphic mappings into canonical algebraic varieties. *Ann. of Math. (2)* **93**, 439–458 (1971).

[15] P. Griffiths and J. Harris, Local differential geometry and algebraic geometry. *Ann. Ec. Norm sup.* (Dec. 1979).

[16] P. Griffiths and J. Harris, On the variety of special linear systems on a general algebraic curve. *Duke Math. J.*, to appear.

[17] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. Wiley, New York, 1978.

[18] S. Iitaka, On D-dimensions of algebraic varieties. *J. Math. Soc. Japan* **23**, 356–373 (1971).

[19] S. Kleiman, Rigorous foundations of Schubert's enumerative calculus, in Mathematical developments arising from Hilbert's problems. In *Proc. of Symposia*, AMS, Vol. 27.

[20] S. Kobayashi, Intrinsic distances, measures and geometric function theory. *Bull. Amer. Math. Soc.* **82**, 357–416 (1976).

[21] S. Kobayashi and T. Ochiai, Mappings into compact complex manifolds with negative first Chern class. *J. Math. Soc. Japan* **23**, 137–148 (1971).

[22] A. Mayer, Families of K-3 surfaces.

[23] S. Mori, Projective manifolds with ample tangent bundles. To appear.

[24] J. Noguchi, Meromorphic mappings into a compact complex space. *Hiroshima Math. J.* **7** (2), 411–425 (1977).

[25] T. Ochiai, On holomorphic curves in algebraic varieties with ample irregularity. *Invent. Math.* **43**, S3–96 (1977).

[26] E. Picard, Sur une propriété des fonctions uniformes, liées par une relation algébrique. *Compt. Rend.* **91**, 724–726 (1880).

[27] B. Saint-Donat, Projective models of K-3 surfaces.

[28] F. Sakai, Symmetry powers of the cotangent bundle and classification of algebraic varieties. In *Proc. Copenhagen Summer Meeting in Algebraic Geometry*, 1978.

[29] H. Wu, A remark on holomorphic sectional curvature. *Indiana Math. J.* **22**, 1103–1108 (1972/73).