

Two Results in the Global Theory of Holomorphic Mappings

PHILLIP A. GRIFFITHS†

HARVARD UNIVERSITY

INTRODUCTION

The theory of *value distributions*, or *Nevanlinna theory*, began with the subtle and far-reaching quantitative refinement by R. Nevanlinna of Picard's theorem (cf. Nevanlinna [12]) concerning the exceptional values of an entire meromorphic function $f: \mathbb{C} \rightarrow \mathbb{P}^1$. The theory was extended by Ahlfors [1] to an analysis of the position of a nondegenerate *holomorphic curve* $f: \mathbb{C} \rightarrow \mathbb{P}^n$ relative to the hyperplanes in the complex projective n space \mathbb{P}^n . Recently the quantitative aspect of Nevanlinna theory was extended in a different direction to nondegenerate *equidimensional* holomorphic mappings $f: \mathbb{C}^n \rightarrow M_m$, where M is an arbitrary smooth projective variety and one is interested in how the image $f(\mathbb{C}^n)$ meets the divisors on M [4]. There has also been considerable work by Stoll concerning the case of a general holomorphic mapping $f: \mathbb{C}^n \rightarrow M_m$ in which the position of $f(\mathbb{C}^n)$ relative to the subvarieties of higher codimension in M is studied (cf. Stoll [14]).

At this point it would seem that further substantial progress in the global study of general holomorphic mappings perhaps depends on understanding the following problems.

(A) To what extent can the basic results of Nevanlinna theory for divisors, such as the *Nevanlinna inequality* $N(a, r) < T(r) + C$ (cf. Nevanlinna [12, p. 175]), be carried over to higher codimension?

(B) What can be said about the position of a holomorphic curve in a general algebraic variety?

Question A is related to the *Bezout problem* discussed by Griffiths [8], and an understanding of B seems necessary for progress on the *Kobayashi metric* [10] of general algebraic varieties.

In this paper we shall give two theorems, one concerning each of the above problems. These results, which are stated in Sections 1 and 6, are by no means definitive but

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rather are an attempt to focus attention on, and perhaps clarify in special cases, problems A and B. With a similar purpose in mind, in Sections 5 and 7 we have given some general remarks and specific questions related to these problems.

1. STATEMENT OF THEOREM I

Let $f: \mathbb{C}^2 \rightarrow \mathbb{P}^2$ be a nondegenerate holomorphic mapping. Given a point $W \in \mathbb{P}^2$, we assume that $f^{-1}(W)$ is discrete and let $n(W, r)$ be the number of points (counted with multiplicities) in $f^{-1}(W) \cap \{z \in \mathbb{C}^2: \|z\| \leq r\}$. Our basic problem is to estimate $n(W, r)$ in terms of quantities independent of W (cf. Griffiths [8] for a general discussion of this question). For the usual reasons arising from Jensen's theorem, it is better to seek an estimate on the *counting function*

$$N(W, r) = \int_0^r n(W, t) (dt/t). \quad (1.1)$$

[Note: To allow for the possibility $f(0) = W$, it is necessary to set $n(W, 0) = \lim_{\varepsilon \rightarrow 0} n(W, \varepsilon)$ and

$$N(W, r) = \int_0^r \{n(W, t) - n(W, 0)\} (dt/t) + n(W, 0) \log r.$$

Generally speaking, we shall leave it to the reader to make the necessary technical adjustments arising from such special cases.]

Let $\omega = dd^c \log \|Z\|^2$ be the standard Kähler metric on \mathbb{P}^2 and $\Omega \geq f^* \omega$. The basic first-order geometric invariants attached to f are the two *order functions*

$$T_2(r) = \int_0^r \left(\int_{\|z\| \leq t} \Omega \wedge \Omega \right) \frac{dt}{t}, \quad T_1(r) = \int_0^r \left(\int_{\|z\| \leq t} \Omega \wedge dd^c \tau \right) \frac{dt}{t}, \quad (1.2)$$

where $\tau = \log \|z\|^2$ is the standard *exhaustion function* for \mathbb{C}^2 . We observe that (cf. Wu [15])

$$T_2(r) = \int_{W \in \mathbb{P}^2} N(W, r) dW \quad (1.3)$$

is the *average* of the counting functions with respect to the normalized ($\int_{\mathbb{P}^2} dW = 1$) invariant measure on \mathbb{P}^2 . Contrary to the one-variable case, we cannot estimate $N(W, r)$ by its average $T_2(r)$, nor even in terms of both $T_2(r)$ and $T_1(r)$ (see Section 5). However, we shall give an estimate on $N(W, r)$ in terms of $T_2(r)$, $T_1(r)$, and another higher-order invariant $S(r)$ to be defined now.

Let \mathbb{P}^1 be the set of lines through the origin in \mathbb{C}^2 . For each $\xi \in \mathbb{P}^1$ we let $\mathbb{C}_\xi \subset \mathbb{C}^2$ be the corresponding line and $f_\xi: \mathbb{C} \rightarrow \mathbb{P}^2$ the *holomorphic curve* (cf. Ahlfors [1] and Chern [5]) given by restricting f to \mathbb{C}_ξ . Each such holomorphic curve has a *dual curve* $f_\xi^*: \mathbb{C} \rightarrow \mathbb{P}^{2*}$, and we let $\Omega_\xi = f_\xi^*(\omega)$, $\Omega_\xi^* = (f_\xi^*)^* \omega$ be the pulled-back metrics. The quantities

$$T_\xi(r) = \int_0^r \left(\int_{\substack{\|z\| \leq t \\ z \in \mathbb{C}_\xi}} \Omega_\xi \right) \frac{dt}{t}, \quad T_\xi^*(r) = \int_0^r \left(\int_{\substack{\|z\| \leq t \\ z \in \mathbb{C}_\xi}} \Omega_\xi^* \right) \frac{dt}{t} \quad (1.4)$$

are the basic order functions regulating the growth of f_ξ . As will be seen, it is possible to estimate $T_\xi^*(r)$ in terms of $T_\xi(r)$, and $T_\xi(r)$ in terms of $T_1(\lambda r)$, $\lambda > 1$. If we set

$$\Omega_\xi^* = h\Omega_\xi, \quad S(r) = \int_{\|z\| \leq r} \log^+ h\Omega^* \wedge dd^c \tau, \quad (1.5)$$

then $S(r)$ is a second-order invariant of the mapping f . Using the notation

$$A(r) < B(r) \quad //$$

to mean that "the stated inequality holds outside an open set $E \subset \mathbb{R}^+$ with $\int_E dt/t < \infty$," our main result is the following theorem.

Theorem I. The counting function is estimated by

$$N(W, r) \leq T_2(r) + S(W, r) + C, \quad (1.6)$$

where the remainder term $S(W, r)$ satisfies

$$\int_0^r S(W, t)(dt/t) < CT_1(\lambda r)^{2+\epsilon} + \int_0^r S(t)(dt/t) \quad //. \quad (1.7)$$

Remark. Essentially this amounts to estimating the growth of $N(W, r)$ in terms of the quantities $T_2(r)$, $T_1(r)$, and $S(r)$, which are independent of W . The geometric interpretation of $S(r)$ will be discussed in Section 5. The main result of Griffiths [8] is of a similar nature, except that the quantity $S(r)$ has a somewhat better geometric interpretation there.

2. THE FIRST MAIN THEOREM; A RESULT OF CHERN-WU

Let $f: \mathbb{C}^2 \rightarrow \mathbb{P}^2$ be a nondegenerate holomorphic mapping. Given $W \in \mathbb{P}^2$, set $W = A \cap B$, where $A, B \in \mathbb{P}^{2*}$ are perpendicular lines defined by orthogonal unit vectors $A, B \in \mathbb{C}^3$. Following the notations of Chern [5] regarding \mathbb{P}^n , the (1, 1) forms

$$\omega = dd^c \log \|Z\|^2, \quad \omega_0 = dd^c \log(|Z, A|^2 + |Z, B|^2) \quad (2.1)$$

($|Z, A|^2 = |\langle A, Z \rangle|^2$ and $|Z, B|^2 = |\langle B, Z \rangle|^2$) are well defined on \mathbb{P}^2 and give, respectively, the usual Kähler metric on \mathbb{P}^2 and the invariant density on the \mathbb{P}_W^1 of lines through $W \in \mathbb{P}^2$. Choose coordinates $Z = (Z_0, Z_1, Z_2)$ in \mathbb{C}^3 such that

$$\langle A, Z \rangle = Z_1, \quad \langle B, Z \rangle = Z_2, \quad W = [1, 0, 0].$$

In terms of the standard affine coordinate system $(w_1, w_2) \rightarrow [1, w_1, w_2]$ around W , we have

$$\omega = dd^c \log(1 + |w_1|^2 + |w_2|^2), \quad \omega_0 = dd^c \log(|w_1|^2 + |w_2|^2). \quad (2.2)$$

Consider the singular (1, 1) form (Levine form)

$$\Lambda_W = \log \left(\frac{|Z|^2}{|Z, A|^2 + |Z, B|^2} \right) (\omega + \omega_0). \quad (2.3)$$

Formally we have

$$dd^c \Lambda_W = (\omega - \omega_0) \wedge (\omega + \omega_0) = \omega \wedge \omega$$

since $\omega_0 \wedge \omega_0 = 0$. Locally around W ,

$$\Lambda_W = \log \left(\frac{1}{|w_1|^2 + |w_2|^2} \right) dd^c \log(|w_1|^2 + |w_2|^2) + (\dots),$$

where (\dots) are less singular terms. From this we recognize the principal part of Λ_W as minus the Bochner-Martinelli kernel. Considering Λ_W as a locally L^1 form, we consequently find the equation of currents (cf. Griffiths and King [9] and Wu [15])

$$dd^c \Lambda_W = \omega \wedge \omega - \delta_W. \quad (2.4)$$

Applying f^* to (2.4) and integrating twice gives the *first main theorem* (FMT) (cf. Griffiths and King [9], Stoll [14], and Wu [15])

$$N(W, r) + \int_{\|z\|=r} f^* \Lambda_W \wedge d^c \tau = T_2(r) + \int_{\|z\| \leq r} f^* \Lambda_W \wedge dd^c \tau + C, \quad (2.5)$$

where $\tau = \log \|z\|^2$. Note that the *Levi form* $dd^c \tau = dd^c \log \|z\|^2$ is the density on the \mathbb{P}^1 of lines through the origin in \mathbb{C}^2 . Using the notation

$$S(W, r) = \int_{\|z\| \leq r} f^* \Lambda_W \wedge dd^c \tau,$$

and noting that $f^* \Lambda_W \wedge d^c \tau \geq 0$ on the sphere $\|z\| = r$, we obtain from (2.5) the inequality

$$N(W, r) \leq T_2(r) + S(W, r) + C. \quad (2.6)$$

As an application of (2.6), we shall derive the theorem of Chern-Wu [15]. For this we use the following lemma.

Lemma 2.1

$$\int_{W \in \mathbb{P}^2} \Lambda_W dW = c\omega, \quad c > 0. \quad (2.7)$$

Proof. The average $\Lambda = \int_{W \in \mathbb{P}^2} \Lambda_W dW$ is an L^1 form on \mathbb{P}^2 which is invariant under the unitary group. Consequently the Laplacian $\Delta \Lambda = 0$ in the sense of currents since \mathbb{P}^2 is a symmetric space. It follows from the regularity theorem that Λ is C^∞ , thus harmonic in the usual sense, and finally that $\Lambda = c\omega$ for some constant $c > 0$. Q.E.D.

Using the lemma and positivity of everything in sight, we have

$$\int_{W \in \mathbb{P}^2} S(W, r) dW = \int_{\|z\| \leq r} \left(\int_{W \in \mathbb{P}^2} \Lambda_W dW \right) \wedge dd^c \tau = c \int_{\|z\| \leq r} \omega \wedge dd^c \tau = cT_1'(r),$$

where $T_1'(r) = dT_1(r)/d \log r$ and $T_1(r)$ is given by (1.2). Suppose that $\int_{f(\mathbb{C}^2)} dW = 1 - \delta$, $\delta \geq 0$. Then integrating (2.6), we obtain

$$T_2(r) = \int_{W \in \mathbb{P}^2} N(W, r) dW \quad [\text{by (1.3)}]$$

$$= \int_{W \in f(\mathbb{C}^2)} N(W, r) dW \quad (\text{obviously})$$

$$\leq (1 - \delta)T_2(r) + cT_1'(r) + C \quad [\text{by (2.6)}].$$

Dividing by $T_2(r)$ gives

$$1 \leq (1 - \delta) + c[T_1'(r)/T_2(r)] + [C/T_2(r)],$$

from which we obtain, by letting $r \rightarrow \infty$, the following theorem.

Theorem (Chern-Wu [15]). If $\lim_{r \rightarrow \infty} [T_1'(r)/T_2(r)] = 0$, then the image $f(\mathbb{C}^2)$ is dense in \mathbb{P}^2 .

Remark. The standard example of a map $f: \mathbb{C}^2 \rightarrow \mathbb{P}^2$ where the image omits an open set is that of *Fatou-Bieberbach*. It seems to me that this example is a reflection of the enormous automorphism group of \mathbb{C}^2 . For instance, for any entire function $h(\zeta)$ the map $(z_1, z_2) \rightarrow (z_1 + h(z_2), z_2)$ is a volume-preserving automorphism of \mathbb{C}^2 .

In outline, the Fatou-Bieberbach example is constructed as follows: Let $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a biholomorphic map having the origin and some other point $z_0 \neq 0$ as *contractive fixed points*. It is easy to find such T , and T may even be taken to be rational. By a fairly easy convergence argument, we may find a local holomorphic coordinate system around zero in which T is linear. Letting $B_\eta = \{z \in \mathbb{C}^2: \|z\| < \eta\}$, we may thus find a biholomorphic mapping $f: B_\eta \rightarrow f(B_\eta) \subset \mathbb{C}^2$ and a linear transformation L on \mathbb{C}^2 such that

$$Tf(z) = fL(z), \quad z \in B_\eta. \quad (2.8)$$

Suppose that the eigenvalues of L are $\leq 1/\lambda < 1$. Then the right-hand side of (2.8) is defined for $z \in B_{\lambda\eta}$, and thus we may define $f(z)$ on $B_{\lambda\eta}$ by $f(z) = T^{-1}fL(z)$. Continuing in this way, f extends to an entire mapping $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ satisfying

$$T^k f = fL^k, \quad k \geq 0. \quad (2.9)$$

It follows from (2.9) that f is one-to-one, and moreover there is a neighborhood U of z_0 such that $f(\mathbb{C}^2) \cap U = \emptyset$. For, if $f(z) \in U$, then

$$\lim_{k \rightarrow \infty} T^k f(z) = z_0, \quad \lim_{k \rightarrow \infty} fL^k(z) = 0,$$

since L is contractive on all of \mathbb{C}^2 . This is the Fatou-Bieberbach example.

3. TWO ESTIMATES FROM THE THEORY OF HOLOMORPHIC CURVES

Let $f: \mathbb{C} \rightarrow \mathbb{P}^2$ be a nondegenerate holomorphic curve. We follow the terminology and notations of the paper by Chern [5] and represent f by a holomorphic homogeneous coordinate vector $Z(\zeta)$, $\zeta \in \mathbb{C}$. The dual curve $f^*: \mathbb{C} \rightarrow \mathbb{P}^{2*}$ is then given by $Z(\zeta) \wedge Z'(\zeta)$, and the pulled back Kähler metrics by

$$\Omega = dd^c \log \|Z(\zeta)\|^2, \quad \Omega^* = dd^c \log \|Z(\zeta) \wedge Z'(\zeta)\|^2. \quad (3.1)$$

For fixed $W \in \mathbb{P}^2$ we let Λ_W be given by (2.3) and seek to estimate the integral

$$\int_0^r \left(\int_{|\zeta| \leq t} f^* \Lambda_W \right) (dt/t). \quad (3.2)$$

For this we shall use the *Ahlfors inequalities* [1,5] in the theory of holomorphic curves, which we now recall.

To begin with, we set

$$v(t) = \int_{|\zeta| \leq t} \Omega, \quad v^*(t) = \int_{|\zeta| \leq t} \Omega^*,$$

and denote by

$$V(r) = \int_0^r v(t)(dt/t), \quad V^*(r) = \int_0^r v^*(t)(dt/t),$$

the *order functions* of f and f^* . The relationship between $V(r)$ and $V^*(r)$ is given by the *second main theorem* (SMT), which is derived from the formulae (cf. Chern [5])

$$\begin{aligned} \Omega &= \frac{\|Z \wedge Z'\|^2}{\|Z\|^4} \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right) = k \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right), \\ \Omega^* &= \frac{\|Z\|^2 \|Z \wedge Z' \wedge Z''\|^2}{\|Z \wedge Z'\|^4} \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right) = k^* \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right). \end{aligned} \quad (3.3)$$

From (3.1) and (3.3) it follows that $dd^c \log k = \Omega^* - 2\Omega$, which leads to the inequalities (cf. Chern [5])

$$V^*(r) \leq 2V(r) + \log \left[(1/2\pi) \int_{|\zeta|=r} k d\theta \right], \quad V^*(r) \leq 2V(r) + \log V(r) \quad //.$$

Similarly, we may estimate the growth of $V(r)$ by that of $V^*(r)$, using the second equation in (3.3).

Setting

$$\rho_W = \|Z\|^2 / (|Z, A|^2 + |Z, B|^2),$$

the Ahlfors inequality we shall use is the estimate

$$\int_0^r \left(\int_{|\zeta| \leq t} \rho_W \Omega \right) \left(\frac{dt}{t} \right) < C_\lambda V(r) + C, \quad \lambda < 1. \quad (3.4)$$

As a first application of this, we write

$$\begin{aligned} f^* \omega &= \Omega = dd^c \log \|Z\|^2, \\ f^* \omega_0 &= \Omega_0 = dd^c \log (|Z, A|^2 + |Z, B|^2), \\ f^* \Lambda_W &= \log \rho_W \Omega + \log \rho_W \Omega_0, \end{aligned}$$

and then we seek to estimate the integral [cf. (2.3)]

$$\int_{r_0}^r \left(\int_{|z| \leq t} \log \rho_W \Omega \right) (dt/t)$$

for large r . To do this, we have by the concavity of the logarithm

$$\begin{aligned} \int_{r_0}^r \left(\int_{|z| \leq t} \log \rho_W \Omega \right) \frac{dt}{t} &= \frac{\log r}{\lambda} \left\{ \frac{1}{\log r} \int_{r_0}^r v(t) \left[\frac{1}{v(t)} \int_{|z| \leq t} \log \rho_W \Omega \right] \frac{dt}{t} \right\} \\ &\leq \frac{\log r \cdot v(r)}{\lambda} \left\{ \frac{1}{\log r} \int_{r_0}^r \log \left(\int_{|z| \leq t} \rho_W \Omega \right) \frac{dt}{t} \right\} \\ &\leq \frac{\log r \cdot v(r)}{\lambda} \log \left\{ \int_{r_0}^r \left(\int_{|z| \leq t} \rho_W \Omega \right) \frac{dt}{t} \right\} \\ &\leq \frac{\log r \cdot v(r)}{\lambda} \log [C_\lambda V(r) + C] \quad (\text{by (3.4)}) \\ &\leq C'_\lambda V(r)^{2+\varepsilon} + C' \quad // \end{aligned}$$

since $v(r) \leq V(r)^{1+\varepsilon}$ // (cf. Nevanlinna [12, p. 253]), $\log r \leq cV(r)$ for large r , and where r_0 has been chosen large enough so that $\log r_0 \geq 1$, $v(r_0) \geq 1$. Combining, we obtain

$$\int_0^r \left(\int_{|z| \leq t} \log \rho_W \Omega \right) (dt/t) < CV(r)^{2+\varepsilon} + C' \quad //, \quad (3.5)$$

where we have chosen a fixed λ (say $\lambda = \frac{1}{2}$).

The problem of estimating

$$\int_0^r \left(\int_{|z| \leq t} \log \rho_W \Omega_0 \right) (dt/t)$$

is more subtle. The basic step is the following lemma.

Lemma 3.1

$$\Omega_0 \leq C\Omega + C^*\Omega^*, \quad C, C^* \text{ constants.} \quad (3.6)$$

Assuming the lemma, we will complete our basic estimate on the integral (3.2). The term

$$C \int_0^r \left(\int_{|z| \leq t} \log \rho_W \Omega \right) (dt/t)$$

is estimated by (3.5). As for the other term, we write

$$\Omega = h^*\Omega^*, \quad h = 1/h^*,$$

$$\int_0^r \left(\int_{|z| \leq t} \log \rho_W \Omega^* \right) \frac{dt}{t} = \frac{1}{\lambda} \int_0^r \left(\int_{|z| \leq t} \log(h^* \rho_W \Omega^*) \right) \frac{dt}{t} + \frac{1}{\lambda} \int_0^r \left(\int_{|z| \leq t} \log h \Omega^* \right) \frac{dt}{t} \quad (3.7)$$

The first integral on the right-hand side of (3.7) may be estimated as before using concavity of the logarithm and the Ahlfors inequality (3.4). This gives

$$\int_0^r \left(\int_{|z| \leq t} \log(h^* \rho_W \Omega^*) \right) (dt/t) < CV^*(r)V(r)^{1+\varepsilon} + C' \quad //, \quad (3.8)$$

Combining (3.3), (3.5)–(3.8) gives our final estimate

$$\int_0^r \left(\int_{|z| \leq t} f^* \Lambda_W \right) \frac{dt}{t} < CV(r)^{2+\varepsilon} + \int_0^r \left(\int_{|z| \leq t} \log^+ h \Omega^* \right) \frac{dt}{t} \quad //, \quad (3.9)$$

Proof of Lemma 3.1. The form $\omega_0 = dd^c \log(|Z, A|^2 + |Z, B|^2)$ has a singularity at $W \in \mathbb{P}^2$, and so it is not immediately apparent that $\Omega_0 = f^* \omega_0$ is locally bounded on \mathbb{C} . However, this may be seen as follows: Let $\tilde{\mathbb{P}}_W^2 \xrightarrow{\pi} \mathbb{P}^2$ be the quadratic transform of \mathbb{P}^2 at W . Then $\pi^{-1}(W) = \mathbb{P}_W^1$ is the set of lines in \mathbb{P}^2 passing through W , and $\tilde{\mathbb{P}}_W^2 - \mathbb{P}_W^1 = \mathbb{P}_W^2 - W$. Since ω_0 is the density on \mathbb{P}_W^1 , $\pi^* \omega_0$ is smooth on $\tilde{\mathbb{P}}_W^2$. There is a unique holomorphic lifting \tilde{f} of f such that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{f}} & \tilde{\mathbb{P}}_W^2 \\ & \searrow f & \downarrow \pi \\ & & \mathbb{P}^2 \end{array}$$

is commutative, and thus $\Omega_0 = \tilde{f}^*(\pi^* \omega_0)$ is C^∞ on \mathbb{C} .

To prove the desired estimate on Ω_0 , we choose affine coordinates (w_1, w_2) around W such that $\omega_0 = dd^c \log(|w_1|^2 + |w_2|^2)$ [cf. (2.2)]. Clearly $\omega_0 \leq C\omega$ outside a neighborhood

$$U = \{(w_1, w_2) : |w_1|^2 + |w_2|^2 < \eta^2\} \quad \text{of } W.$$

Given a holomorphic curve $g(\zeta) = (w_1(\zeta), w_2(\zeta))$ from the disc $|\zeta| < \delta$ into $U \subset \mathbb{P}^2$, it is clear that the size of $g^* \omega_0$ is maximized when the curve passes through W . Locally we may choose w_1, w_2 such that

$$w_1(\zeta) = \alpha \zeta^{1+a} + (\dots), \quad w_2(\zeta) = \beta \zeta^{2+a+b} + (\dots),$$

where (\dots) denotes higher-order terms (cf. Ahlfors [1, p. 6]). Then by (3.3)

$$\begin{aligned} g^* \omega_0 &= \frac{\|g \wedge g'\|^2}{\|g\|^4} \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right) \\ &= \frac{|\alpha\beta|^2(1+b)^2(|\zeta|^{2(1+2a+b)} + \dots)}{|\alpha|^4(|\zeta|^{4+4a} + \dots)} \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right) \\ &= \frac{|\beta|^2(1+b)^2}{|\alpha|^2} (|\zeta|^{2b} + \dots) \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right). \end{aligned} \quad (3.10)$$

Similarly, setting $Z(\zeta) = [1, w_1(\zeta), w_2(\zeta)]$ and using (3.3),

$$\begin{aligned} (g^*)^*\omega &= \frac{\|Z\|^2 \|Z \wedge Z' \wedge Z''\|^2}{\|Z \wedge Z'\|^4} \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right) \\ &= \frac{|\alpha\beta|^2 (1+a)^2 (1+b)^2 (2+a+b)^2 (|\zeta|^{4a+2b} + \dots)}{|\alpha|^4 (1+a)^4 (|\zeta|^{4a} + \dots)} \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right) \\ &= \frac{|\beta|^2 (1+b)^2 (2+a+b)^2 (|\zeta|^{2b} + \dots)}{|\alpha|^2 (1+a)^2} \left(\frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right), \end{aligned} \quad (3.11)$$

where $g^*(\zeta) = Z(\zeta) \wedge Z'(\zeta)$ is the dual curve to $g(\zeta)$. Comparing (3.10) and (3.11) gives $g^*\omega_0(W) \leq g^*\omega^*(W)$, from which the lemma follows. Q.E.D.

4. PROOF OF THEOREM I

Referring to the FMT (2.5) and its Corollary 2.6, we must prove the estimate

$$\int_0^r S(W, t) \frac{dt}{t} < CT_1(\lambda r)^{2+\epsilon} + \int_0^r S(t) \frac{dt}{t} \quad //, \quad \lambda > 1, \quad (4.1)$$

for the remainder term

$$S(W, r) = \int_{\|Z\| \leq r} f^* \Lambda_W \wedge dd^c \tau.$$

Since $dd^c \tau = d\bar{\zeta}$ is the invariant measure on the \mathbb{P}^1 of lines through the origin in \mathbb{C}^2 , we may iterate the integral for the remainder to have

$$S(W, r) = \int_{\xi \in \mathbb{P}^1} \left(\int_{\|z\| \leq r} f_\xi^* \Lambda_W \right) d\bar{\xi}. \quad (4.2)$$

On the other hand, referring to (3.9), we have

$$\int_0^r \left(\int_{\|z\| \leq t} f_\xi^* \Lambda_W \right) \frac{dt}{t} < CT_\xi(r)^{2+\epsilon} + \int_0^r \left(\int_{\|z\| \leq t} \log^+ h\Omega^* \right) \frac{dt}{t} \quad //_\xi, \quad (4.3)$$

where $//_\xi$ means that the exceptional interval E_ξ depends on the line ξ . However, an examination of the proofs of the inequalities in which the exceptional intervals appear shows that we may choose a uniform exceptional interval E_{ξ_0} which works for all ξ in a neighborhood of ξ_0 . Using the compactness of \mathbb{P}^1 , we may thus have an estimate (4.3) where $//$ replaces $//_\xi$. Combining (4.2) and (4.3) gives

$$\int_0^r S(W, t) \frac{dt}{t} < C \int_{\xi \in \mathbb{P}^1} T_\xi(r)^{2+\epsilon} d\bar{\xi} + \int_0^r S(t) \frac{dt}{t} \quad //, \quad (4.4)$$

$$S(t) = \int_{\|z\| \leq t} \log^+ h\Omega^* \wedge dd^c \log \|z\|^2.$$

The proof of Theorem I follows from (4.4) and the following proposition.

Proposition 4.1

$$T_\xi(r) < C_\lambda T_1(\lambda r), \quad \lambda > 1. \quad (4.5)$$

Proof. For each $z \neq 0$ we let $\xi(z)$ be the line joining z to the origin and set

$$T(z, r) = \int_{\substack{w \in \xi(z) \\ \|w\| \leq r\|z\|}} \Omega.$$

Lemma 4.1

$$T(z, r) \text{ is a plurisubharmonic (psh) function of } z \in \mathbb{C}^2. \quad (4.6)$$

Assuming the lemma, we use the *sub-mean-value principle* for psh functions to have

$$\begin{aligned} T(z_0, r) &\leq C_\delta \int_{\|z - z_0\| \leq \delta} T(z, r) d\mu(z) \leq C_\delta \int_{\|z\| \leq \|z_0\| + \delta} T(z, r) d\mu(z) \\ &= C_\delta \int_{\|z\| \leq \|z_0\| + \delta} \left(\int_{\substack{w \in \xi(z) \\ \|w\| \leq r\|z\|}} \Omega \right) d\mu(z) \leq C_\delta \int_{\xi \in \mathbb{P}^1} \left(\int_{\|z\| \leq r(\|z_0\| + \delta)} \Omega \right) d\bar{\xi} \\ &= C_\delta T_1(r\|z_0\| + r\delta) \end{aligned}$$

Choosing $\|z_0\| = 1$, we obtain

$$T_\xi(r) \leq C_\lambda T_1(\lambda r), \quad \lambda = 1 + \delta,$$

which proves (4.5). Q.E.D.

Proof of Lemma 4.1. We consider each line $A \subset \mathbb{P}^2$ as a point $A \in \mathbb{P}^{2*}$, and set

$$n(A, z, t) = \text{number of points } w \text{ satisfying } f(w) \in A, \quad w \in \xi(z), \quad \|w\| \leq t\|z\|,$$

$$N(A, z, r) = \int_0^r n(A, z, t) (dt/t).$$

Then the FMT for holomorphic curves gives (cf. Chern [5])

$$T(z, r) = \int_{A \in \mathbb{P}^{2*}} N(A, z, r) dA.$$

On the other hand, an easy argument (cf. Griffiths and King [9, Section 4]) gives that $N(A, z, r)$ is psh in z . Since the average of psh functions is again psh, we are done. Q.E.D.

5. SOME COMMENTS AND EXAMPLES

(i) The nicest possibility of a generalization of the standard upper bound on the number of zeros of an entire meromorphic function $f(z)$, $z \in \mathbb{C}$, in terms of the growth of f would be an estimate of the following kind: Given a holomorphic mapping $f: \mathbb{C}^2 \rightarrow \mathbb{P}^2$ and $W \in \mathbb{P}^2$ such that $f^{-1}(W)$ is discrete, then

$$N(W, r) < CT_2(r) + C'T_1(r) + C'' \quad //, \quad (5.1)$$

However, any such inequality (5.1) is *false*, for the following reason. Let $f_1, f_2 \in \mathcal{O}(\mathbb{C}^2)$ be the two functions given by Cornalba and Shiffman [6]. Setting

$$M(f_i, r) = \max_{\|z\| \leq r} \log |f_i(z)|$$

and making appropriate choices in their example, we have

$$M(f_i, r) \leq C_\varepsilon r^\varepsilon, \quad \varepsilon > 0, \quad i = 1, 2. \quad (5.2)$$

(i.e., f_1, f_2 are of finite order zero). On the other hand, the number $n(r)$ of common zeros of f_1, f_2 in the ball $\|z\| \leq r$ may be assumed to satisfy

$$n(r) \geq C e^{e^r}. \quad (5.3)$$

Setting $f = (f_1, f_2)$, we obtain a holomorphic mapping $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2 \subset \mathbb{P}^2$, and we claim that no estimate (5.1) holds when $W = (0, 0) \in \mathbb{C}^2 \subset \mathbb{P}^2$.

To see this, we first observe that

$$T_1(r) = \int_{\xi \in \mathbb{P}^1} T_\xi(r) d\xi$$

satisfies

$$T_1(r) \leq C_\varepsilon r^\varepsilon, \quad \varepsilon > 0, \quad (5.4)$$

because this inequality is true for the order functions $T_\xi(r)$ for all $\xi \in \mathbb{P}^1$ (cf. Nevanlinna [12, p. 175]). On the other hand, letting Φ be the Euclidean volume form,

$$T_2(r) \leq \int_0^r \left(\int_{\|z\| \leq t} f^* \Phi \right) \frac{dt}{t} = \int_0^r \left(\int_{\|z\| \leq t} |J|^2 \Phi \right) \frac{dt}{t},$$

where $J(z)$ is the Jacobian determinant of $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$. From (5.2) we find an estimate

$$|J(z)|^2 \leq C e^{2r^\varepsilon},$$

which leads to an inequality

$$T_2(r) \leq C e^{r^\varepsilon}. \quad (5.5)$$

Clearly (5.3)–(5.5) show that (5.1) cannot hold.

In conclusion, it would seem that the problem of estimating the size of $f^{-1}(W)$ in terms of the growth of f has one of the following two possibilities (cf. Griffiths [8]).

(A) The “size” of $f^{-1}(W)$ should take into account not only the number of points in $f^{-1}(W)$, but also their relative position in \mathbb{C}^2 .

(B) The “size” of $f^{-1}(W)$ means the number of points, and any estimate on $f^{-1}(W)$ involves higher-order invariants of f .

(ii) It is interesting to consider the Bezout problem in the light of the recent results of Pan and Skoda [13]. Before doing this let us recall the essentials of the one-variable theory, restricting our consideration to the finite-order case [12] for simplicity.

Let $f(z) \in \mathcal{O}(\mathbb{C})$ be an entire holomorphic function with

$$f(0) = 1, \quad M(f, r) = \max_{|z| \leq r} \log |f(z)|,$$

and $n(r)$ the number of zeros of f in $|z| \leq r$. For any discrete set of points $D = \{z_1, z_2, \dots\}$ we let $n(D, r)$ be the number of z_i with $|z_i| \leq r$. Then the three basic results are as follows.

(A) $n(r) \leq CM(f, 2r)$ (upper bound estimate).

(B) Given D , there exists $f \in \mathcal{O}(\mathbb{C})$ satisfying $\{f = 0\} = D, M(f, r) \leq Cn(D, r)$ (existence).

(C) Given D and any two functions f, \tilde{f} as in (B), then $f = e^P \tilde{f}$, where P is a polynomial of degree $\leq \text{ord}(D)$ (uniqueness).

Of these, (C) is perhaps the most interesting and most useful in applications (to number theory, elliptic functions, etc.). The results (A)–(C) all generalize to divisors in \mathbb{C}^n [11].

If we look at points in \mathbb{C}^2 , then the direct analog of (A) is false (Cornalba–Shiffman). On the other hand, given $D = \{z_1, z_2, \dots\}$ a discrete set of points of finite order in \mathbb{C}^2 , then Pan has shown that D is given by the common zeros of three functions f_1, f_2, f_3 of finite order. Skoda proves the same result in general [13]. It seems quite reasonable, although it has not been proven yet, that D is given by two functions of finite order, since this is true without the growth conditions. So far as I know, no analog of (C) has been discussed yet.

Now it seems reasonable to combine (B) and (C) and discuss the growth of an ideal \mathcal{I} , meaning not only the generators but also the relations. Thus \mathcal{I} might be said to have finite order if there are certain preferred sets of generators of \mathcal{I} given by entire functions of finite order, and if the relations between the sets of generators are also of finite order. However, this is speculative, and the only thing which seems certain is that there is more to the story than what is presently known.

(iii) In the Bezout-type estimate given by Griffiths [8] there was a geometric interpretation of the terms corresponding to $S(r)$ in Theorem I involving the inflection points of the analytic curve $V \subset \mathbb{C}^3$. In the case of Theorem I there does not seem to be any such immediate geometric interpretation, but we can give a typical local description of how f looks around points giving a large contribution to $S(r)$. Namely, the map

$$(z, w) \xrightarrow{f} (z^{n+1}, w + z^{n+2})$$

has $f^{-1}(0, 0)$ isolated and with multiplicity $n + 1$. Taking the lines $w = \text{const}$ to play the role of the lines through the origin in \mathbb{C}^2 , the local contribution to $S(r)$ is roughly

$$\int \log[|w|^2 |z|^{2n} / (|w|^2 + |z|^{2n})^2] dz d\bar{z} dw d\bar{w},$$

and the logarithm is infinite at $z = w = 0$ if $n > 0$.

6. STATEMENT AND PROOF OF THEOREM II

Let M be a compact, complex manifold of dimension two (M is an analytic surface) and $D \subset M$ an effective divisor; i.e., $D = \sum_i n_i C_i$, where the C_i are irreducible curves and $n_i > 0$. Consider a holomorphic mapping

$$f: \mathbb{C} \rightarrow M - D. \quad (6.1)$$

Definition. A variation F of f is given by a holomorphic mapping $F: \mathbb{C} \times D_\varepsilon \rightarrow M - D$, where $D_\varepsilon = \{w \in \mathbb{C}: |w| < \varepsilon\}$ and $F(z, 0) = f(z)$.

The holomorphic mapping f is *isolated* if for any variation F of f the Jacobian determinant $J(F) \equiv 0$. Intuitively, f is isolated if under any variation of f the image set does not move.

We recall that D has *simple normal crossings* if $D = C_1 + \cdots + C_N$, where the C_i are smooth, irreducible curves meeting transversely. Let $[D]$ be the line bundle determined by D and K_M the canonical bundle of M . Using the notations of Carlson and Griffiths [4], our second main result is the following theorem.

Theorem II. If D has simple normal crossings and $c_1([D]) + c_1(K_M) > 0$, then any holomorphic mapping $f: \mathbb{C} \rightarrow M - D$ is isolated.

Proof. This is a simple consequence of the volume form Ω on $M - D$ constructed by Carlson and Griffiths [3, §2] and the Ahlfors lemma for volume forms [10]. We first recall that Ω has the curvature properties

$$\text{Ric } \Omega > 0, \quad \text{Ric } \Omega \wedge \text{Ric } \Omega \geq \Omega. \quad (6.2)$$

Next we let $P(\rho_1, \rho_2)$ be the bicylinder $|z_i| < \rho_i$, $i = 1, 2$, in \mathbb{C}^2 and

$$\Theta(\rho_1, \rho_2) = \rho_1^2 \rho_2^2 \Phi / (\rho_1^2 - |z_1|^2)^2 (\rho_2^2 - |z_2|^2)^2$$

be the Poincaré volume form on $P(\rho_1, \rho_2)$, where Φ is the flat Euclidean volume form. The Ahlfors lemma says that given any pseudo-volume form Ψ on $P(\rho_1, \rho_2)$ satisfying the curvature conditions (6.2), then $\Psi \leq \Theta(\rho_1, \rho_2)$. In particular, if $\Psi(0) \neq 0$, then

$$\rho_1 \rho_2 \leq [\Phi(0)/\Psi(0)]. \quad (6.3)$$

Now let $F: \mathbb{C} \times D_\varepsilon$ be a variation of f . If F is nontrivial, then we may assume that $J(F)(0) \neq 0$ and apply the Ahlfors lemma to $\Psi = F^* \Omega$ on $D_\varepsilon \times D_\varepsilon$. The estimate (6.3) gives $r\varepsilon < \infty$, which is a contradiction. Q.E.D.

7. SOME COMMENTS AND QUESTIONS REGARDING HOLOMORPHIC CURVES

(i) One of the most attractive questions in the global theory of holomorphic mappings is the study of the position of a holomorphic curve $f: \mathbb{C} \rightarrow M$ in a general algebraic variety M . In case $M = \mathbb{P}^n$ and we are interested in the position of f relative to the linear hyperplanes in \mathbb{P}^n , the theorem of Picard-Borel and subsequent quantitative refinement by Ahlfors [1] give a beautiful understanding of the problem. However, for general M or even for hypersurfaces in \mathbb{P}^n , very little is known.

To formulate a sensible problem, we restrict ourselves to the case $\dim M = 2$.

Problem 7.1. In case D has simple normal crossings and $c_1([D]) + c_1(K_M) > 0$, does the image of a holomorphic mapping $f: \mathbb{C} \rightarrow M - D$ lie in an algebraic curve (cf. Green [7] for examples)?

Remarks. In case $M = \mathbb{P}^2$ and $D = L_1 + \cdots + L_N$ is a sum of lines, simple normal crossings means that the $\{L_i\}$ are in *general position*, and the inequality on Chern classes means that $N \geq 4$. In this case the answer to the problem is affirmative, by the theorem of Borel [1].

In general, we can use the conditions on D together with Carlson's trick [3] to reduce Problem 7.1 to the case where D is zero. Namely, the hypotheses on D are basically those which allow one to construct a finite covering $\tilde{M} \rightarrow M$ branched only along D and where $c_1(K_{\tilde{M}}) > 0$; i.e., \tilde{M} is a *canonical algebraic surface*. The mapping f then lifts to $\tilde{f}: \mathbb{C} \rightarrow \tilde{M}$.

Thus let us assume that M is already a canonical algebraic surface and ask whether the image of $f: \mathbb{C} \rightarrow M$ lies in an algebraic curve. If this is to be true, then we should have some a priori idea in which curve C the image lies, or at least a bound on the degree of C . For example, in the case of the $f: \mathbb{C} \rightarrow \mathbb{P}^2 - \{4 \text{ lines}\}$, the image $f(\mathbb{C})$ lies in a line, and indeed in one of the diagonals of the quadrilateral given by the four lines (see Fig. 1). In general, however, even the following problem in algebraic geometry does not seem to be known.

Problem 7.2. Let M be a canonical algebraic surface. Then, are there a finite number of rational and elliptic curves on M ?

(ii) A problem closely related to this one deals with *Kobayashi metric* [10] on an analytic surface M . By the theorem of Royden, the Kobayashi metric is given by a length function $F(x, \xi)$, $x \in M$, $\xi \in T_x(M)$, defined on the holomorphic tangent space $T_x(M)$ at each point $x \in M$. Letting $D_r = \{z \in \mathbb{C}: |z| < r\}$, the definition of $F(x, \xi)$ is $F(x, \xi) = \inf_f (1/r_f)$, where $f: D_{r_f} \rightarrow M$ is a holomorphic mapping satisfying the conditions $f(0) = x$, $f_* \partial/\partial z = \xi$. A beautiful theorem of Bloch [2] gives the Kobayashi metric on $\mathbb{P}^2 - \{L_1 + L_2 + L_3 + L_4\}$ as follows: $F(x, \xi) > 0$ unless x lies on a diagonal of the quadrilateral given by the $\{L_i\}$ and ξ is tangent to this diagonal (Fig. 1; the pictured vectors are the only ones having zero length for the Kobayashi metric).

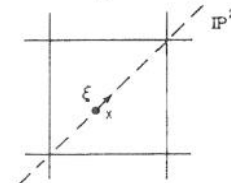


Fig. 1

Problem 7.3. Is the Kobayashi metric $F(x, \xi)$ on a canonical algebraic surface positive on a Zariski open set? In particular, assuming an affirmative answer to Problem 7.2, is $F(x, \xi) > 0$ unless ξ is tangent to one of the finitely many rational and elliptic curves on M ?

As a closing remark, it seems to me that a proof of any of these questions will necessitate relating the local higher-order geometry of an analytic curve in M to the global structure of M ; i.e., some sort of *Plücker formulas* for an analytic curve in M are needed.

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