Two Results in the Global Theory of Holomorphic Mappings

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INTRODUCTION

The theory of value distribution, or Nevanlinna theory, began with the subtle and far-reaching quantitative refinement by R. Nevanlinna of Picard's theorem (cf. Nevanlinna [12]) concerning the exceptional values of an entire meromorphic function \( f: \mathbb{C} \to \mathbb{P}^1 \). The theory was extended by Ahlfors [1] to an analysis of the position of a nondegenerate holomorphic curve \( f: \mathbb{C} \to \mathbb{P}^n \) relative to the hyperplanes in the complex projective \( n \) space \( \mathbb{P}^n \). Recently the quantitative aspect of Nevanlinna theory was extended in a different direction to nondegenerate equidimensional holomorphic mappings \( f: \mathbb{C}^n \to M_n \), where \( M \) is an arbitrary smooth projective variety and one is interested in how the image \( f(\mathbb{C}^n) \) meets the divisors on \( M \) [4]. There has also been considerable work by Stoll concerning the case of a general holomorphic mapping \( f: \mathbb{C} \to M_n \) in which the position of \( f(\mathbb{C}^n) \) relative to the subvarieties of higher codimension in \( M \) is studied (cf. Stoll [14]).

At this point it would seem that further substantial progress in the global study of general holomorphic mappings perhaps depends on understanding the following problems.

(A) To what extent can the basic results of Nevanlinna theory for divisors, such as the Nevanlinna inequality \( N(a, r) < T(r) + C \) (cf. Nevanlinna [12, p. 175]), be carried over to higher codimension?

(B) What can be said about the position of a holomorphic curve in a general algebraic variety?

Question A is related to the Bezout problem discussed by Griffiths [8], and an understanding of B seems necessary for progress on the Kobayashi metric [10] of general algebraic varieties.

In this paper we shall give two theorems, one concerning each of the above problems. These results, which are stated in Sections 1 and 6, are by no means definitive but rather are an attempt to focus attention on, and perhaps clarify in special cases, problems A and B. With a similar purpose in mind, in Sections 5 and 7 we have given some general remarks and specific questions related to these problems.

1. STATEMENT OF THEOREM 1

Let \( f: \mathbb{C} \to \mathbb{P}^2 \) be a nondegenerate holomorphic mapping. Given a point \( W \in \mathbb{P}^2 \), we assume that \( f^{-1}(W) \) is discrete and let \( n(W, r) \) be the number of points (counted with multiplicities) in \( f^{-1}(W) \cap (z \in \mathbb{C}^2 : \|z\| < r) \). Our basic problem is to estimate \( N(W, r) \) in terms of quantities independent of \( W \) (cf. Griffiths [8] for a general discussion of this question). For the usual reasons arising from Jensen's theorem, it is better to seek an estimate on the counting function

\[
N(W, r) = \int_0^r n(W, t) \frac{dt}{t}.
\]

[Note: To allow for the possibility \( f(0) = W \), it is necessary to set \( n(W, 0) = \lim_{t \to 0} n(W, t) \) and

\[
N(W, r) = \int_0^r (n(W, t) - n(W, 0)) \frac{dt}{t} + n(W, 0) \log r.
\]

Generally speaking, we shall leave it to the reader to make the necessary technical adjustments arising from such special cases.]

Let \( \omega = -dd^* \log \|z\|^2 \) be the standard Kähler metric on \( \mathbb{P}^2 \) and \( \Omega = f^* \omega \). The basic first-order geometric invariants attached to \( f \) are the two order functions

\[
T_{1}(r) = \int_{\mathbb{C}^2} \left( \int_{|z| < r} (\Omega \wedge \Omega) \right) \frac{dt}{t}, \quad T_{2}(r) = \int_{\mathbb{C}^2} \left( \int_{|z| < r} (\Omega \wedge d\Omega) \right) \frac{dt}{t}, \quad (1.2)
\]

where \( r(t) = \log \|z\|^2 \) is the standard exhaustion function for \( \mathbb{C}^2 \). We observe that (cf. Wu [15])

\[
T_{3}(r) = \int_{\mathcal{W}} \frac{NdW}{D} \quad (1.3)
\]

is the average of the counting functions with respect to the normalized \( (f^*dW = 1) \) invariant measure on \( \mathbb{P}^2 \). Contrary to the one-variable case, we cannot estimate \( N(W, r) \) by its average \( T_{3}(r) \), nor even in terms of both \( T_{1}(r) \) and \( T_{2}(r) \) (see Section 5). However, we shall give an estimate on \( N(W, r) \) in terms of \( T_{1}(r) \), \( T_{2}(r) \), and another higher-order invariant \( S(r) \) to be defined now.

Let \( \mathbb{P}^2 \) be the set of lines through the origin in \( \mathbb{C}^2 \). For each \( \xi \in \mathbb{P}^2 \) we let \( C_{\xi} = \mathbb{C}^2 \) be the corresponding line and \( f_{\xi}: \mathbb{C} \to \mathbb{P}^2 \) the holomorphic curve (cf. Ahlfors [1] and Chern [5]) given by restricting \( f \) to \( C_{\xi} \). Each such holomorphic curve has a dual curve \( f_{\xi}^*: \mathbb{C} \to \mathbb{P}^2 \), and we let \( \Omega_{\xi} = f_{\xi}^*(\omega) \). The quantities

\[
T_{3}(r) = \int_{\mathbb{C}^2} \left( \int_{|z| < r} (\Omega_{\xi} \wedge \Omega_{\xi}) \right) \frac{dt}{t}, \quad T_{4}(r) = \int_{\mathbb{C}^2} \left( \int_{|z| < r} (\Omega_{\xi} \wedge d\Omega_{\xi}) \right) \frac{dt}{t} \quad (1.4)
\]
are the basic order functions regulating the growth of $f_2$. As will be seen, it is possible
to estimate $T_{2,2}(r)$ in terms of $T_{2,2}(r)$, and $T_{2,2}(r)$ in terms of $T_{1,1}(r)$, $r \geq 1$. If we set
\begin{equation}
\Omega^* = h \Omega_1, \quad S(r) = \int_{|z| = r} \log^+ h \Omega^* \cdot d d^* \tau,
\end{equation}
then $S(r)$ is a second-order invariant of the mapping $f$. Using the notation
\begin{equation}
A(r) < B(r),
\end{equation}
to mean that the stated inequality holds outside an open set $E \subset \mathbb{R}^n + \frac{1}{r < \omega^*}$,
our main result is the following theorem.

**Theorem I.** The counting function is estimated by
\begin{equation}
N(W, r) \leq T_2(r) + S(W, r) + C,
\end{equation}
where the remainder term $S(W, r)$ satisfies
\begin{equation}
\frac{1}{r} \int_{T_{1,1} \geq r} S(W, t) dt / t < C T_{1,1}(2r)^{2 + s} + \int_{r} S(W, t) dt / t.
\end{equation}

**Remark.** Essentially this amounts to estimating the growth of $N(W, r)$ in terms of
the quantities $T_2(r)$, $T_1(r)$, and $S(r)$, which are independent of $W$. The geometric
interpretation of $S(r)$ will be discussed in Section 5. The main result of Griffiths [8]
is of a similar nature, except that the quantity $S(r)$ has a somewhat better geometric
interpretation there.

## 2. THE FIRST MAIN THEOREM: A RESULT OF CHERN-WU

Let $f: \mathbb{C} \to \mathbb{P}^2$ be a nondegenerate holomorphic mapping. Given $W \in \mathbb{P}^2$, set
$W = A \cap B$, where $A, B \in \mathbb{P}^2$ are perpendicular lines defined by orthogonal unit vectors $A, B \in C^2$. Following the notations of Chern [5] regarding $\mathbb{P}^2$, the (1, 1) form
\begin{equation}
\omega = d d^* \log |Z|^2, \quad \omega_0 = d d^* \log |Z| \cdot (|A|^2 + |B|^2)
\end{equation}
is well defined on $\mathbb{P}^2$ and give, respectively, the usual Kähler metric on $\mathbb{P}^2$ and the invariant density on the $L_1$ form of lines through $W \in \mathbb{P}^2$. Choose coordinates $Z = (Z_0, Z_1, Z_2)$ in $C^3$ such that
\begin{equation}
\langle A, Z \rangle = Z_1, \quad \langle B, Z \rangle = Z_2, \quad W = [1, 0, 0].
\end{equation}

In terms of the standard affine coordinate system $(w_1, w_2) \to [1, w_1, w_2]$ around $W$, we have
\begin{equation}
\omega = d d^* \log(1 + |w_1|^2 + |w_2|^2), \quad \omega_0 = d d^* \log |w_1|^2 + |w_2|^2).
\end{equation}

Consider the singular (1, 1) form (Levi form)
\begin{equation}
\Lambda_W = \log \left( \frac{|Z|^2}{|Z|^2 + |A|^2 + |B|^2} \right) (\omega + \omega_0).
\end{equation}

Formally we have
\begin{equation}
d d^* \Lambda_W = (\omega - \omega_0) \wedge (\omega + \omega_0) \equiv \omega \wedge \omega
\end{equation}
since $\omega_0 \wedge \omega_0 = 0$. Locally around $W$,
\begin{equation}
\Lambda_W = \log \left( \frac{1}{|w_1|^2 + |w_2|^2} \right) d d^* \log |w_1|^2 + |w_2|^2) + \cdots
\end{equation}
where (\cdots) are less singular terms. From this we recognize the principal part of $\Lambda_W$ as minus the Bochner–Martinelli kernel. Considering $\Lambda_W$ as a locally $L_1$ form, we consequently find the equation of currents (cf. Griffiths and King [9] and Wu [13])
\begin{equation}
d d^* \Lambda_W = \omega \wedge \omega - \delta_W.
\end{equation}

Applying $f^*$ to (2.4) and integrating twice gives the first main theorem (FMT) (cf. Griffiths and King [9], Stoll [14], and Wu [15])
\begin{equation}
N(W, r) \leq \int_{|z| \leq r} f^* \Lambda_W \wedge d d^* \tau = T_2(r) \int_{|z| \leq r} f^* \Lambda_W \wedge d d^* \tau + C,
\end{equation}
where $\tau = \log |z|^2$. Note that the Lefschetz form $d d^* \tau = d d^* \log |z|^2$ is the density on the $P^1$ of lines through the origin in $C^2$. Using the notation
\begin{equation}
S(W, r) = \int_{|z| \leq r} f^* \Lambda_W \wedge d d^* \tau,
\end{equation}
and noting that $f^* \Lambda_W \wedge d d^* \tau \geq 0$ on the sphere $|z| = r$, we obtain from (2.5) the inequality
\begin{equation}
N(W, r) \leq T_2(r) + S(W, r) + C.
\end{equation}

As an application of (2.6), we shall derive the theorem of Chern–Wu [15]. For this we use the following lemma.

**Lemma 2.1**
\begin{equation}
\int_{S^1} f^* \Lambda_W \wedge d d^* \tau = c \omega, \quad c > 0.
\end{equation}

**Proof.** The average $A = \int_{S^1} f^* \Lambda_W \wedge d d^* \tau$ is an $L_1$ form on $\mathbb{P}^2$ which is invariant under the unitary group. Consequently the Laplacian $\Delta_A = 0$ in the sense of currents since $\mathbb{P}^2$ is a symmetric space. It follows from the regularity theorem that $A = c^{1/2}$, thus harmonic in the usual sense, and finally that $\Lambda = c \omega$ for some constant $c > 0$.

Q.E.D.

Using the lemma and positivity of everything in sight, we have
\begin{equation}
\int_{S^1} S(W, r) \wedge d d^* \tau = \int_{|z| \leq r} \left( \int_{S^1} f^* \Lambda_W \wedge d d^* \tau \right) \wedge d d^* \tau = c \int_{|z| \leq r} \omega \wedge d d^* \tau = c T_1(r),
\end{equation}
where $T_1(r) = d T(r) / d \log r$ and $T_1(r)$ is given by (1.2). Suppose that $\int_{f(C)} d W = 1 - \delta, \quad \delta > 0$. Then integrating (2.6), we obtain
\begin{equation}
T_2(r) = \int_{S^1} N(W, r) \wedge d d^* \tau \quad \text{by (1.3)}
\end{equation}
\begin{equation}
\leq \int_{S^1} N(W, r) \wedge d d^* \tau \quad \text{(obviously)}
\end{equation}
\begin{equation}
\leq (1 - \delta) T_2(r) + c T_1(r) + C \quad \text{by (2.6)}.
\end{equation}
Dividing by $T_2(r)$ gives

$$1 \leq (1 - \delta) + \varepsilon \left( T_1'(r)/T_2(r) \right) + [C/T_2(r)],$$

from which we obtain, by letting $r \to \infty$, the following theorem.

**Theorem (Chern–Wu [15]).** If $\lim_{r \to \infty} [T_1'(r)/T_2(r)] = 0$, then the image $f(C^2)$ is dense in $\mathbb{P}^2$.

**Remark.** The standard example of a map $f: \mathbb{C}^2 \to \mathbb{P}^2$ where the image omits an open set is that of Fatou–Bieberbach. It seems to me that this example is a reflection of the enormous automorphism group of $\mathbb{C}^2$. For instance, for any entire function $h(z)$ the map $(z_1, z_2) \to (z_1 + h(z_2), z_2)$ is a volume-preserving automorphism of $\mathbb{C}^2$.

In outline, the Fatou–Bieberbach example is constructed as follows: Let $T: \mathbb{C}^2 \to \mathbb{C}^2$ be a biholomorphic map having the origin and some other point $z_0 \neq 0$ as contractive fixed points. It is easy to find such $T$ and $T$ may even be taken to be rational. By a fairly easy convergence argument, we may find a local holomorphic coordinate system around zero in which $T$ is linear. Letting $B_\eta = \{z \in \mathbb{C}^2 : ||z|| < \eta \}$, we may thus find a biholomorphic mapping $f: B_\eta \to \mathbb{C}^2$ and a linear transformation $L$ on $\mathbb{C}^2$ such that

$$Tf(z) = fL(z), \quad z \in B_\eta. \quad \text{(2.8)}$$

Suppose that the eigenvalues of $L$ are $\leq |z| < 1$. Then the right-hand side of (2.8) is defined for $z \in B_\eta$, and we can thus define $f(z)$ on $B_\eta$ by $f(z) = T^{-1} fL(z)$. Continuing in this way, $f$ extends to an entire mapping $f: \mathbb{C}^2 \to \mathbb{C}^2$ satisfying

$$T^* f = fL^k, \quad k > 0. \quad \text{(2.9)}$$

It follows from (2.9) that $f$ is one-to-one, and moreover there is a neighborhood $U$ of $z_0$ such that $f(C^2) \cap U = \emptyset$. For, if $f(z) \in U$, then

$$\lim_{k \to \infty} T^k f(z) = z_0, \quad \lim_{k \to \infty} fL^k(z) = 0,$$

since $L$ is contractive on all of $\mathbb{C}^2$. This is the Fatou–Bieberbach example.

### 3. TWO ESTIMATES FROM THE THEORY OF HOLOMORPHIC CURVES

Let $f: \mathbb{C} \to \mathbb{P}^2$ be a nondegenerate holomorphic curve. We follow the terminology and notations of the paper by Chern [5] and represent $f$ by a holomorphic homogeneous coordinate vector $Z(\zeta), \zeta \in \mathbb{C}$. The dual curve $f^*: \mathbb{C} \to \mathbb{P}^2$ is then given by $Z(\zeta) \wedge Z'(\zeta)$, and the pulled back Kähler metrics by

$$\Omega = dd^c \log ||Z(\zeta)||^2, \quad \Omega^* = dd^c \log ||Z(\zeta) \wedge Z'(\zeta)||^2. \quad \text{(3.1)}$$

For fixed $W \in \mathbb{P}^2$ we let $\Lambda_W$ be given by (2.3) and seek to estimate the integral

$$\int_{0}^{1} \left( \int_{||\zeta||^2 < r} f^* \Lambda_W \right)(d\zeta/dt). \quad \text{(3.2)}$$

For this we shall use the Ahlfors inequalities [1,5] in the theory of holomorphic curves, which we now recall.

To begin with, we set

$$\sigma(t) = \int_{[t]} \Omega, \quad \sigma^*(t) = \int_{[t]} \Omega^*,$$

denote by

$$V(r) = \int_{0}^{r} \sigma(t)(dt/t), \quad V^*(r) = \int_{0}^{r} \sigma^*(t)(dt/t),$$

the order functions of $f$ and $f^*$. The relationship between $V(r)$ and $V^*(r)$ is given by the second main theorem (SMT), which is derived from the formulæ (cf. Chern [5])

$$\Omega = \frac{1}{2\pi} \Omega^* \left( \frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right),$$

$$\Omega^* = \frac{1}{2\pi} \Omega \left( \frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta} \right). \quad \text{(3.3)}$$

From (3.1) and (3.3) it follows that $dd^c \log k = \Omega^* - 2\Omega$, which leads to the inequalities (cf. Chern [5])

$$V^*(r) \leq 2V(r) + \log \left( (1/2\pi) \int_{[t]} k d\zeta \right), \quad V(r) \leq 2V^*(r) + \log V(r).$$

Similarly, we may estimate the growth of $V(r)$ by that of $V^*(r)$, using the second equation in (3.3). Setting

$$\rho_w = ||Z(\zeta)||^2(1/2\pi) ||Z'(\zeta)||^2,$$

the Ahlfors inequality we shall use is the estimate

$$\int_{0}^{1} \left( \int_{||\zeta||^2 < r} \rho_w \Omega \right)(d\zeta/t) \leq C V(r) + C, \quad t < 1. \quad \text{(3.4)}$$

As a first application of this, we write

$$f^* \omega_0 = \Omega = dd^c \log ||Z||^2,$$

$$f^* \omega_0 = \Omega_0 = dd^c \log (||Z||^2 + ||Z||^2),$$

$$f^* \Lambda_w = \log \rho_w \Omega + \log \rho_w \Omega_0.$$
and then we seek to estimate the integral [cf. (2.3)]
\[ \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt \]
for large \( r \). To do this, we have by the concavity of the logarithm
\[ \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt \leq \frac{\log r}{\lambda} \left( \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt \right) \]
\[ \leq \frac{\log r \cdot r(t)}{\lambda} \left( \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt \right) \]
\[ \leq \frac{\log r \cdot r(t)}{\lambda} \left( \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt \right) \]
\[ \leq \frac{\log r \cdot r(t)}{\lambda} \log(C_2 V(r) + C) \] (by (3.4))
\[ \leq C_2 V(r)^{1+\epsilon} + C \]
(3.5)

on Since \( r(t) \leq V(r)^{1+\epsilon} \) (cf. Nevanlinna [12, p. 253]), \( r \leq c V(r) \) for large \( r \), and where \( r_0 \) has been chosen large enough so that \( \log r_0 \geq 1, t(r_0) \geq 1 \). Combining, we obtain
\[ \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt < C V(r)^{1+\epsilon} + C \]
(3.6)

where we have chosen a fixed \( \lambda \) (say \( \lambda = \frac{1}{2} \)).

The problem of estimating
\[ \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt \]
is more subtle. The basic step is the following lemma.

**Lemma 3.1**

\[ \Omega_0 \leq C \Omega + C^* \Omega, \quad C, C^* \] constants.

(3.7)

Assuming the lemma, we will complete our basic estimate on the integral (3.2). The term
\[ C \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt \]
is estimated by (3.5). As for the other term, we write
\[ \Omega = h^* \Omega^*, \quad h = 1/h^* \]
\[ \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho_w \Omega \right) dt \\ = \frac{\log r}{\lambda} \left( \int_{r_0}^{r} \left( \int_{|z|<t} \log h^* \rho_w \Omega^* \right) dt \right) \]
(3.7)

The first integral on the right-hand side of (3.7) may be estimated as before using concavity of the logarithm and the Ahlfors inequality (3.4). This gives
\[ \int_{r_0}^{r} \left( \int_{|z|<t} \log h^* \rho_w \Omega^* \right) dt \leq C V(r)^{1+\epsilon} + C \]
(3.8)

Combining (3.3), (3.5)–(3.8) gives our final estimate
\[ \int_{r_0}^{r} \left( \int_{|z|<t} \log \rho \Omega \right) dt < C V(r)^{1+\epsilon} + C \]
(3.9)

**Proof of Lemma 3.1.** The form \( \omega_0 = dd^c \log(|Z|, A| |Z|, B|) \) has a singularity at \( W \subset \mathbb{P}^2 \), and so it is not immediately apparent that \( \Omega_0 = f^* \omega_0 \) is locally bounded on \( C \). However, this may be seen as follows: Let \( \mathbb{P}^2_w \) be the quadratic transform of \( \mathbb{P}^2 \) at \( W \). Then \( \pi^* \mathbb{P}^2 = \mathbb{P}^2 W \) is the set of lines in \( \mathbb{P}^2 \) passing through \( W \), and \( \mathbb{P}^2_w \). Since \( \omega_0 \) is the density on \( \mathbb{P}^2 \), \( \pi^* \omega_0 \) is smooth on \( \mathbb{P}^2_w \).

There is a unique holomorphic lifting \( f \) of \( f \) such that the diagram

\[ \begin{array}{ccc}
C & \xrightarrow{f} & \mathbb{P}^2_w \\
\downarrow & & \downarrow \\
\mathbb{P}^2 & \xrightarrow{\pi} & W
\end{array} \]

is commutative, and thus \( \Omega_0 = f^* \pi^* \omega_0 \) is \( C^* \) on \( C \).

To prove the desired estimate on \( \Omega_0 \), we choose affine coordinates \( (w_1, w_2) \) around \( W \) such that \( \omega_0 = dd^c \log(|w_1|^2 + |w_2|^2) \) [cf. (2.2)]. Clearly \( \omega_0 \leq C \omega_0 \) outside a neighborhood
\[ U = \{(w_1, w_2) : |w_1|^2 + |w_2|^2 < \eta^2\} \]

of \( W \).

Given a holomorphic curve \( g(z) = (w_1(z), w_2(z)) \) from the disc \( |z| < \delta \) into \( U \subset \mathbb{P}^2 \), it is clear that the size of \( g^* \omega_0 \) is maximized when the curve passes through \( W \). Locally we may choose \( w_1, w_2 \) such that
\[ w_1(z) = x_1^{1+\epsilon} + (\cdots), \quad w_2(z) = x_2^{1+\epsilon} + (\cdots), \]
where \( (\cdots) \) denotes higher-order terms (cf. Ahlfors [1, p. 6]). Then by (3.3)
\[ g^* \omega_0 = \left| g \wedge g^* \right|^2 \left( \frac{1}{2\pi} \sum_{k=1}^{n} d\zeta_k \wedge d\zeta_k^* \right) \]
(3.10)
Similarly, setting \( Z(\zeta) = [1, w_2(\zeta), w_3(\zeta)] \) and using (3.3),
\[
(g^*)^w = \frac{|Z|^2 |Z' \wedge Z'\wedge Z|^2}{|Z \wedge Z'|^2} \left( \frac{-1}{2\pi} d\zeta \wedge d\zeta \right)
\]
\[
= \left| \frac{\beta^2(1 + a)(1 + b)^2(1 + a + b)^2(1 + a^2 + b^2 + \cdots)}{\beta^2(1 + a)^2(1 + a)^2} \right| \left( \frac{-1}{2\pi} d\zeta \wedge d\zeta \right)
\]
\[
= \frac{\beta^2(1 + a)^2(1 + a + b)^2(1 + b^2 + \cdots)}{|Z|^2 |Z \wedge Z'|^2} \left( \frac{-1}{2\pi} d\zeta \wedge d\zeta \right),
\]
(3.11)
where \( \phi^* = Z(\zeta) \wedge Z'(\zeta) \) is the dual curve to \( g(\zeta) \). Comparing (3.10) and (3.11) gives \( \phi^* w_0(W) \leq \phi^* w_0(W) \), from which the lemma follows. Q.E.D.

4. PROOF OF THEOREM 1

Referring to the FMT (2.5) and its Corollary 2.6, we must prove the estimate
\[
\int_0^r \int S(W, t) \frac{dt}{t} < C T_2(r) \zeta + \int S(t) \frac{dt}{t}, \quad \lambda > 1,
\]
(4.1)
for the remainder term
\[
S(W, t) = \int_{|z'| < |z|} f^* \Lambda \wedge \delta z.
\]
Since \( \delta z = dz \) is the invariant measure on the \( P^1 \) of lines through the origin in \( C^1 \), we may iterate the integral for the remainder to have
\[
S(W, t) = \int_{|z'| < |z|} \left( \int_{|z| < |z'|} f^* \Lambda \right) dz.
\]
On the other hand, referring to (3.9), we have
\[
\int_0^r \int_{|z'| < |z|} f^* \Lambda \wedge \delta z = C T_2(r) \zeta + \int_0^r \log^{-\lambda} \Omega \wedge \delta z + \int_0^r S(t) \frac{dt}{t}, \quad \lambda > 1,
\]
(4.3)
where \( \delta z \) means that the exceptional interval \( E_\zeta \) depends on the line \( \zeta \). However, an examination of the proofs of the inequalities in which the exceptional intervals appear shows that we may choose a uniform exceptional interval \( E_\zeta \) which works for all \( \zeta \) in a neighborhood of \( \zeta_0 \). Using the compactness of \( P^1 \), we may thus have an estimate (4.3) where \( \delta z \) replaces \( \delta z_\zeta \). Combining (4.2) and (4.3) gives
\[
\int_0^r S(W, t) \frac{dt}{t} < C T_2(r) \zeta + \int_0^r S(t) \frac{dt}{t}, \quad \lambda > 1,
\]
(4.4)
\[
S(t) = \int_{|z'| < |z|} \log^{-\lambda} \Omega \wedge \delta z \log |z|^2.
\]
The proof of Theorem 1 follows from (4.4) and the following proposition.

5. SOME COMMENTS AND EXAMPLES

(i) The nicest possibility of a generalization of the standard upper bound on the number of zeros of an entire meromorphic function \( f(\zeta), \zeta \in C \), in terms of the growth of \( f \) would be an estimate of the following kind: Given a holomorphic mapping \( f : C^1 \rightarrow P^1 \) and \( W \in P^1 \) such that \( f^{-1}(W) \) is discrete, then
\[
N(W, r) < C T_2(r) + C'T_1(r) + C^r
\]
(5.1)
However, any such inequality (5.1) is false, for the following reason. Let \( f_1, f_2 \in \mathcal{O}(\mathbb{C}) \) be the two functions given by Cornalba and Schifflman [6]. Setting
\[
M(f_1, r) = \max_{|z| \leq r} \log|f(z)|
\]
and making appropriate choices in their example, we have
\[
M(f_1, r) \leq C_r r^t, \quad r > 0, \quad t \geq 1, \quad i = 1, 2.
\]
(i.e., \( f_1, f_2 \) are of finite order zero). On the other hand, the number \( n(r) \) of common zeros of \( f_1, f_2 \) in the ball \( |z| \leq r \) may be assumed to satisfy
\[
n(r) \geq C e^{r^t}.
\]
Setting \( f = (f_1, f_2) \), we obtain a holomorphic mapping \( f: \mathbb{C}^2 \to \mathbb{C} \subset \mathbb{P}^2 \), and we claim that no estimate (5.1) holds when \( W = (0, 0) \in \mathbb{C}^2 \subset \mathbb{P}^2 \).

To see this, we first observe that
\[
T_1(r) = \int_{\mathbb{R}^2} T_4(r) d\xi
\]
satisfies
\[
T_1(r) \leq C_r r^t, \quad r > 0,
\]
because this inequality is true for the order functions \( T_4(r) \) for all \( \xi \in \mathbb{P}^1 \) (cf. Nevanlinna [12, p. 125]). On the other hand, letting \( \Phi \) be the Euclidean volume form,
\[
T_2(r) \leq \int_{\mathbb{R}^2} f^* \Phi d\xi = \int_{\mathbb{R}^2} J^* \Phi d\xi,
\]
where \( J(z) \) is the Jacobian determinant of \( f: \mathbb{C}^2 \to \mathbb{C}^2 \). From (5.2) we find an estimate
\[
|J(z)|^2 \leq C e^{r^t},
\]
which leads to an inequality
\[
T_2(r) \leq C e^{r^t}.
\]
Clearly (5.3)–(5.5) show that (5.1) cannot hold.

In conclusion, it would seem that the problem of estimating the size of \( f^{-1}(W) \) in terms of the growth of \( f \) has one of the following two possibilities (cf. Griffiths [8]):

(A) The "size" of \( f^{-1}(W) \) should take into account not only the number of points in \( f^{-1}(W) \), but also their relative position in \( \mathbb{C}^2 \).

(B) The "size" of \( f^{-1}(W) \) means the number of points, and any estimate on \( f^{-1}(W) \) involves higher-order invariants of \( f \).

(iii) In the Bezout-type estimate given by Griffiths [8] there was a geometric interpretation of the terms corresponding to \( S(r) \) in Theorem I involving the intersection points of the analytic curve \( V \subset \mathbb{C}^3 \). In the case of Theorem I there does not seem to be any such immediate geometric interpretation, but we can give a typical local description of how \( f \) looks around points giving a large contribution to \( S(r) \). Namely, the map
\[
(z, w) \mapsto (z^{n-1}, w + z^{n-2})
\]
has \( f^{-1}(0, 0) \) isolated and with multiplicity \( n + 1 \). Taking the lines \( w = c \) to const to play the role of the lines through the origin in \( \mathbb{C}^3 \), the local contribution to \( S(r) \) is roughly
\[
\int \log(|w|^2 + |z|^2)^{n/2} dz \, d\xi \, dw,
\]
and the logarithm is infinite at \( z = w = 0 \) if \( n > 0 \).

6. STATEMENT AND PROOF OF THEOREM II

Let \( M \) be a compact, complex manifold of dimension two (\( M \) is an analytic surface) and \( D \subset M \) an effective divisor; i.e., \( D = \sum n_i C_i \), where the \( C_i \) are irreducible curves and \( n_i > 0 \). Consider a holomorphic mapping
\[
f: \mathbb{C} \to M - D.
\]
Definition. A variation $F$ of $f$ is given by a holomorphic mapping $F: \mathbb{C} \times D_2 \rightarrow M - D$, where $D_2 = \{ w \in \mathbb{C} : |w| < \varepsilon \}$ and $F(x, 0) = f(x)$.

The holomorphic mapping $f$ is isolated if for any variation $F$ of $f$ the Jacobian determinant $J(F) = 0$. Intuitively, $f$ is isolated if under any variation of $f$ the image set does not move.

We recall that $D$ has simple normal crossings if $D = C_1 + \cdots + C_k$, where the $C_i$ are smooth, irreducible curves meeting transversely. Let $|D|$ be the line bundle determined by $D$ and $K_D$ the canonical bundle of $M$. Using the notations of Carlson and Griffiths [4], our second main result is the following theorem.

**Theorem II.** If $D$ has simple normal crossings and $c_1(|D|) + c_1(K_D) > 0$, then any holomorphic mapping $f: \mathbb{C} \rightarrow M - D$ is isolated.

**Proof.** This is a simple consequence of the volume form $\Omega$ on $M - D$ constructed by Carlson and Griffiths [3, 82] and the Ahlfors lemma for volume forms [10]. We first recall that $\Omega$ has the curvature properties

$$\text{Ric } \Omega > 0, \quad \text{Ric } \Omega \wedge \text{Ric } \Omega \geq 0. \quad (6.2)$$

Next we let $P(\rho_1, \rho_2)$ be the bicylinder $|z_i| < \rho_i$, $i = 1, 2$, in $\mathbb{C}^2$ and

$$\Theta(\rho_1, \rho_2) = \rho_1^2 \rho_2^2 \Phi((\rho_1^2 - |z_1|^2)^2 + (\rho_2^2 - |z_2|^2)^2)$$

be the Poincaré volume form on $P(\rho_1, \rho_2)$, where $\Phi$ is the flat Euclidean volume form. The Ahlfors lemma says that given any pseudo-volume form $\Psi$ on $P(\rho_1, \rho_2)$ satisfying the curvature conditions (6.2), then $\Psi \leq \Theta(\rho_1, \rho_2)$. In particular, if $\Psi(0) = 0$, then

$$\rho_1 \rho_2 \leq \Theta(0, 0). \quad (6.3)$$

Now let $F: \mathbb{C} \times D_2$ be a variation of $f$. If $F$ is nontrivial, then we may assume that $J(F)(0) \neq 0$ and apply the Ahlfors lemma to $\Psi = F^* \Omega$ on $D_1 \times D_2$. The estimate (6.3) gives $\rho \rho < \infty$, which is a contradiction. Q.E.D.

**7. SOME COMMENTS AND QUESTIONS REGARDING HOLOMORPHIC CURVES**

(i) One of the most attractive questions in the global theory of holomorphic mappings is the study of the position of a holomorphic curve $f: \mathbb{C} \rightarrow M$ in a general algebraic variety $M$. In case $M = \mathbb{P}^n$ and we are interested in the position of $f$ relative to linear hyperplanes in $\mathbb{P}^n$, the theorem of Picard-Borel and subsequent quantitative refinement by Ahlfors [1] give a beautiful understanding of the problem. However, for general $M$ or even for hypersurfaces in $\mathbb{P}^n$, very little is known.

(ii) To formulate a sensible problem, we restrict ourselves to the case dim $M = 2$.

**Problem 7.1.** In case $D$ has simple normal crossings and $c_1(|D|) + c_1(K_D) > 0$, does the image of a holomorphic mapping $f: \mathbb{C} \rightarrow M - D$ lie in an algebraic curve (cf. Green [7] for examples)?

**Remarks.** In case $M = \mathbb{P}^n$ and $D = L_1 + \cdots + L_k$ is a sum of lines, simple normal crossings means that the $(L_i)$ are in general position, and the inequality on Chern classes means that $N > 4$. In this case the answer to the problem is affirmative, by the theorem of Borel [1].

In general, we can use the conditions on $D$ together with Carlson's trick [3] to reduce Problem 7.1 to the case where $D$ is zero. Namely, the hypotheses on $D$ are basically those which allow one to construct a finite covering $\tilde{M} \rightarrow M$ branched only along $D$ and where $c_1(K_{\tilde{M}}) > 0$; i.e., $\tilde{M}$ is a canonical algebraic surface. The mapping $f$ then lifts to $\tilde{f}: \mathbb{C} \rightarrow \tilde{M}$.

Thus let us assume that $M$ is an already canonical algebraic surface and ask whether the image of $f: \mathbb{C} \rightarrow M$ lies in an algebraic curve. If this is to be true, then we should have some a priori idea in which curve $C$ the image lies, or at least a bound on the degree of $C$. For example, in the case of the $f: \mathbb{C} \rightarrow \mathbb{P}^2$ (4 lines), the image $f(C)$ lies in a line, and indeed in one of the diagonals of the quadrilateral given by the four lines (see Fig. 1). In general, however, even the following problem in algebraic geometry does not seem to be known.

**Problem 7.2.** Let $M$ be a canonical algebraic surface. Then, are there a finite number of rational and elliptic curves on $M$?

(ii) A problem closely related to this one deals with the Kobayashi metric [10] on an analytic surface $M$. By the theorem of Rovenskii, the Kobayashi metric is given by a length function $F(x, \xi), x \in M, \xi \in T_x(M)$, defined on the holomorphic tangent space $T_x(M)$ at each point $x \in M$. Letting $D_x = \{ z \in \mathbb{C} : |z| < \rho_x \}$, the definition of $F(x, \xi)$ is $F(x, \xi) = \inf \{ f(\rho) \}$, where $f, D_x \rightarrow M$ is a holomorphic mapping satisfying the conditions $f(0) = x, f(\rho) \rightarrow \xi$. A beautiful theorem of Bloch [2] gives the Kobayashi metric on $\mathbb{P}^2$ (4 lines) $- (L_1 + L_2 + L_3 + L_4)$ as follows: $F(x, \xi) > 0$ unless $\xi$ is tangent to one of the finitely many rational and elliptic curves on $M$.

As a closing remark, it seems to me that a proof of any of these questions will necessitate relating the local higher-order geometry of an analytic curve in $M$ to the global structure of $M$; i.e., some sort of Plücker formulas for an analytic curve in $M$ are needed.

![Fig. 1](image-url)
REFERENCES

8. P. Griffiths, On the Bezzout problem for entire analytic sets (to be published in *Ann. of Math.*).