Chapter I

VARIATION OF HODGE STRUCTURE

Phillip Griffiths
Written by Loring Tu

§1. *Hodge structures*

Let X be a compact Kähler manifold (e.g., a smooth projective variety). A C^∞ form on X decomposes into (p,q)-components according to the number of dz's and $d\bar{z}$'s. Denoting the C^∞ n-forms and the C^∞ (p,q)-forms on X by $A^n(X)$ and $A^{p,q}(X)$ respectively, we have the decomposition

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X) .$$

The cohomology $H^{p,q}(X)$ is defined to be

$$\begin{aligned} H^{p,q}(X) &= \{\text{closed } (p,q)\text{-forms}\}/\{\text{exact } (p,q)\text{-forms}\} \\ &= \{\phi \in A^{p,q}(X) : d\phi = 0\}/dA^{n-1}(X) \cap A^{p,q}(X) . \end{aligned}$$

THEOREM 1 (Hodge Decomposition Theorem). *Let* X *be a compact Kähler manifold. Then in each dimension* n *the complex de Rham cohomology of* X *can be written as a direct sum*

$$H^n_{DR}(X,C) = \bigoplus_{p+q=n} H^{p,q}(X) .$$

REMARK 2. One can define a decreasing filtration on $A^n(X)$ by

$$F^pA^n(X) = A^{n,0}(X) \oplus \cdots \oplus A^{p,n-p}(X)$$

and a decreasing filtration on $H^n_{DR}(X)$ by

$$F^pH^n_{DR}(X) = H^{n,0}(X) \oplus \cdots \oplus H^{p,n-p}(X) .$$

The group $F^pH^n_{DR}(X)$ may also be described as

$$F^pH^n_{DR}(X) = \{\phi \in F^pA^n(X) : d\phi = 0\}/dA^{n-1} \cap F^pA^n(X) .$$

It has been found useful to extract the contents of the Hodge decomposition theorem into a definition, for it is not only the complex cohomology of a compact Kähler manifold that possesses such a decomposition.

DEFINITION 3. A *Hodge structure of weight* n, denoted $\{H_Z, H^{p,q}\}$, is given by a lattice H_Z of finite rank together with a decomposition on its complexification $H = H_Z \otimes C$:

$$H = \bigoplus_{p+q=n} H^{p,q}$$

such that

$$H^{p,q} = \overline{H^{q,p}} .$$

Here by a lattice of finite rank we mean simply a finitely generated free Abelian group.

Alternatively a Hodge structure of weight n can be given by a lattice H_Z of finite rank together with a decreasing filtration on its complexification $H = H_Z \otimes C$:

$$H = F^0 \supset F^1 \supset \cdots \supset F^n$$

such that

$$H \simeq F^p \oplus \overline{F}^{n-p+1}$$

The two definitions are equivalent, for given a decomposition $H = \oplus H^{p,q}$, one defines the filtration by

$$F^p = H^{n,0} \oplus \cdots \oplus H^{p,n-p} ,$$

and given a filtration $\{F^p\}_{p=0,\cdots,n}$, one defines the decomposition by

$$H^{p,q} = F^p \cap \overline{F}^q .$$

It is not difficult to check that these constructions satisfy the requisite properties. We may therefore denote a Hodge structure of weight n either by $\{H_Z, H^{p,q}\}$ or by $\{H_Z, F^p\}$. The $H^{p,q}$ are called the *Hodge components of* H and the filtration $\{F^p\}$ the *Hodge filtration* of H.

REMARK 4. By thinking heuristically of F^p as forms possessing at least p dz's, the various superscripts become more intelligible. For example, $\overline{F}^q$ would be forms possessing at least q $d\overline{z}$'s. Since the total weight is $p+q=n$, $F^p \cap \overline{F}^q$ consists of forms having *precisely* p dz's and q $d\overline{z}$'s. Similarly, $\overline{F}^{n-p+1}$ consists of forms having at least $n-(p-1)$ $d\overline{z}$'s, or equivalently at most $p-1$ dz's; consequently, $F^p \oplus \overline{F}^{n-p+1}$ encompasses all n-forms.

EXAMPLES OF HODGE STRUCTURES. (a) (Hodge [8]). Let X be a compact Kähler manifold. For any integer n take

$$H_Z = H^n(X, Z)/\text{torsion} .$$

Then

$$H = H_Z \otimes C = H^n_{DR}(X, C) = \bigoplus_{p+q=n} H^{p,q}(X) ,$$

and $\{H_Z, H^{p,q}(X)\}$ is a Hodge structure of weight n.

(b) (Deligne [3]). As in (a) but with X any smooth complete abstract algebraic variety over C. (Such a variety need not be Kähler, since it may admit no embedding into a projective space.)

A *polarized algebraic variety* is a pair (X, ω) consisting of an algebraic variety X together with the first Chern class ω of a positive line bundle on X. Let

$$L : H^n(X, C) \to H^{n+2}(X, C)$$

be multiplication by ω. We recall below two fundamental theorems of Lefschetz.

THEOREM 5 (Hard Lefschetz Theorem). *On a polarized algebraic variety* (X, ω) *of dimension* d,

$$L^k : H^{d-k}(X,\mathbb{C}) \to H^{d+k}(X,\mathbb{C})$$

is an isomorphism for every positive integer $k \leq d$.

Thus

$$L^{d-n} : H^n(X,\mathbb{C}) \to H^{2d-n}(X,\mathbb{C})$$

is an isomorphism. The *primitive cohomology* $P^n(X,\mathbb{C})$ is defined to be the kernel of L^{d-n+1}. (For the geometric interpretation of this definition, see [6, p. 122].)

THEOREM 6 (Lefschetz Decomposition Theorem). *On a polarized algebraic variety* (X, ω), *we have for any integer* n *the following decomposition*:

$$H^n(X,\mathbb{C}) \simeq \bigoplus_{k=0}^{\left[\frac{n}{2}\right]} L^k P^{n-2k}(X,\mathbb{C}) .$$

It follows that the primitive cohomology groups determine completely the full complex cohomology.

Let (X, ω) be a polarized algebraic variety. Define

$$H_{\mathbb{Z}} = P^n(X,\mathbb{C}) \cap H^n(X, \mathbb{Z})$$

and

$$H^{p,q} = P^n(X,\mathbb{C}) \cap H^{p,q}(X) .$$

Then $\{H_{\mathbb{Z}}, H^{p,q}\}$ is a Hodge structure of weight n. On this Hodge structure there is a bilinear form

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$$

given by

$$Q(\phi,\psi) = (-1)^{n(n-1)/2} \int_X \phi \wedge \psi \wedge \omega^{d-n} .$$

This bilinear form makes $\{H_{\mathbb{Z}}, H^{p,q}\}$ into a *polarized Hodge structure* in the following sense.

DEFINITION 7. A *polarized Hodge structure of weight* n, denoted $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ or $\{H_{\mathbb{Z}}, F^p, Q\}$, is given by a Hodge structure of weight n together with a bilinear form

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z} ,$$

which is symmetric for n even and skew-symmetric for n odd, satisfying the two *Hodge-Riemann bilinear relations*:

(8) $\qquad Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = n-p$ and $q' = n-q$,

(9) $\qquad (\sqrt{-1})^{p-q} Q(\psi, \overline{\psi}) > 0$ for any nonzero element ψ in $H^{p,q}$.

We define the *Weil operator* $C : H \to H$ by

$$C|_{H^{p,q}} = (\sqrt{-1})^{p-q} .$$

For example, $C(dz) = idz$ and $c(d\overline{z}) = -id\overline{z}$. In terms of the Hodge filtration $\{F^p\}$ the bilinear relations are

(10) $\qquad Q(F^p, F^{n-p+1}) = 0$,

(11) $\qquad Q(C\psi, \overline{\psi}) > 0$ for any nonzero element ψ in H.

This bilinear form Q is called a *polarization* on the Hodge structure.

Two basic constructions and their relation with cycles

We can associate to a Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}\}$ of weight n one of two objects depending on whether n is even or odd.

(i) If $n = 2m$, then the *Hodge group* is

$$H_Z^{m,m} = H_Z \cap H^{m,m}.$$

An element of the Hodge group is called a *Hodge class*. The rank of the Hodge group $H_Z^{1,1}$ is called the *Picard number* of X.

To motivate what is to follow, we recall the construction of the Jacobian of a curve. Let C be a curve and γ a 1-cycle on C. Integration of holomorphic 1-forms over γ

$$\omega \mapsto \int_\gamma \omega, \quad \omega \in H^0(C, \Omega^1),$$

defines a linear functional on $H^0(C, \Omega^1)$. Thus there is a map $H_1(C, Z) \to (H^0(C, \Omega^1))^*$. The Jacobian of C is

$$J(C) = \frac{(H^0(C, \Omega^1))^*}{H_1(C, Z)}.$$

Making the identifications

$$(H^0(C, \Omega^1))^* \simeq (H^{1,0})^* \simeq H^{0,1}$$

and

$$H_Z \simeq H_1(C, Z),$$

we can write

$$J(C) = H_Z \backslash H^{0,1}.$$

Now we come to the construction of the intermediate Jacobian.

(ii) If $n = 2m-1$, then

$$H = \underbrace{H^{2m-1,0} \oplus \cdots \oplus H^{m,m-1}}_{F^m} \oplus H^{m-1,m} \oplus \cdots \oplus H^{0,2m-1}$$

and the *intermediate Jacobian* of H is

$$\begin{aligned} J &= H_Z \backslash (\text{second half of Hodge decomposition}) \\ &= H_Z \backslash H/F^m. \end{aligned}$$

These two constructions, the Hodge group and the intermediate Jacobian, are closely related to the study of algebraic cycles on a smooth variety. Let X be a smooth algebraic variety. Two cycles Z_1 and Z_2 on X are *algebraically equivalent* if, roughly speaking, one can be deformed into the other via an algebraic family of cycles on X. To be more precise, there is an algebraic variety S and an algebraic cycle T in $S \times X$ such that Z_1 and Z_2 are the restrictions of T to two fibers of the projection $\pi: S \times X \to S$ (see Figure 1). It may not be possible to choose T effective even when Z_1 and Z_2 are.

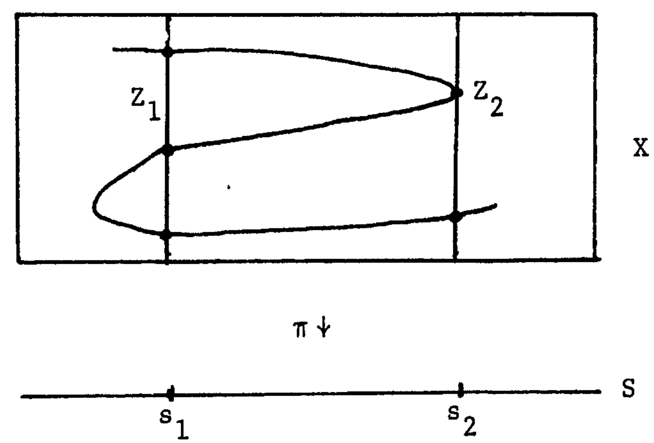

Figure 1

Two cycles are *rationally equivalent* if they are algebraically equivalent with a chain of P^1's as the parameter space S (Figure 2).

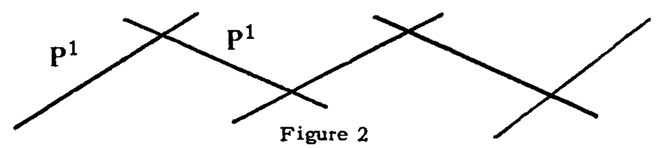

Figure 2

Let $\mathcal{Z}^m(X)$ be the Abelian group generated by the codimension m reduced and irreducible subvarieties of X. An element of $\mathcal{Z}^m(X)$ is a *codimension* m *algebraic cycle.* Denoting by $\mathcal{Z}^m_r(X)$, $\mathcal{Z}^m_a(X)$, and $\mathcal{Z}^m_h(X)$ the codimension m cycles rationally equivalent to zero, algebraically equivalent to zero, and homologically equivalent to zero respectively, there is a sequence of inclusions

$$\mathcal{Z}^m_r(X) \subset \mathcal{Z}^m_a(X) \subset \mathcal{Z}^m_h(X) \subset \mathcal{Z}^m(X) .$$

If Z is a cycle of codimension m on a variety X of dimension d, one can associate to the homology class of Z a fundamental class $f([Z]) \in H^{m,m}_{\mathbf{Z}}$ as follows. Let Z_{reg} be the regular points of Z. Integration over Z_{reg} defines a linear functional on $H^{2(d-m)}(X)$:

$$\psi \mapsto \int_{Z_{reg}} \psi , \qquad \psi \in A^{2(d-m)}(X) . \tag{12}$$

By Poincaré duality, this linear functional $f(Z)$ may be identified with an element of $H^{2m}(X, \mathbf{Z})$. Because on a variety of dimension d a $2d$-form must be of type (d,d), the form ψ in (12) can be taken to be in $A^{d-m,d-m}(X)$. Hence

$$f(Z) \in (H^{d-m,d-m}(X))^* \simeq H^{m,m} .$$

If Z is homologous to zero, then by Stokes' theorem, the integral of a closed form over Z is zero. We have, therefore, a map

$$f : \mathcal{Z}^m(X)/\mathcal{Z}^m_h(X) \to H^{m,m}_{\mathbf{Z}} ,$$

called the *fundamental class map.*

Next we take up the relation between cycles and intermediate Jacobians. Given a smooth variety X of dimension d, the m^{th} intermediate Jacobian $J^m(X)$ is defined to be the intermediate Jacobian of the cohomology $H^{2m-1}(X)$:

$$J^m(X) = H^{m-1,m} \oplus \dots \oplus H^{0,2m-1}(X)/H_{\mathbf{Z}} .$$

By Poincaré duality there is a canonical identification

$$J^m(X) = (F^{d-m+1} H^{2d-2m+1}(X))^*/\Lambda^* ,$$

where Λ^* is the image of the map

$$\alpha : H_{2d-2m+1}(X, \mathbf{Z}) \to (H^{2d-2m+1}(X))^*$$

given by integration. Our choice of the superscripts is dictated by the definition of the *Abel-Jacobi map* for codimension m cycles:

$$u : \mathcal{Z}^m_h(X)/\mathcal{Z}^m_r(X) \to J^m(X) .$$

If Z is a cycle homologous to zero in X, then $Z = \partial\Gamma$ for some chain Γ of dimension $2(d-m)+1$. We define $u(Z)$ by

$$u(Z)(\psi) = \int_\Gamma \psi \quad \text{for all } \psi \text{ in } A^{2(d-m)+1}(X). \tag{13}$$

If Γ is not a manifold, this integral is taken in the sense of currents. Liebermann ([9] and [10]) showed that all this makes sense.

An element of $H(X, \mathbf{Q}) \cap H^{m,m}$ is called a *rational Hodge class.* By tensoring with $\mathbf{Q}$, the fundamental class map can be defined over $\mathbf{Q}$. The Hodge conjecture asks whether every rational Hodge class is the fundamental class of some algebraic cycle with rational coefficients. (There are counterexamples for torsion integral classes. See Atiyah and Hirzebruch, ''Analytic cycles on complex manifolds,'' Topology 1 (1962), 25-45.)

The codimension one case of the Hodge conjecture is answered by the following theorem of Lefschetz.

THEOREM 1.14 (Lefschetz Theorem on (1,1)-Classes). *Let* X *be a smooth projective variety. Then every integral* $(1,1)$*-class on* X *is the fundamental class of a divisor on* X.

Before considering some low-dimensional examples of the Hodge group and the intermediate Jacobian, we want to identify two of the intermediate Jacobians. On a smooth variety X of any dimension d there is the exponential sequence

$$0 \longrightarrow Z \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0 .$$

From the associated long exact sequence

$$\cdots \to H^1(X,Z) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \to H^2(X,Z) \to \cdots ,$$

we see that $H^1(X,\mathcal{O})/H^1(X,Z)$ is the group of the isomorphism classes of the line bundles of first Chern class zero. This is by definition the *Picard variety* $\mathrm{Pic}^0(X)$ of X. Note that it is also the intermediate Jacobian $J^1(X)$:

$$J^1(X) \simeq \mathrm{Pic}^0(X) . \tag{15}$$

On the other hand, since

$$H^{2d-1}(X) = H^{d,d-1} \oplus H^{d-1,d} ,$$

the intermediate Jacobian

$$\begin{aligned} J^d(X) &= H^{d-1,d}/H_Z \\ &= H^0(X,\Omega^1)^*/H_1(X,Z) . \end{aligned}$$

This last group is by definition the *Albanese variety* $\mathrm{Alb}(X)$ of X. Hence

$$J^d(X) = \mathrm{Alb}(X) . \tag{16}$$

EXAMPLE 17. (a) For a smooth curve C the intermediate Jacobian J^1 is the Jacobian of the curve, rational equivalence is linear equivalence, and the Abel-Jacobi map for J^1 is the usual Abel-Jacobi map

$$u : \mathrm{Div}^0(C) \to J(C) .$$

Note that in this case,

$$J^1 = J(C) = \mathrm{Pic}^0(C) = \mathrm{Alb}(C) .$$

(b) Let F be a smooth surface.

For $H^1(F)$ the intermediate Jacobian is the Picard variety:

$$J^1(F) \simeq \mathrm{Pic}^0(F) .$$

For $H^2(F)$ the Lefschetz theorem on (1,1)-classes completely settles the nature of the Hodge classes.

For $H^3(F)$ the intermediate Jacobian is the Albanese variety:

$$J^2(F) \simeq \mathrm{Alb}(F) .$$

(c) Let V be a threefold. For $H^1(V)$ and $H^5(V)$ the intermediate Jacobians $J^1(V)$ and $J^3(V)$ are again the Picard and the Albanese varieties respectively, of which we have some degree of understanding. The group $H^2(V)$ is taken care of by the Lefschetz theorem on (1,1)-classes. Since $H^4(V)$ is isomorphic to $H^2(V)$ by the Hard Lefschetz theorem, the Hodge conjecture holds for $H_Z^{2,2}$. Indeed, if ϕ is an integral (2,2)-class, then $\phi = \omega \cdot \xi$ for some rational (1,1)-class ξ. By the Lefschetz theorem on (1,1)-classes, ξ is a multiple of the fundamental class of a divisor S in V. But then an integral multiple of ϕ is the fundamental class of a hyperplane section of S.

Thus the first mysterious group is $H^3(V)$ of a threefold V. If $h^{3,0}(V) = 0$, then

$$H^3(V) = H^{2,1} \oplus H^{1,2}$$

behaves very much like a Hodge structure of weight one. But if $h^{3,0}(V) \neq 0$, then the associated intermediate Jacobian $J^2(V)$ so far eludes understanding. We mention here one interesting special case.

SPECIAL CASE. Let $V \subset \mathbf{P}^4$ be a smooth quintic threefold. (Threefolds of lower degree all have $h^{3,0} = 0$.) Using Schubert calculus, for example, one can show that V has 2875 lines, some counted with multiplicity. Because $H^2(V) = \mathbf{Z}$, the difference of any two lines is homologous to zero:

$$L_{ij} = L_i - L_j \in H^2(V) .$$

It is known that if V is a general quintic threefold, then $u(L_{ij}) \neq 0$, where u is the Abel-Jacobi map.

OPEN QUESTIONS. Are the only relations given by $u(L_{ij}) + u(L_{ji}) = 0$? Can the configuration $\{u(L_{ij})\} \subset J^2(V)$ be determined using infinitesimal variation of Hodge structure (cf. Chapters III, XII below)?

We remark that, by considering higher degree rational curves on V, Herb Clemens has drawn a very surprising conclusion (cf. Chapter XVI below).

Another question is to explicitly compute out the Abel-Jacobi mapping in one nontrivial example with $h^{3,0} \neq 0$; e.g., the threefold $\sum_{i=0}^{4} x_i^d = 0$ $(d \geqq 5)$ in $\mathbf{P}^4$.

(d) For a fourfold X, apart from H^1, H^2, H^6, and H^7, which we can take care of as before, virtually nothing is known about the rest.

PROBLEM. Try to understand the algebraic subvarieties of

$$X = F_1 \times F_2 \quad \text{(a product of two surfaces)}$$

and

$$X = \mathbf{C}^4/\text{lattice} \quad \text{(an Abelian variety)} .$$

§2. *Classifying spaces*

Let $H_{\mathbf{Z}}$ be a fixed lattice, n an integer, Q a bilinear form on $H_{\mathbf{Z}}$, which is symmetric if n is even and skew-symmetric if n is odd, and $\{h^{p,q}\}$ a collection of integers such that $p+q = n$ and $\Sigma h^{p,q} = \operatorname{rank} H_{\mathbf{Z}}$. As before, denote by H the complexification $H_{\mathbf{Z}} \otimes \mathbf{C}$.

DEFINITION 18. With the notations above, the *classifying space* D for the polarized Hodge structures of type $\{H_{\mathbf{Z}}, Q, h^{p,q}\}$ is the set of all collections of subspaces $\{H^{p,q}\}$ of H such that

$$H = \bigoplus_{p+q=n} H^{p,q}, \quad \dim H^{p,q} = h^{p,q} ,$$

and on which Q satisfies the two bilinear relations (8) and (9).

Set $f^p = h^{n,0} + \cdots + h^{p,n-p}$. In terms of filtrations, D is the set of all filtrations

$$H = F^0 \supset F^1 \supset \cdots \supset F^n, \quad \dim F^p = f^p ,$$

on which Q satisfies the bilinear relations (10) and (11).

A priori D is only a set, but in fact it can be given the structure of a complex manifold. The simplest way to do this is to represent D as an open subset of a homogeneous algebraic variety.

DEFINITION 19. The *compact dual* $\check{D}$ of D is the subspace of $\Pi_{p=0}^{n} G(f^p,H)$ consisting of the filtrations $\{F^p\}$ on H satisfying the first bilinear relation (10). Here $G(f^p,H)$ denotes the Grassmannian of f^p-dimensional subspaces of H.

Because the first bilinear relation (10) is a set of algebraic equations, the compact dual is clearly an algebraic variety. We will show below that it is in fact a complex manifold.

In connection with a polarized Hodge structure there are three basic Lie groups:

$$\begin{aligned} G_{\mathbf{Z}} &= \operatorname{Aut}(H_{\mathbf{Z}},Q) \\ &= \{g : H_{\mathbf{Z}} \to H_{\mathbf{Z}} | Q(g\phi, g\xi) = Q(\phi,\xi) \ \text{for all } \phi, \xi \text{ in } H_{\mathbf{Z}}\} , \end{aligned}$$

$$G_{\mathbf{R}} = \operatorname{Aut}(H_{\mathbf{R}},Q) ,$$

$$G_{\mathbf{C}} = \operatorname{Aut}(H,Q) .$$

EXERCISE 20. Show that G_C acts transitively on the compact dual $\check{D}$ and that G_R acts transitively on the classifying space D.

Because the group G_C acts transitively on $\check{D}$, the variety $\check{D}$ is smooth. Let B be the stabilizer of a point in $\check{D}$. Then

$$\check{D} \simeq G_C/B .$$

Since D is an open subset of $\check{D}$, it is also a complex manifold. By Exercise 20,

$$D \simeq G_R/V ,$$

where $V = G_R \cap B$ is the stabilizer of a point in D.

We will find it useful to have the infinitesimal versions of these basic Lie groups. So let $\{H_0^{p,q}\}$ in D be chosen as the base point, which we regard as the reference Hodge structure. The Lie algebra $\mathfrak{g}_R$ of G_R has the following description:

$$\mathfrak{g}_R = \{Y \in \mathrm{Hom}(H_R,H_R) | Q(Y\psi,\eta)+Q(\psi,Y\eta) = 0 \text{ for all } \psi,\ \eta \text{ in } H_R\} .$$

Analogous descriptions hold for the Lie algebras $\mathfrak{g}_Z$ and $\mathfrak{g}_C$ of G_Z and G_C. Here $\mathfrak{g}_Z = \mathfrak{g}_C \cap \mathrm{Hom}(H_Z,H_Z)$; note that

$$\mathfrak{g}_C = \mathfrak{g}_Q \otimes C .$$

We can give the space $\mathrm{Hom}(H,H)$ a Hodge structure of weight zero by setting

$$\mathrm{Hom}(H,H)^{r,-r} = \bigoplus_{p+q=n} \mathrm{Hom}(H^{p,q},H^{p+r,q-r})$$

and

$$\mathrm{Hom}(H,H)^{r,-r} = \overline{\mathrm{Hom}(H,H)^{-r,r}} .$$

Since $\mathfrak{g}_C$ is a rationally defined subspace of $\mathrm{Hom}(H,H)$, it inherits a

Hodge structure of weight zero from $\mathrm{Hom}(H,H)$:

$$\mathfrak{g}_C^{r,-r} = \mathfrak{g}_C \cap \mathrm{Hom}(H,H)^{r,-r} = \overline{\mathfrak{g}_C^{-r,r}} .$$

If $\mathfrak{b}$ is the Lie algebra of the complex stability group B, then

$$\mathfrak{b} = \bigoplus_{r \geq 0} \mathfrak{g}_C^{r,-r} ,$$

for these are precisely the infinitesimal automorphisms that leave the reference Hodge filtration fixed. If $\mathfrak{h}$ is the Lie algebra of the real stability group H, then

$$\mathfrak{h} = \mathfrak{b} \cap \mathfrak{g}_R .$$

Over the compact dual $\check{D}$ we have the universal subbundles $\mathcal{F}^p \to \check{D}$. These are holomorphic vector bundles. Their quotient bundles

$$\mathcal{H}^{p,q} = \mathcal{F}^p/\mathcal{F}^{p+1}$$

are also holomorphic vector bundles. We call the restriction of these $\mathcal{H}^{p,q}$ to the classifying space D, the *Hodge bundles*. The Hodge bundle $\mathcal{H}^{p,q}$ is a vector bundle over D whose fiber at the point $\{F^p\}$ is $F^p/F^{p+1} = H^{p,q}$. Note that $\mathcal{F}^p$ has the C^∞ decomposition

$$\mathcal{F}^p = \mathcal{H}^{n,0} \oplus \dots \oplus \mathcal{H}^{p,n-p} .$$

Because of the second bilinear relation, each Hodge bundle $\mathcal{H}^{p,q}$ has a G_R-invariant metric, $(\sqrt{-1})^{p-q}Q(\ ,\bar{\ })$, making it into a Hermitian vector bundle. As is well known, any Hermitian vector bundle has a canonical connection and therefore a curvature. We shall comment on the curvature of the Hodge bundles in the second lecture.

We now turn to the tangent bundle of D. First recall that over the Grassmannian $G(k,H)$ there is a canonical isomorphism

$$T(G(k,H)) \simeq \mathrm{Hom}(S,Q) ,$$

where S and Q are the universal subbundle and quotient bundle respectively. One way of giving this isomorphism is by the following recipe. Suppose $F(t)$ is an arc in $G(k,H)$ with initial point $F \in G(k,H)$ and initial vector $\xi \in T_F(G(k,H))$. For any v in F, let $v(t)$ be a curve in $F(t)$ with $v(0) = v$. Then the homomorphism $\bar{\xi} : F \to H/F$ corresponding to the tangent vector ξ is given by

$$\bar{\xi}(v) = \frac{d}{dt} v(t)\Big|_{t=0} \pmod{F} .$$

Denoting by $\mathcal{H}$ the trivial bundle with fiber H over $\check{D}$, we therefore have

$$T(\check{D}) \subset \bigoplus_{p=1}^{n} \mathrm{Hom}(\mathcal{F}^p, \mathcal{H}/\mathcal{F}^p)$$

$$= \bigoplus_{p=1}^{n} \mathrm{Hom}(\mathcal{H}^{n,0} \oplus \cdots \oplus \mathcal{H}^{p,n-p}, \mathcal{H}^{p-1,n-p+1} \oplus \cdots \oplus \mathcal{H}^{0,n}) .$$

Similarly, the tangent bundle $T(D)$ is also contained in this direct sum of homomorphisms of Hodge bundles. Because each Hodge bundle has a $G_{\mathbf{R}}$-invariant Hermitian metric, the classifying space D also has a $G_{\mathbf{R}}$-invariant Hermitian metric, which we denote by ds_D^2.

REMARK 21. In fact, this $G_{\mathbf{R}}$-invariant metric on the classifying space D is induced by the Killing form on the Lie algebra $\mathfrak{g}_{\mathbf{R}}$.

Examples of classifying spaces

EXAMPLE 22 (Weight One). For $n = 1$,

$$H = H^{1,0} \oplus H^{0,1} , \qquad H^{1,0} = \overline{H^{0,1}} ,$$

and Q is a skew-symmetric bilinear form. Let $g = \dim H^{1,0}$. Then each filtration $H^{1,0} \subset H$ is an element of the Grassmannian $G(g,H)$. Relative to a suitable basis for H, the skew form Q is represented by the matrix

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} .$$

So D is the set of all g-dimensional isotropic subspaces of $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ on $\mathbf{C}^{2g}$. (An *isotropic* subspace of a skew form J is a subspace V such that $J(V,V) \equiv 0$.) These are precisely the *maximal* isotropic subspaces.

The classifying space D may be identified with the Siegel upper half space

$$\mathbf{H}_g = \{g \text{ by } g \text{ complex matrices } Z = X + iY \mid Z \text{ is symmetric and } Y \text{ is positive definite}\} ,$$

as follows. Each element of $G(g,H)$ is represented by a $2g$ by g matrix Ω, up to the equivalence relation

$$\Omega \sim B\Omega A , \quad B \in Sp(2g,\mathbf{Z}) , \quad A \in GL(g,\mathbf{C}) .$$

The matrix Ω can be brought to the normal form

$$\Omega = \begin{bmatrix} I \\ Z \end{bmatrix} ,$$

where Z is a g by g complex matrix. By the first bilinear relation Z is symmetric. By the second bilinear relation $\mathrm{Im}\, Z$ is positive definite.

§3. *Variation of Hodge structure*

Let $\pi : \mathfrak{X} \to S$ be a family of smooth polarized projective varieties. By this we mean that there is a commutative diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "\pi"'] & \subset S \times \mathbf{P}^N \arrow[dl] \\ S & \end{tikzcd}$$

and that the fibers of $\pi : X \to S$ are smooth projective varieties. By associating to each fiber X_s the Hodge decomposition of its primitive cohomology, we get locally maps from open subsets of S into the classifying space D. Because one can identify the integer cohomology $H^n(X_s, Z)$/torsion of a fiber with a fixed lattice H_Z only up to the action of the monodromy group Γ, it is in general not possible to get a map from S into D. Instead, what one has is a map into the quotient of D by the action of the monodromy group

$$\phi : S \to \Gamma \backslash D ,$$

called the *period map*. It is shown in [4] that the period map is holomorphic.

A priori the differential of the period map

$$\phi_* : T(S) \to T(D) \subset \bigoplus_{p=1}^{n} \bigoplus_{r=1}^{p} (\mathrm{Hom}(\mathcal{H}^{p,q}, \mathcal{H}^{p-r,q+r}))$$

goes into the full tangent space of D at each point. However, by [4] we do know that in fact ϕ_* shifts the Hodge filtration by only one; that is, ϕ_* maps into the subbundle $\overset{n}{\underset{p=1}{\oplus}} \mathrm{Hom}(\mathcal{H}^{p,q}, \mathcal{H}^{p-1,q+1})$.

To formalize the essential properties of the period map, we digress for a minute to discuss differential systems.

DEFINITION 23. A *differential system* on a complex manifold X is given in one of two equivalent ways:

(i) either by a holomorphic subbundle $T_h(X)$ of the holomorphic tangent bundle $T(X)$, or

(ii) by an ideal I in the complex $\Omega^*(X)$ of holomorphic forms on X generated by a collection of 1-forms and their exterior derivatives.

An ideal such as in (ii) is called a *differential ideal*.

Given (i) we define I to be the differential ideal generated by

$$\begin{aligned} \psi \in T_h(X)^{\perp} &= \text{annihilator of } T_h(X) \\ &= \{\psi \in \Omega^1(X) | \psi(v) = 0 \text{ for all } v \in T_h(X)\} \end{aligned}$$

and

$$d\psi \in \Omega^2(X) .$$

Conversely, given a differential ideal I we define the holomorphic subbundle $T_h(X)$ to be the kernel of all the 1-forms in I. (This requires a constant rank assumption.)

DEFINITION 24. An *integral manifold* of a differential ideal I on X is a holomorphic map of complex manifolds, $\phi : S \to X$, such that $\phi^* I = 0$.

This is equivalent to saying that

$$\phi_* T(S) \subset T_h(X) .$$

In checking integrality it suffices to check it on 1-forms, since if ψ is a 1-form and $\phi^*\psi = 0$, then $\phi^* d\psi = d\phi^*\psi = 0$.

DEFINITION 25. The *horizontal differential system* on D is

$$T_h(D) = \{\xi \in T(D) | \xi(F^p) \subset F^{p-1}\} .$$

Because integrality is a local condition and because the period map ϕ is locally liftable to D, we may restate the horizontality of the period map in the following form: the period map of a family of polarized algebraic varieties is an integral manifold of the horizontal differential system on D.

DEFINITION 26. An *integral element* of a differential ideal I on a complex manifold X is a subspace $E \subset T_x(X)$ for some x in X such that

$$(*) \qquad \psi_\alpha|_E = 0$$

and

$$(**) \qquad d\psi_\alpha|_E = 0$$

for all generators ψ_α of degree 1 in I.

Note that (*) are linear equations and (**) are quadratic equations on a Grassmannian. In the theory of differential equations a system is said to be *involutive* if, roughly speaking, one cannot obtain new equations by differentiating the system. In other words, given (1) a system of differen- equations, one looks at (2) the set of all solutions, and then at (3) the system of all differential equations annihilating the solutions in (2). If (3) = (1), the system is said to be *involutive*.

OPEN QUESTION 27. Let $\phi : S \to \Gamma\backslash D$ be the period map of a family of polarized algebraic varieties and let I be the differential ideal determined by

$$\phi_* : T(S) \to T(\Gamma\backslash D)\,.$$

Are the differential equations of I an involutive system?

DEFINITION 28. A *variation of Hodge structure* is a map $\phi : S \to \Gamma\backslash D$, where S is a complex manifold and Γ is a subgroup of $G_{\mathbf{Z}}$, such that ϕ is

(i) locally liftable,

(ii) holomorphic,

(iii) an integral manifold of the horizontal differential system I_h.

REMARK 29. Let $\tilde{S}$ be the universal covering of S. Then Condition (i) is equivalent to the existence of a map $\tilde{\phi} : \tilde{S} \to D$ which makes the following diagram commutative:

$$\begin{tikzcd} \tilde{S} \arrow[r, dashed, "\tilde{\phi}"] \arrow[d] & D \arrow[d] \\ S \arrow[r, "\phi"] & \Gamma\backslash D\,. \end{tikzcd}$$

In terms of filtrations a variation of Hodge structure may be thought of as a family of variable Hodge filtrations, varying with s,

$$F_s^p = H_s^{n,0} \oplus \cdots \oplus H_s^{p,n-p}$$

on a fixed vector space H such that

(i) for each s the filtration is defined up to the action of an element of $\Gamma \subset G_{\mathbf{Z}}$,

(ii) $\partial F_s^p / \partial \bar{s} \subset F_s^p$,

(iii) $\partial F_s^p / \partial s \subset F_s^{p-1}$.

Let $\mathfrak{M}_g$ be the moduli space of smooth curves of genus g and $\Gamma_g = Sp(g, \mathbf{Z})$ the symplectic group. In Example 22, we noted that the classifying space for the polarized Hodge structures of weight 1 can be identified with the Siegel upper half space $\mathcal{H}_g$. Hence, by associating to each curve its Hodge filtration $H^{1,0} \subset H^1$, we obtain a map

$$\phi : \mathfrak{M}_g \to \Gamma_g \backslash \mathcal{H}_g\,.$$

However, because $\mathfrak{M}_g$ is not smooth and ϕ is not locally liftable around curves with automorphisms, it is not a variation of Hodge structure as we have defined the term. To take care of this situation, we introduce the notion of an *extended variation of Hodge structure.*

DEFINITION 30. Let S be any variety, possibly singular. A map $\phi : S \to \Gamma\backslash D$ is called an *extended variation of Hodge structure* if there is a smooth dense Zariski open set $S' \subset S$ such that $\phi|_{S'} : S' \to \Gamma\backslash D$ is a variation of Hodge structure.

By the local liftability property, given a variation of Hodge structure, we have the *monodromy representation*

$$\rho : \pi_1(S) \to \Gamma\,.$$

For an extended variation of Hodge structure, there is the monodromy representation

$$\rho : \pi_1(S') \to \Gamma\,,$$

which, of course, depends on the open subset S'. But it turns out that the group $\rho(\pi_1(S')) \subset \Gamma$ is independent of $S' \subset S$. This is called the *monodromy group* of the extended variation of Hodge structure.

Speaking in general terms the global Torelli problem asks whether the Hodge decomposition of a variety determines the isomorphism class of the variety. The idea for proving such a statement would be to try to associate a geometric object to a Hodge structure, for example, the theta divisor of a Jacobian in the case of curves. In higher dimensions this cannot be done. What seems more amenable is the generic global Torelli theorem, which would say that the period map ϕ from some sort of moduli space $\mathfrak{M}$ onto its image $\phi(\mathfrak{M}) \subset \Gamma\backslash D$ has degree one; this means ϕ is generically one-to-one, but there could be points in $\Gamma\backslash D$ whose inverse images have more than one point. For an exposé of the recent progress in the pathology of the period map, see Chapter VIII of this volume. Generic global Torelli theorems are discussed in Chapters IX - XIII.

The period map and Hodge bundles

Let $\phi: S \to \Gamma\backslash D$ be a variation of Hodge structure with monodromy representation $\rho: \pi_1(S) \to \Gamma$. We view the universal covering $\tilde{S}$ of S as a principal $\pi_1(S)$-bundle over S. Then the monodromy representation ρ induces a lattice bundle $\mathcal{H}_Z$ with group $\pi_1(S)$ over S, defined as the quotient of $\tilde{S} \times H_Z$ by the equivalence relation

$$(\tilde{s}g, \psi) \sim (\tilde{s}, g\psi), \quad \text{where} \quad g \in \pi_1(S) .$$

The complexification of $\mathcal{H}_Z$ is a complex vector bundle over S. Because $\mathcal{H}$ has the same transition functions as $\mathcal{H}_Z$, it is locally constant and hence holomorphic. In the following, we will identify the holomorphic vector bundles over S with the locally free sheaves of $\mathcal{O}_S$-modules. Associated to the locally constant sheaf $\mathcal{H}$ is a (1,0)-connection ∇, relative to which the sections of $\mathcal{H}_Z$ are locally constant.

At each point $s \in S$, the period map $\phi: S \to \Gamma\ D$ defines a filtration

$$\{0\} \subset \mathcal{F}^n_s \subset \cdots \subset \mathcal{F}^1_s \subset \mathcal{F}^0_s = \mathcal{H}_s ,$$

giving rise to a sequence of holomorphic subbundles $\mathcal{F}^p$ of $\mathcal{H}$. The

Hodge bundle $\mathcal{H}^{p,q}$ is defined to be the quotient bundle

$$\mathcal{H}^{p,q} = \mathcal{F}^p/\mathcal{F}^{p+1} .$$

There is a C^∞ (*not* holomorphic) decomposition

$$\mathcal{H} = \bigoplus_{p+q=n} \mathcal{H}^{p,q}, \qquad \mathcal{H}^{p,q} = \overline{\mathcal{H}^{q,p}} .$$

In this context, the infinitesimal period relation becomes

$$\nabla\mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1 .$$

In summary, a variation of Hodge structure $\phi: S \to \Gamma\backslash D$ gives the data $\{\mathcal{H}_Z, \mathcal{F}^p, \Delta, S\}$, where $\mathcal{H}_Z$ is a sheaf of lattices, $\mathcal{F}^p$ a filtration on $\mathcal{H} = \mathcal{H}_Z \otimes \mathcal{O}_S$, and $\nabla: \mathcal{H} \to \mathcal{H} \otimes \Omega^1$ the connection whose locally constant sections are $\mathcal{H}_Z \otimes C$ and which satisfies the infinitesimal period relation. Conversely, a set of data such as this determines a period map $\phi: S \to \Gamma\backslash D$. So a variation of Hodge structure can be given in either of these two equivalent ways.

EXAMPLE 31. Let $\pi: \mathfrak{X} \to S$ be a family of polarized algebraic varieties of dimension d all of whose fibers are smooth. Set

$$\mathcal{H}_Z = \mathcal{R}^n \pi_* Z ,$$

$$\mathcal{F}^p_s = F^p H^n_{DR}(X_s, C) .$$

Define ∇ to be differentiation under the integral sign; this means,

$$\frac{d}{ds} \int_\gamma \omega(s) = \int_\gamma \nabla_{\frac{\partial}{\partial s}} \omega(s) ,$$

where $\gamma \in H_n(X_s, Z)$ is a family of cycles. Then $\{\mathcal{H}, \mathcal{F}^p, \nabla, S\}$ is a variation of Hodge structure.

Normal functions

Given a variation of Hodge structure $\{\mathcal{H}_{\mathbf{Z}}, \mathcal{H}^{p,q}, \nabla, S\}$, the connection induces by the infinitesimal period relations a map

$$(*) \qquad D : \mathcal{H}/\mathcal{F}^p \to (\mathcal{H}/\mathcal{F}^{p-1}) \otimes \Omega^1 .$$

A section ν of $\mathcal{H}/\mathcal{F}^p$ with $D\nu = 0$ is said to be *quasi-horizontal.* The quasi-horizontal sections of course include the horizontal ones (those for which $\nabla\nu = 0$).

In case the variation of Hodge structure has odd weight $n = 2m-1$, we can define a family of intermediate Jacobians by setting

$$\mathcal{J} = \mathcal{H}_{\mathbf{Z}} \backslash \mathcal{H}/\mathcal{F}^m \to S .$$

Because $\nabla\mathcal{H}_{\mathbf{Z}} = 0$, (*) induces a map on $\mathcal{J}$:

$$D : \mathcal{J} \to (\mathcal{H}/\mathcal{F}^{m-1}) \otimes \Omega^1 .$$

The kernel of D,

$$\mathcal{J}_h = \{\nu \in \mathcal{J} | D\nu = 0\} ,$$

is called the *sheaf of normal functions.* The sections of $\mathcal{J}_h$ are the *normal functions.* These are discussed in Chapters XIV - XVII.

EXAMPLE 32. Let $\mathfrak{X} \to S$ be a family of algebraic varieties and $\mathcal{Z} \subset \mathfrak{X}$ a codimension m cycle such that each intersection

$$\mathcal{Z} \cdot X_s = Z_s \in \mathcal{Z}^m_h(X_s)$$

is homologous to zero. Define

$$\nu(s) = u_{X_s}(Z_s) \in J(X_s)$$

where $u_{X_s} : \mathcal{Z}^m_h(X_s) \to J(X_s)$ is the Abel-Jacobi map on X_s. It is known that $\nu : S \to \mathcal{J}$ is holomorphic and assumes values in $\mathcal{J}_h$ (see [5]). It is called the *normal function associated to* $\mathcal{Z}$.

DEFINITION 33. A polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ is said to be *unimodular* if $\det Q = \pm 1$.

REMARK. The intermediate Jacobian of a unimodular polarized Hodge structure of odd weight is a principally polarized complex torus.

PROBLEM 34 (Beauville). Let ϕ be a unimodular polarized Hodge structure. Suppose ϕ can be written as a direct sum of unimodular polarized Hodge structures:

$$\phi = \oplus \phi_\nu .$$

Is this decomposition unique?

REMARK. The answer is no if the Hodge structures are not assumed unimodular.

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PHILLIP GRIFFITHS
MATHEMATICS DEPARTMENT
HARVARD UNIVERSITY
CAMBRIDGE, MA 02138

LORING TU
MATHEMATICS DEPARTMENT
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218

Chapter II
CURVATURE PROPERTIES OF THE HODGE BUNDLES

Phillip Griffiths
Written by Loring Tu

We consider a polarized variation of Hodge structure $\phi : S \to \Gamma\backslash D$, which we think of locally as a variable polarized Hodge decomposition on a fixed vector space:

$$\begin{cases} H = \bigoplus\limits_{p+q=n} H_s^{p,q} \\ F_s^p = H_s^{n,0} \oplus \cdots \oplus H_s^{p,n-p}, \end{cases}$$

where s varies over the variety S. (To be strictly correct, s should be in the universal covering $\tilde{S}$, for otherwise it may not be possible to have the fixed vector space H. Locally the description just given is fine.) We have

$$\frac{\partial H_s^{p,q}}{\partial \bar{s}} \subseteq H_s^{p+1,q-1} \oplus H_s^{p,q}$$

and by conjugation,

$$\frac{\partial H_s^{p,q}}{\partial s} \subseteq H_s^{p,q} \oplus H_s^{p-1,q+1} ;$$

or in terms of filtrations,

$$\frac{\partial F_s^p}{\partial \bar{s}} \subseteq F_s^p$$