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ON THE VARIETY OF SPECIAL LINEAR SYSTEMS ON A GENERAL ALGEBRAIC CURVE

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0. Introduction

(a) On a smooth algebraic curve C of genus g we consider a divisor D of degree d. A classical problem is to determine the dimension $h^0(D)$ of the vector space $H^0(D)$ of rational functions having poles only on D, or equivalently the dimension $r(D)$ of the complete linear system $|D| = \mathbb{P}(H^0(D))$ of effective divisors linearly equivalent to D. Denoting by K the canonical divisor, the Riemann-Roch formula

$$r(D) = d - g + h^0(K - D)$$

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gives the answer in case D is non-special—i.e., $h^0(K - D) = 0$—but says little for special divisors, which, as is well known, are usually the divisors that are relevant to specific geometric problems. As a first answer to the question

what special linear systems may exist on C?

we have Clifford's theorem

$$r(D) \leqslant d/2.$$

In a sense this answer is sharp, for on a hyperelliptic curve and for every d and r with $r \leqslant d/2 \leqslant g - 1$ there exist divisors that satisfy $\deg D = d$ and $r(D) = r$. Hyperelliptic curves, however, are exceptional for $g \geqslant 3$, and so we still have the question

what special linear systems exist on a general curve of genus g?

More precisely, we can ask how many linear systems of a given degree and dimension there are on a general curve C in the following sense: Effective divisors of degree d are parametrized by the dth symmetric product C_d of C, and linear equivalence classes of degree d by the Jacobian $J(C)$. Letting C_d^r and W_d^r denote, respectively, the locus of divisors D of degree d with $r(D) \geqslant r$ and the image of C_d^r in the Jacobian, we may ask:

On a general curve, what are the dimensions of C_d^r *and* W_d^r?

It appears that the answer to this question was first suggested by Brill and Noether [1], who asserted that on a general curve of genus g the dimension of W_d^r should be

$$\begin{aligned}\rho &= g - (r + 1)(g - d + r)\\ &= h^0(K) - h^0(D)h^0(K - D),\end{aligned}$$

and the dimension of C_d^r equal to $\rho + r$.

Assuming this is correct further interesting geometric questions arise: for example, in those cases where $\rho = 0$ so that W_d^r is a finite set of points we may ask, how many? For general $\rho \geqslant 0$, since we have nice descriptions of that part of the cohomology ring generated by analytic subvarieties for both the Jacobian and symmetric product of a curve with general moduli, we may ask

What are the cohomology classes of the loci $W_d^r \subset J(C)$ *and* $C_d^r \subset C_d$?

Here we recall that the analytic cohomology of $J(C)$, where C has general moduli, is generated by the class θ of the theta divisor $\theta \sim W_{g-1}^0$, and the analytic cohomology ring of C_d is generated by the pullback $\tilde{\theta} = \pi^*\theta$ of θ under the natural map $\pi : C_d \to J$ together with the class x of the image X_p of C_{d-1} in C_d under the map $D \to D + p$. With a view toward using the answer to this question for solving enumerative problems, we may ask how C_d^r sits in C_d relative to other naturally defined subvarieties; e.g.,

Does C_d^r meet the general subvariety
$X^{r+\rho}_{p_1, \ldots, p_{r+\rho}} = C_{d-r+\rho} + p_1 + \cdots + p_{r+\rho}$ *transversely?*

The answer to the above question is contained in the

MAIN THEOREM. (I) *For any curve C of genus g*

$$\dim W_d^r \geqslant \rho \qquad \dim C_d^r \geqslant \rho + r.$$

(II) *For a general curve* (a)

$$\dim W_d^r = \rho \qquad \dim C_d^r = \rho + r,$$

(b) *The fundamental classes w_d^r of W_d^r and c_d^r of C_d^r are given by*

$$w_d^r = \lambda(g, r, d)\theta^{g-\rho}$$

$$c_d^r = \lambda(g, r, d) \sum_{\alpha=0}^{r} \frac{(-1)^\alpha (g - d + r - 1 + \alpha)!}{\alpha!\,(r - \alpha)!} x^\alpha \tilde{\theta}^{g-r-\rho-\alpha}$$

where

$$\lambda(g, r, d) = \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}.$$

(c) *If $p_1, \ldots, p_{r+\rho}$ are general points of a general curve, then the intersection $X^{r+\rho}_{p_1, \ldots, p_{r+\rho}} \cap C_d^r$ is transverse, consisting of*

$$\frac{g!\,(\rho + r)!}{\rho!\,(g - d + 2r + \rho)!} \prod_{i=0}^{r-1} \frac{i!}{(g - d + r + i)!}$$

distinct points.

As mentioned before, the roots of this theorem are to be found in the classic paper by Brill and Noether [2], who proved (I) assuming that W_d^r is non-empty. This was deduced as a consequence of the above formula for the fundamental class w_d^r, which was proved in Kempf [6] and Kleiman-Laksov [8, 9]. In this formula it is assumed that $\dim W_d^r = \rho$, and the multiplicities of the various irreducible components of W_d^r are counted. Our main contribution is to establish (II) which together with [6, 8, 9] implies that on a general curve W_d^r and C_d^r have the predicted dimension ρ and that all irreducible components occur with multiplicity one. Here the case $r = 1$ was proved by Laksov [10], Lax [11], and

Martens [12] and the case $r = 2$ recently by Arbarello and Cornalba. From the point of view of this paper a crucial step was furnished by Kleiman [7], who revived an old idea of Severi [14] to use a degeneration method of Castelnuovo [3] to establish the dimension statement (IIa). In Kleiman's paper it is rigorously proved that the dimension count would follow from a transversality statement concerning certain Shubert cycles associated to generic chords to a rational normal curve, and in this paper we shall prove the transversality result which we shall call the Castelnuovo-Severi-Kleiman conjecture. This proof will be given in section 2, after we have given in section 1 a proof of the reduction of (IIa) to the Castelnuovo-Severi-Kleiman conjecture; in section 3 we prove the refined transversality/multiplicity-one assertions in IIb and c.

Throughout this paper, we will assume the results of Kempf-Kleiman-Laksov, specifically the assertion that "The locus $W_d^r \subset J(C)$ (respectively $C_d^r \subset C_d$) supports the cohomology class given in IIb," which in particular implies that $W_d^r \neq \emptyset$.

(b) There are a number of statements one can make about the geometry of a general linear system D of dimension $\geqslant r$ and degree d on a general curve C of genus g, purely on the basis of part IIa of our main theorem and the behavior of the function $\rho(g, r, d)$. In the following, then, D will denote a general point of $W_d^r(C)$.

(1) $|D|$ *has dimension exactly* r. This just amounts to saying that

$$\begin{aligned}\dim W_d^{r+\delta} &= g - (r + \delta + 1)(g - d + r + \delta)\\ &= g - (r+1)(g - d + r) - \delta(g - d + 2r + 1 + \delta)\\ &< g - (r+1)(g - d + r) = \dim W_d^r.\end{aligned}$$

(2) $|D|$ *is not composite*, i.e., D cannot be written as the sum of two effective divisors D_1, D_2 with $r(D_1) + r(D_2) = r$. To see this, we simply note that for any $r_1 + r_2 = r$, $d_1 + d_2 = d$,

$$\begin{aligned}\dim W_{d_1}^{r_1} + \dim W_{d_2}^{r_2} &= g - (r_1 + 1)(g - d_1 + r_1) + g - (r_2 + 1)(g - d_2 + r_2)\\ &= g - (r+1)(g - d + r) - r_1(d_2 - r_2) - r_2(d_1 - r_1)\\ &< \dim W_d^r\end{aligned}$$

since (as long as $d_i > 0$ and $g > 0$), $d_i > r_i$. Applying this in particular in case $r_1 = 0$, we may conclude that

(3) $|D|$ *has no base points.*

(4) *For* $r \geqslant 2$, *the map* $\phi_D : C \to \mathbf{P}^r$ *given by* D *is a birational embedding*. This follows not so much from our main Theorem as from the statement that *a general curve* C *of genus* $g \geqslant 2$ *cannot be expressed as a multiple cover of any curve* C' *of genus* $g' \geqslant 1$. This is readily seen from a count of parameters: the curve C'

will depend on $3g' - 3$ parameters, and the m-sheeted covering $C \to C'$ depends on b parameters, where

$$b = 2g - 2 - m(2g' - 2)$$

is the number of branch points of the cover. Thus if $m \geqslant 2$, C will depend on

$$2g - 2 - \left(m - \frac{3}{2}\right)(2g' - 2) \leqslant 2g - 2 < 3g - 3$$

parameters, and so cannot be general.

We see from this that if ϕ_D maps C m-fold onto a curve $C_0 \subset \mathbf{P}^r$, then C_0 must be rational, and $|D|$ composite with a pencil, contradicting (2).

In consequence of the above, we have the

COROLLARY. *For $r \geqslant 2$ and $g \geqslant (r+1)(g-d+r)$, a general curve of genus g may be birationally embedded as a nondegenerate curve of degree exactly d in $\mathbf{P}^r$.*

This corollary may be applied to obtain information on Chow varieties, as follows:

Suppose that Ξ is an irreducible component of the Chow variety of algebraic curves of fixed degree d in $\mathbf{P}^r$. Assume, moreover, that the curve C_ξ corresponding to a general point $\xi \in \Xi$ has the properties (i) *C_ξ is non-degenerate, and* (ii) *C_ξ is a general curve of (arithmetic) genus $g \geqslant 2$. Then the Brill-Noether number*

$$\rho = g - (r+1)(g-d+r) \geqslant 0,$$

and the dimension of Ξ is given by

$$\dim \Xi = r^2 + 2r + \rho + 3g - 3.$$

When $r = 3$—i.e., for space curves—this is

$$\dim \Xi = 4d.$$

Finally, Ξ is reduced (more precisely, the corresponding component of the Hilbert scheme is reduced).

The last statement will be proved in [1].

We mention here two further questions one may ask in regard to the "general linear system" D above. First, one may ask for more information about the map ϕ_D: in particular, is it the case that for $r \geqslant 3$ ϕ_D is a biregular embedding; and, more generally, for C general, what will be the codimension in W_d^r of the loci $W_d^{r'}, W_d^{r''}$ of linear systems $|D|$ possessing a node, and a cusp, respectively? [Some investigation tends to indicate that the naive estimates:

$$\operatorname{codim} W_d^{r'} \subset W_d^r = r - 2$$

$$\operatorname{codim} W_d^{r''} \subset W_d^r = r - 1$$

hold—suggesting in particular that ϕ_D is indeed an embedding when $r \geqslant 3$.] Another series of questions one may pose concern the multiplicative structure of the linear system $|D|$, e.g., the ranks of the maps

$$\phi_k : \mathrm{Sym}^k H^0(D) \to H^0(kD)$$

and

$$\mu_0 : H^0(D) \otimes H^0(K - D) \to H^0(K).$$

It has been proved that for D as above, the second map μ_0 does have maximal rank (to appear [1]); as for the first collection of maps—whose rank in effect describes the postulation of the image curve $\phi_D(C) \subset \mathbf{P}^r$—little is at present known.

Finally, we mention here some restatements and/or special cases of part IIc of our main theorem. First of all, in case $\rho = 0$—i.e., $g = (r+1)(g-d+r)$—we see that by IIb a general curve C of genus g may be realized as a nondegenerate curve of degree d in $\mathbf{P}^r$ in exactly $g!\,\lambda(g,r,d)$ ways. More generally, in case $\rho > 0$, we may ask further that a given collection $p_1, \ldots, p_{r+\rho}$ of points of C be mapped into a hyperplane in $\mathbf{P}^r$; IIc tells us that for $p_1, \ldots, p_{r+\rho} \in C$ general, there exist exactly

$$\frac{(\rho + r)!\; g!}{\rho!\,(g - d + 2r + \rho)!} \prod_{i=0}^{r-1} \frac{1!}{(g - d + r + i)!}$$

nondegenerate maps $C \to \mathbf{P}^r$ carrying $\{p_\alpha\}$ into a hyperplane.

Taking $r = 1$, we obtain in particular the statements:

On a general curve of genus $g = 2k$, *there exist exactly* $(2k)!/(k!(k+1)!)$ *rational functions of degree* $k+1$, *modulo linear functional transformations* $f \to (af+b)/(cf+d)$ (cf. [10]), and in general:

For $p_1, \ldots, p_{\rho+1}$ *general points on a general curve* C *of genus* $g = 2k - \rho$, *there exist exactly* $(\rho+1)(2k-\rho)!/[(k+1)!\,(k-\rho)!]$ *rational functions of degree* $k+1$ *with poles* (*resp. zeroes*) *at* $p_1, \ldots, p_{\rho+1}$, *modulo transformations of the form* $f \to af + b$ (*resp.* $f \to f/(cf+d)$).

(c) The most direct approach to the problem of determining the dimension of C_d^r is to simply write down the conditions that a divisor $D = p_1 + \cdots + p_d$ on C move in an r-dimensional linear system. The Riemann-Roch formula suggests that we examine the following matrix:
If z_i is a local coordinate around p_i and $\omega_1, \ldots, \omega_g$ a basis for the holomorphic differentials on C, then locally around p_i

$$\omega_\alpha = f_{i,\alpha}(z_i)\, dz_i.$$

According to the Riemann-Roch formula, $r(D) \geqslant r$ if, and only if, the Brill-Noether matrix

$$\Omega(D) = \begin{bmatrix} f_{11}(p_1) & \cdots & f_{1g}(p_1) \\ \vdots & & \vdots \\ f_{d1}(p_d) & \cdots & f_{dg}(p_d) \end{bmatrix}$$

has rank $\leqslant d-r$. (We remark that we are assuming that the points p_i are distinct, but it is well known that the symmetric product C_d is a smooth manifold and our considerations may be extended to cover the case when some of the p_i coincide.) More precisely, the ith row of the Brill-Noether matrix gives the homogeneous coordinates of the point p_i on the canonical curve $C \subset \mathbf{P}^{g-1}$. Then

$$\operatorname{rank} \Omega(D) = \dim \overline{p_1, \ldots, p_d} - 1$$

where $\overline{p_1, \ldots, p_d}$ is the linear span of the points $p_i \in \mathbf{P}^{g-1}$. (This is defined in case some of the points p_i are coincident, in part e of this section.)

Now the matrix $\Omega(D)$ defines a holomorphic mapping

$$\Omega : \Delta \to M_{d,g}$$

from a coordinate neighborhood of Δ in C_d to the space of $d \times g$ matrices, and since the locus $M_r \subset M_{d,g}$ of matrices having rank $\leqslant d-r$ has codimension $r(g-d+r)$ we may conclude that if the inverse image $C_d^r = \Omega^{-1}(M_r)$ is non-empty then its codimension is at most that of M_r in $M_{d,g}$; i.e.,

Every component of C_d^r has dimension $\geqslant d - r(g-d+r) = \rho + r$.

This is the original argument of Brill-Noether, and it suggests that

Every component of W_d^r has dimension $\geqslant \rho$.

(To completely justify this reasoning there is one technical point: While it seems reasonable that the general fibre of $C_d^r \to W_d^r$ has dimension exactly equal to r, until the dimension of C_d^r or W_d^r is actually computed it is a priori possible that $W_d^r = W_d^{r+1}$, in which case the fibres of $C_d^r \to W_d^r$ all have dimension $\geqslant r+1$. This turns out not to cause trouble, as our approach will circumvent the matter.)

The next step in the analysis of C_d^r was taken by Kempf [6] and Kleiman-Laksov [8] and [9]. In the second Kleiman-Laksov paper the idea is to compute the expected classes c_d^r and w_d^r by globalizing the above argument and, finding the classes non-zero, to conclude that neither is empty. To explain this briefly, we note that a change of holomorphic coordinates on C_d has the effect of multiplying on the left the Brill-Noether matrix by the Jacobian matrix of the coordinate change. Consequently, $\Omega(D)$ may be interpreted as the local expression of a global section of $T^*(C_d) \otimes \mathbf{C}^g$ where $T^*(C_d)$ is the cotangent bundle of C_d. Now the locus where a given collection of sections of a vector bundle E have a given rank is described by a polynomial in the Chern classes of E by a special case of Porteous' formula (cf. page 415 of [5]). Computing everything out in this case gives:

The locus C_d^r supports the cohomology class

$$\lambda(g,r,d) \sum_{\alpha=0}^{r} (-1)^{\alpha} \frac{(g-d+r-1+\alpha)!}{\alpha!\,(r-\alpha)!} x^{\alpha} \tilde{\theta}^{g-r-\rho-\alpha}$$

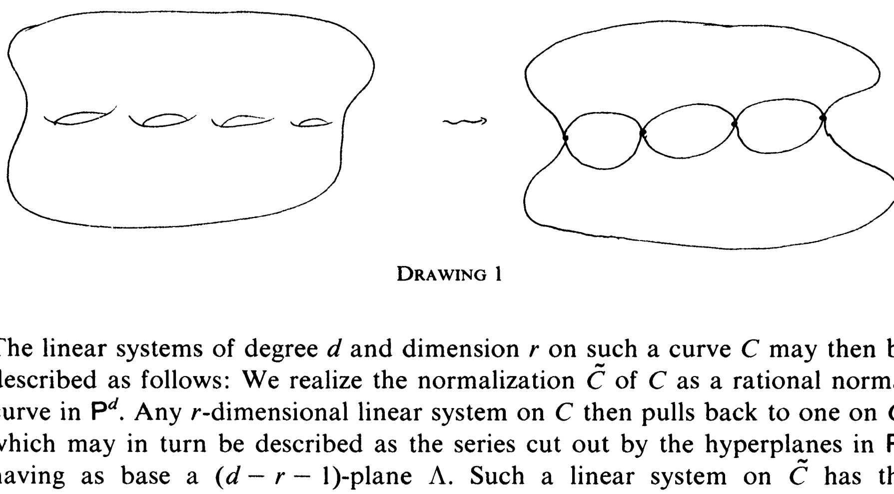
and is therefore of dimension at least $\rho + r$. Similarly, W_d^r supports the class $\lambda(g, r, d)\theta^{g-\rho}$ and is thus of dimension $\geqslant \rho$.

(d) We now come to the problem of establishing the dimension count

$$\dim W_d^r = \rho \qquad (*)$$

for a general curve C. The difficulty here is how to deal with a "general curve." It is certainly not the case that (*) is true for every curve: branched covers of $\mathbf{P}^1$ with small sheet number and smooth plane curves of degree $\geqslant 5$ provide examples where $\rho < 0$ but W_d^r is non-empty. On the other hand, by upper-semicontinuity of dimension and the irreducibility of the moduli of curves it will suffice to prove (*) for just a single smooth curve. This help is more apparent than real: when g is large, for virtually every curve that we can actually write down (e.g., complete intersections, plane curves with few singularities, etc.) the dimension of W_d^r is too large.

One way out of this apparent bind was suggested—albeit indirectly—by Castelnuovo [3]. He was trying to compute the degree of W_d^r in case $\rho = 0$—i.e., $g = (r + 1)(g - d + r)$—and had the beautiful idea of specializing not to a particular smooth curve but to a generic curve of arithmetic genus g having g nodes—what we shall hereafter call a *Castelnuovo canonical curve* (here is the picture of the Reimann surface of one of arithmetic genus 4).



DRAWING 1

The linear systems of degree d and dimension r on such a curve C may then be described as follows: We realize the normalization $\tilde{C}$ of C as a rational normal curve in $\mathbf{P}^d$. Any r-dimensional linear system on C then pulls back to one on $\tilde{C}$, which may in turn be described as the series cut out by the hyperplanes in $\mathbf{P}^d$ having as base a $(d - r - 1)$-plane Λ. Such a linear system on $\tilde{C}$ has the additional property that every divisor of the system (that is, every hyperplane through Λ) containing one of the points (p_α, q_α) $(\alpha = 1, \ldots, g)$ lying over a node of C contains the other as well. Equivalently, Λ meets the chord $\overline{p_\alpha q_\alpha}$ joining the points of $\tilde{C}$ lying over each node of C. Consequently, the linear systems g_d^r on C correspond to $(d - r - 1)$-planes $\Lambda \subset \mathbf{P}^d$ meeting each of g chords to a rational normal curve. Now the subvariety $\sigma_r(l)$ of the Grassmannian $\mathbf{G} = \mathbf{G}(d - r - 1, d)$ consisting of $(d - r - 1)$-planes meeting a line $l \subset \mathbf{P}^d$ is a Schubert cycle of codimension r on $\mathbf{G}$, and so the expected dimension of the intersection $\bigcap_{\alpha=1}^{g} \sigma_r(\overline{p_\alpha q_\alpha})$ is

$$\begin{aligned}\dim \mathsf{G} - rg &= (r+1)(d-r) - rg \\ &= g - (r+1)(g-d+r) \\ &= \rho.\end{aligned}$$

Thus we hope for exactly ∞^ρ such $(d-r-1)$-planes Λ meeting each of g chords to a rational normal curve. In case $\rho = 0$, if we *assume* that the cycles $\sigma_r(\overline{p_\alpha q_\alpha})$ intersect transversely on the Grassmannian then we may compute their intersection number in G; this number, Castelnuovo suggests, should be the degree of W_d^r on a general curve.

Interestingly, Castelnuovo in his paper was interested solely in guessing the degree of W_d^r; at that time the statement $\dim W_d^r = \rho$ on a general curve was considered an established theorem.[1] Indeed, it was not until Anhang G in [1] that Severi first pointed out (in public) that the statement had in fact never been satisfactorily proved and, looking for a means to remedy the situation, suggested that Castelnuovo's degeneration might afford a proof. The argument would proceed in two parts: One would first use a specialization to conclude that, if the dimension of W_d^r on a general curve were strictly larger than ρ, then the same would be true also for a general Castelnuovo canonical curve C_0. Then one could rule out this possibility by representing $W_d^r(C_0)$ as the intersection of a collection of Schubert cycles as above and analyzing that intersection. Unfortunately, Severi was unable to complete either part of his two-step proof, and his idea lay dormant until Kleiman [7], using the techniques of modern algebraic geometry, analyzed with complete rigor the behavior of linear systems on suitably degenerating curves and established the first half of Severi's projected argument. The upshot is that parts IIa and b of the main theorem were reduced to the following statement concerning the intersection of Schubert cycles:

CASTELNUOVO-SEVERI-KLEIMAN CONJECTURE. *If, for any chord l to a rational normal curve $C \subset \mathsf{P}^d$ we denote by $\sigma(l) \subset \mathsf{G}(k,d)$ the Schubert cycle of k-planes Λ that meet l, then for generic chords the intersection*

$$\sigma(l_1) \cap \cdots \cap \sigma(l_g)$$

[1] It is interesting to go back and look at the original paper [3] of Castelnuovo, in which the idea of studying linear systems on a general curve by specializing to a g-nodal curve first appears. Again, it must be borne in mind that to Castelnuovo parts I and IIa of the main theorem above were proven theorems; he was attempting to establish, in effect, a special case ($\rho = 0$) of part IIb. Nonetheless, he seems to be aware of at least some of the difficulties involved in making precise the sort of argument he suggests: in a footnote to the article in Atti Acc. Lincei (which was altered in the version published in his *Memoire Scelte*) he points out that his argument is based "more on intuition (and on various examples) than on a true mathematical proof". He concludes by saying that "we allow ourselves to use a principle as yet unproved in order to solve a difficult problem, because we believe that such efforts may be of use to science, as long as we say explicitly what is suggested and what is proved."

is almost everywhere transverse. In particular, it has no multiple components and its dimension is

$$\dim(\sigma(l_1) \cap \cdots \cap \sigma(l_g)) = (k+1)(d-k) - g(d-k-1).$$

The purpose of the present paper is to complete Severi's program by giving a proof of this conjecture and thereby concluding the main theorem as stated above. Our proof shows that the conjecture is in fact valid for any non-degenerate irreducible local piece of complex-analytic arc in P^d.

In section 1 we will examine the behavior of linear systems under a Castelnuovo specialization and give an argument, within the framework and from the point of view of this paper, of the reduction of the dimension-theoretic part of the main theorem to the Castelnuovo-Severi-Kleiman conjecture. Our analysis is more elementary but yields somewhat less information that that in [7]; it is included here for the sake of completeness.

In section 2 we give a proof of the conjecture. Very briefly the idea is to degenerate further by carefully letting the g chords come together. Put another way, the main difficulty in the problem is what is meant by a general curve? Since all rational normal curves are the same, the $3g-3$ moduli have become the $2g-3$ moduli of g generic chords. By letting the chords come together we eliminate all moduli or, more geometrically, we degenerate the Castelnuovo g-nodal curve to a rational curve having one nasty but rigid singularity.

Finally, in section 3 we use the transversality statement to conclude the last part of the main theorem.

In writing up the arguments in each of the two steps of the proof we have in each instance given what were to us the crucial examples—that is, those special cases that when properly understood showed how the general argument should go.

(e) We now give some notations and terminology.

Throughout we shall use the word "curve" to mean a smooth non-hyperelliptic curve of genus $g \geqslant 3$. We shall use standard algebro-geometric notations—such as $|D|$ for a linear system of divisors, K for the canonical line bundle, etc.—all of which are the same as in [5]. Additionally we make the following conventions:

(i) $\mathsf{G}(k,d)$ is the Grassmannian of P^k's in P^d; it is the same as the Euclidean Grassmannian $G(k+1, d+1)$ of C^{k+1}'s in C^{d+1};

(ii) In section 2 we shall use $l(p,q)$ to denote the Schubert cycle $\sigma_{d-k-1}(\overline{pq})$ of k-planes meeting the line $\overline{pq}$; and

(iii) Finally, we shall frequently use without mention the well-known fact that any rational mapping of a smooth curve to a Grassmannian is in fact holomorphic: we embed the Grassmannian in projective space, and then the map is given on the punctured disc $t \neq 0$ by

$$f(t) = [f_0(t), \ldots, f_d(t)]$$

where the $f_\nu(t)$ are holomorphic for $t \neq 0$ and meromorphic at $t = 0$. We may write $f_\nu(t) = t^\lambda g_\nu(t)$ where all $g_\nu(t)$ are holomorphic at $t = 0$ and some $g_\nu(0) \neq 0$, and then

$$f(t) = [\, g_0(t), \ldots, g_d(t) \,]$$

is holomorphic at the origin.

In addition, we want to introduce here one notion that will be used repeatedly in the remainder of this paper: the multiplicity $m_p(C \cdot \Lambda)$ of intersection of a curve $C \subset \mathbf{P}^n$ with a linear subspace $\Lambda \subset \mathbf{P}^n$ at a point p. This may be defined either as the intersection multiplicity at p of C with a general hyperplane $H \subset \mathbf{P}^n$ containing Λ, or as the index at points over p of the pullback $\pi^* I_\Lambda$ of the ideal of $\Lambda \subset \mathbf{P}^n$ to the normalization $\tilde{C} \xrightarrow{\pi} C$. The equivalence may be seen by noting that the equation of a general $H \supset \Lambda$ generates $\pi^* I_\Lambda$ at each point of $\tilde{C}$ lying over p. The intersection multiplicity clearly has the following properties:

(i) If $\Lambda' \supset \Lambda$, then $m_p(\Lambda' \cdot C) \geqslant m_p(\Lambda \cdot C)$

(ii) If Λ' is a general plane of its dimension containing Λ, then $m_p(\Lambda' \cdot C) = m_p(\Lambda \cdot C)$, and

(iii) If Λ_t is a family of planes in $\mathbf{P}^n$, the function

$$m(t) = m_p(C \cdot \Lambda_t)$$

is upper-semicontinuous.

In what follows, we will often use the notion of intersection multiplicity implicitly. Thus, for example, by a *d-secant plane* to a curve $C \subset \mathbf{P}^n$ we will mean a plane Λ such that

$$\sum_p m_p(C \cdot \Lambda) = d;$$

if we fix a plane $\Lambda \subset \mathbf{P}^n$ and a hyperplane $H \supset \Lambda$ not passing through any of the singularities of C, "the divisor cut on C by H residual to Λ" will mean the divisor

$$\sum_p (m_p(C \cdot H) - m_p(C \cdot \Lambda)) \cdot p.$$

Finally by the *span* $\overline{m_1 p_1, \ldots, m_\alpha p_\alpha}$ of a linear combination $\sum m_\alpha p_\alpha$ of points $p_\alpha \in C$ we will mean the smallest plane $\Lambda \subset \mathbf{P}^n$ such that for all α

$$m_{p_\alpha}(C \cdot \Lambda) \geqslant m_\alpha;$$

or, equivalently, the intersection of the hyperplanes satisfying this condition. It is worth pointing out that if the points p are not inflectionary on C—as for example will always be the case if C is a rational normal curve—then this is just the span of the osculating $(m_1 - 1)$-plane to C at p_1, the osculating $(m_2 - 1)$-plane to C at p_2, and so on.

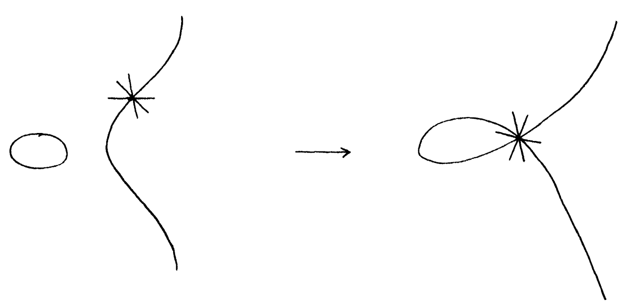
It is a pleasure to thank Enrico Arbarello and Maurizio Cornalba, with whom we worked on the problem from a cohomological viewpoint, related to the second interpretation given above of the Brill-Noether number. These results,

which among other things give the infinitesimal behavior of special linear systems under a variation of structure of the curve, will appear in a monograph that is currently under preparation. In addition, conversations with Steve Kleiman, Ron Donagi, and Mark Green have been extremely helpful.

Finally, we would like to thank the referee of this paper for a number of very valuable suggestions.

1. Reduction of the dimension count to the conjecture

(a) We consider here the specialization argument. The basic principle we would like to apply states that under any proper holomorphic map $f: X \to Y$, the dimension of the fibre is an upper-semicontinuous function on Y. Now the loci $W_d^r(C)$ of linear systems on a variable smooth curve C fit together nicely to form a closed subvariety $\mathcal{J}_d^r$ of the Jacobian bundle $\mathcal{J}$ over the moduli space of smooth curves. (This is intuitively clear; since these are just heuristic considerations we will not give a formal proof.) Thus there is no difficulty in applying this principle to smooth curves: the dimension of $W_d^r(C_0)$ for any smooth curve is consequently greater than or equal to dim $W_d^r(C)$ for a generic curve of the same genus. To use this line of argument with respect to a Castelnuovo canonical curve, however, means that we need to complete $\mathcal{J}_d^r$ to a variety proper over a subvariety of moduli containing Castelnuovo canonical curves. Now, as is fairly well known, the Jacobian bundle extends naturally to these degenerate curves; line bundles and/or linear systems on Castelnuovo canonical curves are parameterized by its generalized Jacobian $J(C_0) \cong (\mathbf{C}^*)^g$, which fits nicely into the family of Jacobians over moduli of smooth curves. The family $\mathcal{J}_d^r$ likewise extends to a closed subvariety of the extended Jacobian bundle over Castelnuovo curves, if we take as fibre $W_d^r(C_0)$ over such a curve the set of linear systems as described above—i.e., linear systems on the normalization $\tilde{C} \cong \mathbf{P}^1$ of C_0, every divisor of which contains one of the two points p_α, q_α if and only if it contains the other. The problem is that the generalized Jacobian of C_0, and likewise the fibre of $\mathcal{J}_d^r$ over C_0, is not compact, so that the principle of upper semicontinuity does not hold. For a concrete example one may imagine a family of elliptic curves acquiring a node



DRAWING 2

As generic linear system we take the g_2^1 of lines through a point $p \in C$, where under the specialization $C \to C_0$ the point p tends to the node p_0 of C_0. The points residual to p_0 in the pencil of lines through p_0 do not describe a linear system in the above sense on the nodal cubic. Of course this particular example can be overcome, but the phenomenon it illustrates requires attention. Put another way, if C_t is a family of smooth curves specializing to a Castelnuovo canonical curve C_0 and $|D_t| = \{D_{t,\lambda}\}_{\lambda \in \mathrm{P}^r}$ a linear system on C_t for $t \neq 0$, then the point $u(D_t)$ in the Jacobian $J(C_t)$ will tend to infinity—i.e., may not have a limit—in the generalized Jacobian $J(C_0)$ in case the limiting system of divisors $\{D_{0,\lambda}\}$ has a base point at one or more of the nodes of C_0. This raises the possibility that an entire component of $W_d^r(C_t)$ may disappear in the limit.

One way around this difficulty is by constructing a suitable compactification of the generalized Jacobian $J(C_0)$ to a complete variety $\overline{J(C_0)}$ such that the corresponding family $\bar{\mathfrak{J}}$ is proper over the compactified moduli space of curves. One may then close up $\mathfrak{J}_d^r$ so as to insure the existence of a specialization $W_d^r(C) \to W_d^r(C_0)$, and then the new data obtained in the limit must be analyzed. This is the method employed by Kleiman [7], and in a sense is the most natural approach to the problem.

The approach taken here will be based on synthetic geometry: rather than compactifying the family $W_d^r(C)$ for singular curves C as above, we will compactify the locus of associated special secant planes to the canonical curves. In fact, since we are then dealing with subvarieties of the Grassmannian the compactification already exists, and it is our task to interpret geometrically what we get. This whole approach is motivated by the geometric version of the Riemann-Roch theorem, that says that a divisor $D = \sum p_i$ on a canonical curve $C \subset \mathrm{P}^{g-1}$ moves in an r-dimensional linear system if and only if the span of the points p_i is a $(d-r-1)$-plane. Every divisor D with $h^0(D) = r+1$ thus corresponds to a d-secant $(d-r-1)$-plane to C, and since every linear system of degree d and dimension r contributes ∞^r such planes we see that part IIa of the main theorem stated in the introduction will follow once we establish that

the locus $\Sigma \subset \mathrm{G}(d-r-1, g-1)$ of d-secant $(d-r-1)$-planes to a general canonical curve $C \subset \mathrm{P}^{r-1}$ has dimension $\rho + r = g - (r+1)(d-g+r) + r$. (*)

The remainder of this chapter will be devoted to showing that (*) follows from the Castelnuovo-Severi-Kleiman conjecture. To do this, we let C_0 be a general Castelnuovo canonical curve, $\{C_t\}$ any 1-parameter family of canonical curves (generally smooth) degenerating to C_0,[2] and consider the intermediate statement

The locus $\Sigma \subset \mathrm{G}(d-r-1, g-1)$ of d-secant $(d-r-1)$-planes to C_0 not passing through any of the nodes C_0 has dimension at most $\rho + r$. (**)

[2]The existence of such a family, given C_0, is established in [7].

We now argue in two steps: we first show, in parts (b) and (c), that

$$\mathrm{CSK} \Rightarrow (**);$$

and then in part (d) that

$$(**) \Rightarrow (*).$$

(b) For the first part we shall recount some definitions pertaining to Castelnuovo canonical curves. Such a curve C_0 is obtained by identifying g pairs of points p_α and q_α on the Riemann sphere $\mathbf{P}^1$. A rational function that is regular at the nodes is given by a meromorphic function $f(t)$ on $\mathbf{P}^1$ such that $f(p_\alpha)$ and $f(q_\alpha)$ are finite and equal. We will be concerned primarily with divisors D on C_0 that are supported away from the nodes; two such divisors D and D' are linearly equivalent if there is a rational function f on C_0 regular at the nodes with $(f) = D' - D$. A linear system on C_0 is a family of effective divisors of the form $\{D + (f)\}$ where f varies in a vector space of rational functions on C_0. As in the case of smooth curves we may describe linear systems geometrically as follows. If $C_0 \subset \mathbf{P}^r$ is an embedding and $\Lambda \subset \mathbf{P}^r$ any linear subspace that is disjoint from the nodes then the residual intersections of C_0 with the hyperplanes in $\mathbf{P}^r$ containing Λ form a linear system.

An abelian differential on C_0 is a meromorphic 1-form ω on $\mathbf{P}^1$ that has at most first order poles at the points p_α, q_α and satisfies

$$\mathrm{Res}_{p_\alpha}(\omega) + \mathrm{Res}_{q_\alpha}(\omega) = 0.$$

There are g linearly independent abelian differentials on C_0: If t is a Euclidean coordinate on $\mathbf{P}^1$ they may be taken to be

$$\omega_\alpha = \frac{dt}{(t-\lambda_\alpha)(t-\lambda'_\alpha)} \qquad \alpha = 1, \dots, g$$

where $\lambda_\alpha = t(p_\alpha)$ and $\lambda'_\alpha = t(q_\alpha)$.

As for smooth curves we may consider the canonical mapping

$$\phi_K : C_0 \to \mathbf{P}^{g-1}$$

given by

$$t \to [\omega_1(t), \dots, \omega_g(t)].$$

This map extends across the nodes, and again as for smooth curves the canonical mapping will be an embedding unless C_0 is a hyperelliptic Castelnuovo canonical curve. (These may be described as follows: In terms of a suitable Euclidean coordinate t we have

$$t(p_\alpha) = -t(q_\alpha) = \lambda_\alpha$$

for all α, then the differentials are

$$\omega_\alpha = \frac{dt}{t^2 - \lambda_\alpha^2}$$

and the canonical mapping ϕ_K factors through the involution $t \to -t$). Since for a general collection of $2g \geqslant 6$ points p_α on $\mathbf{P}^1$ there are no automorphisms permuting the p_α, we see that again just as for smooth curves a general Castelnuovo canonical curve of genus greater than 2 is non-hyperelliptic. We shall assume this to be the case, so that ϕ_K is a biregular embedding.

For a divisor $D = s_1 + \cdots + s_d$ supported away from the nodes we define the index of speciality $i(D)$ to be the dimension of the space $H^0(K - D)$ of abelian differentials on C_0 that vanish at the points s_i. Equivalently, if $\overline{D} \subset \mathbf{P}^{g-1}$ is the linear span of the points $\phi_K(s_i)$ then

$$\dim \overline{D} = g - 1 - i(D).$$

For Castelnuovo canonical curves we have the elementary

Riemann-Roch theorem: The dimension of the complete linear system $|D|$ of divisors linearly equivalent to D is given by

$$r(D) = d - g + i(D)$$

or, geometrically,

$$\dim \overline{D} = d - 1 - r(D).$$

Proof. Let t be a Euclidean coordinate on $\mathbf{P}^1$ with $t(p_\alpha) = \lambda_2$, $t(q_\alpha) = \lambda'_\alpha$, and $t(s_i) = \mu_i$. We take the abelian differentials $\omega_1, \ldots, \omega_g$ as above. A rational function on $\mathbf{P}^1$ with poles at the points s_i may be written as

$$f(t) = b_0 + \sum \frac{b_i}{t - \mu_i}\,.$$

This function will be a rational function on C_0 if and only if

$$f(\lambda_\alpha) = f(\lambda'_\alpha) \qquad \alpha = 1, \ldots, g.$$

This is equivalent to

$$\begin{aligned} 0 &= \sum_i \frac{b_i}{\lambda_i - \mu_i} - \sum_i \frac{b_i}{\lambda'_\alpha - \mu_i} \\ &= \sum_i \frac{b_i(\lambda_\alpha - \lambda'_\alpha)}{(\mu_i - \lambda_\alpha)(\mu_i - \lambda'_\alpha)} \qquad \alpha = 1, \ldots, g; \end{aligned}$$

i.e., we must have

$$\sum_i \frac{b_i}{(\mu_i - \lambda_\alpha)(\mu_i - \lambda'_\alpha)} = 0 \qquad \alpha = 1, \ldots, g$$

which is the same as

$$\sum_i b_i \omega_\alpha(p_i) = 0 \qquad \alpha = 1, \ldots, g. \qquad \square$$

The geometric Riemann-Roch theorem may be restated as follows:

If $C_0 \subset \mathbf{P}^{g-1}$ is any Castelnuovo canonical curve and $D = p_1 + \cdots + p_d$ a divisor on C_0 supported away from the nodes, then the complete linear system $|D|$ has dimension r if and only if the points p_i span a $\mathbf{P}^{d-r-1}$.

(c) We can now easily establish the

LEMMA 1. *Assuming the Castelnuovo-Severi-Kleiman conjecture, the family of d-secant $(d - r - 1)$-planes to a general Castelnuovo canonical curve $C_0 \subset \mathbf{P}^{g-1}$ not passing through any of the nodes of C_0 has dimension at most $\rho + r$.*

Proof. By the geometric Riemann-Roch theorem for Castelnuovo canonical curves, a divisor of degree d on C_0 supported away from the nodes and spanning a $(d - r - \epsilon - 1)$-plane in $\mathbf{P}^{g-1}$ moves in an $(r + \epsilon)$-dimensional linear system. Again we realize the normalization $\tilde{C} \cong \mathbf{P}^1$ of C_0 as a standard rational normal curve of degree d in $\mathbf{P}^d$; then our linear system is cut out by the system of hyperplanes $H \subset \mathbf{P}^d$ containing a fixed $(d - r - \epsilon - 1)$-plane Λ that meets each of the g chords to $\tilde{C}_0$ joining pairs of points lying over the nodes of C_0. Assuming the Castelnuovo-Severi-Kleiman conjecture, the family of such linear systems is parametrized by the Λ's as above, and it has dimension exactly

$$g - (r + \epsilon + 1)(g - d + r + \epsilon).$$

Accordingly the family of all such divisors on C_0 has dimension

$$g - (r + \epsilon + 1)(g - d + r + \epsilon) + r + \epsilon = \rho + r - \epsilon(g - d + 2r + \epsilon).$$

Finally, since every such divisor gives rise to $\infty^{\epsilon(g-d+r)}$ d-secant $(d - r - 1)$-planes and every d-secant $(d - r - 1)$-plane to C_0 supported away from the nodes contains such a divisor for some $\epsilon \geq 0$, we see that the dimension of the family of such planes is at most

$$\max_{\epsilon \geq 0} \rho + r - \epsilon(r + \epsilon) = \rho + r. \qquad \square$$

As a corollary to this lemma we have the slightly stronger

LEMMA 1′. *If $p_1, \ldots, p_\delta$ are among the nodes of a generic Castelnuovo canonical curve C_0, then the family of $(d - r - 1)$-planes through $p_1, \ldots, p_\delta$ and meeting C_0 in a total of $d + \delta$ points (counting multiplicity) has dimension at most $g - (r + 1)(g - d + r) + r - \delta$.*

Proof. The basic remark is that if C_0 is generic—so that the nodes on C_0 correspond to g pairs of generic points on $\mathbf{P}^1$—then projection from the $\mathbf{P}^{\delta-1}$ spanned by the points $p_1, \ldots, p_\delta$ maps C_0 to a generic canonical curve $\overline{C}_0 \subset \mathbf{P}^{g-1-\delta}$ of genus $g-\delta$. The projection also maps the $(d+\delta)$-secant $(d-r-1)$-planes through $p_1, \ldots, p_\delta$ bijectively to $(d-\delta)$-secant $(d-r-\delta-1)$-planes to $\overline{C}_0$,[3] and applying lemma 1 this family has dimension at most

$$g-\delta-(r+1)((g-\delta)-(d-\delta)+r)+r=g-(r+1)(g-d+r)+r-\delta.$$

(d) With this established, we can now show that IIa in the main theorem follows from the Castelnuovo-Severi-Kleiman conjecture. The conclusion of the following discussion is stated at the end of this section.

To set up, let C_t be a family of canonical curves over a curve with local parameter t, with C_t smooth for $t \neq 0$ and C_0 a Castelnuovo canonical curve. Let $\Sigma_t \subset \mathrm{G}(d-r-1,\ g-1)$ be the locus of d-secant $(d-r-1)$-planes to C_t, and Σ the closure of the incidence correspondence

$$\Sigma = \overline{\{(t,\Lambda) : \Lambda \in \Sigma_t, t \neq 0\}}\ .$$

Σ_0 will denote the fibre of Σ over 0.

At the same time, we consider the secant planes associated to the dual series; that is, we let $\Sigma_t' \subset \mathrm{G}(e-s-1,\ g-1)$ be the locus of $e=(2g-2-d)$-secant $(e-s-1)=(g-r-2)$-planes to C_t

$$\Sigma' = \overline{\{(t,\Lambda') : \Lambda \in \Sigma_t, t \neq 0\}}$$

and Σ_0' the fiber of Σ' over 0. We note that since $\dim W_d^r = \dim W_e^s$, we have

$$\dim \Sigma_t \geqslant g-(r+1)(g-d+r)+r$$

and

$$\dim \Sigma_t' \geqslant g-(s+1)(g-e+s)+s$$
$$= g-(g-d+r)(r+1)+g-d+r-1$$

and that it will suffice to prove equality in either of these inequalities.

[3] That for any $\Lambda \in \mathbf{P}^{g-1}$ through $p_1, \ldots, p_\delta$,

$$\#\big(\pi(\Lambda)\cdot \overline{C}_0\big) = \#(\Lambda \cdot C_0) - 2\delta$$

follows from the fact that the intersection multiplicity of the plane $\mathbf{P}^{\delta-1} = \overline{p_1, \ldots, p_\delta}$ with C at each p_i is *exactly* 2. (This in turn follows from the fact that the projection map—the map given, if you like, by the linear system of differentials on C_0 vanishing at $p_1, \ldots, p_\delta$—embeds the normalization $\overline{C_0}$ of C_0 at $p_1, \ldots, p_\delta$ as a Castelnuovo canonical curve in $\mathbf{P}^{g-\delta-1}$).

We see first that, counting multiplicity, every plane $\Lambda \in \Sigma_0$ is at least d-secant to C_0. Consequently if a generic $\Lambda \in \Sigma_0$ did not pass through any of the nodes of C, we would be done: by the principle of upper-semicontinuity and lemma 1 we would have for generic t

$$\begin{aligned}\dim \Sigma_t &\leqslant \dim \Sigma_0 \\ &\leqslant g-(r+1)(g-d+r)+r.\end{aligned}$$

The question then is, can it happen that *every* plane $\Lambda \in \Sigma_0$ passes through one or more of the nodes of C_0; and if so, what can we say about the remaining points of intersection of Λ with C_0?

An example should help to clarify the sort of phenomena that may be expected. Consider a smooth canonical curve $C \subset \mathbf{P}^3$ of genus 4. Such a curve is the transverse intersection of a quadric surface $Q \subset \mathbf{P}^3$ with a cubic S; assume that Q is of rank 3, i.e., a cone over a plane conic. C will then possess one g_3^1, with trisecant lines the lines of the cone Q. Now let the cubic surface S vary in a pencil $\{S_t\}$, with S_0 simply tangent to Q at four points so that the curve $C_0 = Q \cap S_0$ is Castelnuovo. This is a nice degeneration: as $C_t = Q \cap S_t$ moves toward C_0, the actual locus Σ_t of trisecant lines remains fixed; in the limit the generic line avoids the nodes of C_0, while four lines pass through the nodes and meet C_0 once elsewhere.

Suppose, on the other hand, we take S_t a pencil with S_0 passing simply through the vertex of C_0 and tangent at three other points. Again, C_0 acquires four ordinary nodes; but now we see that, in the limit, all the trisecant lines pass through the node of C at the vertex of Q. Thus it is possible—at least in some families—that the planes of Σ_0 all contain a node of C_0 (note, however, that by virtue of the requirement that Q be singular this is not a generic degeneration, and C_0 not a generic Castelnuovo canonical curve). We observe one peculiar phenomenon: while in the first case the four trisecant lines passing through the nodes of C_0 met C_0 residually in only one point, in the second case, where every trisecant line passes through a node, the general one still meets C_0 residually in two points.

This last observation turns out to be the crucial one: in general, if the general plane $\Lambda \in \Sigma_0$ contains δ nodes $p_1, \ldots, p_\delta$ of C_0, then the total intersection number of a general $\Lambda \in \Sigma_0$ will be at least $d+\delta$, that is, Λ will still meet C_0 $d-\delta$ times away from the nodes (or with higher multiplicity at the nodes). Informally, we may say that each node "counts only once" in the sense that, as Λ_t tends to Λ_0, the node p_i will absorb (i.e., be the limiting position of) only one of the d points of intersection $\Lambda_t \cap C$. To see this, we have to look at the behavior of the two families Σ_t and Σ_t' together.

The proof of the geometric Riemann-Roch theorem ([5] p. 248) shows that Σ_t and Σ_t' are closely intertwined:

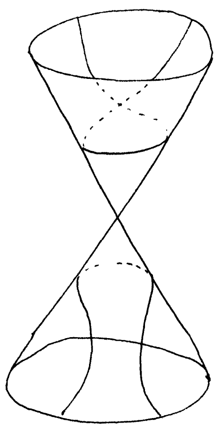


FIGURE 1

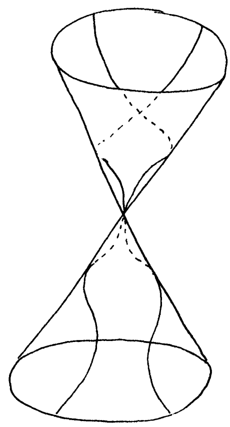


FIGURE 2

Precisely, we see that every hyperplane $H \subset \mathbf{P}^{g-1}$ containing a plane $\Lambda \in \Sigma_t$ will also contain a unique plane $\Lambda' \in \Sigma'_t$, and vice-versa. This being a closed condition on the pair of subvarieties $\Sigma_t \subset \mathbf{G}(d-r-1, g-1)$ and $\Sigma'_t \subset \mathbf{G}(g-r-2, g-1)$, it holds as well for the two limit families Σ_0 and Σ'_0. Note that the irreducible components $\{\Sigma_{0,\alpha}\}$ and $\{\Sigma'_{0,\alpha}\}$ of Σ_0 and Σ'_0 are thereby in one-to-one correspondence,[4] with every hyperplane containing a plane Λ of one component of Σ_0 containing a plane Λ' of the corresponding component of Σ'_0.

Now, suppose that every plane Λ of some irreducible component $\Sigma_{0,\alpha}$ of Σ_0 passes through some of the nodes, say $\{p_1, \dots, p_\delta\}$, of C_0. If Λ' is any plane of the corresponding component $\Sigma'_{0,\alpha}$ of Σ'_0, then it follows that every hyperplane through Λ' contains some $\Lambda \in \Sigma_{0,\alpha}$ and hence contains $\{p_1, \dots, p_\delta\}$; *consequently every* $\Lambda' \in \Sigma'_{0,\alpha}$ *must also contain* $\{p_1, \dots, p_\delta\}$.

Now, for every pair of planes $\Lambda_t \in \Sigma_t, \Lambda'_t \in \Sigma'_t$ lying in a hyperplane $H_t \subset \mathbf{P}^{g-1}$, the sum of their points of intersection with C_t comprise the hyperplane section $H_t \subset C_t$, and the same is true of a pair of planes

[4]Explicitly, this may be seen by considering the incidence correspondence

$$I \subset \Sigma_0 \times \mathbf{P}^{g-1*} \times \Sigma'_0$$

defined by

$$I = \{(\Lambda, H, \Lambda') : \Lambda, \Lambda' \subset H\};$$

the fibers of the projections $I \to \Sigma_0, I \to \Sigma'_0$ being projective spaces $\mathbf{P}^{g-d+r-1}$ and $\mathbf{P}^r$ respectively, the irreducible components of I are simultaneously in one-to-one correspondence with those of Σ_0 and Σ'_0.

$\Lambda \in \Sigma_{0,\alpha}$, $\Lambda' \in \Sigma'_{0,\alpha}$ lying in a hyperplane H_0.[5] We have then

$$\sum_{p \in H \cdot C_0} m_p(H \cdot C_0) = \#(H \cdot C_0)$$
$$= 2g - 2,$$

and so to count the total intersection numbers $\bar{d}$ and $\bar{e}$ of general planes $\Lambda \in \Sigma_{0,\alpha}$ and $\Lambda' \in \Sigma'_{0,\alpha}$ with C_0 we must relate the multiplicities $m_p(\Lambda \cdot C_0)$ and $m_p(\Lambda' \cdot C_0)$ to $m_p(H \cdot C_0)$ at every point $p \in H \cdot C_0$.

To do this, let us first assume that p is any base point of the family $\{\Lambda \in \Sigma_{0,\alpha}\}$, or equivalently of the residual family $\{\Lambda' \in \Sigma'_{0,\alpha}\}$. Suppose that m_p is the intersection multiplicity at $p \in C_0$ with a *general* $\Lambda \in \Sigma_{0,\alpha}$—i.e.,

$$m_p = \min_{\Lambda \in \Sigma_{0,\alpha}} (m_p(\Lambda \cdot C_0)),$$

and define m'_p similarly; suppose that $m'_p \geqslant m_p$. If $\Lambda \in \Sigma_{0,\alpha}$ is a general plane, $H \subset \mathbf{P}^{g-1}$ a general hyperplane containing Λ, and $\Lambda' \in \Sigma'_{0,\alpha}$ a plane of the residual family lying in H, then

$$m_p = m_p(\Lambda \cdot C_0) = m_p(H \cdot C_0)$$
$$m_p(H \cdot C_0) \geqslant m_p(\Lambda' \cdot C_0) \geqslant m'_p.$$

It follows that

$$m_p = m'_p = m_p(\Lambda \cdot C_0) = m_p(\Lambda' \cdot C_0) = m_p(H \cdot C_0).$$

In particular, for *any base point* p

$$m_p(\Lambda \cdot C_0) + m_p(\Lambda' \cdot C_0) \geqslant m_p(H \cdot C),$$

and *in case the base point* p *is a node*

$$m_p(\Lambda \cdot C_0) + m_p(\Lambda' \cdot C_0) = 2m_p(H \cdot C_0)$$
$$\geqslant m_p(H \cdot C_0) + 2. \qquad (*)$$

Now let Λ, H, and Λ' be as above and consider any point $q \in H \cdot C_0$ other than a base point of the families $\Sigma_{0,\alpha}$ or $\Sigma'_{0,\alpha}$. If $q \in \Lambda$, then since H is a general hyperplane through Λ

$$m_q(\Lambda \cdot C_0) = m_q(H \cdot C_0),$$

[5] The point here is simply that, since the incidence correspondence

$$I = \{(t, p) : p \in H_t \cap C_t\}$$

is flat over the t-curve, the complement of the fiber I_0 over $t = 0$ is dense in I; the subvariety

$$J = \{(t, p) : p \in (\Lambda \cup \Lambda') \cap C_t\},$$

being closed, then contains I_0.

while if $q \in \Lambda'$ then since $q \notin \Lambda$ and H is general through Λ we have by Bertini's theorem

$$m_p(H \cdot C_0) = 1.$$

Since a priori $m_p(H \cdot C_0) \geqslant m_p(\Lambda' \cdot C_0)$ this gives

$$m_p(\Lambda' \cdot C_0) = m_p(H \cdot C_0),$$

and consequently: *the relation*

$$m_p(\Lambda \cdot C) + m_p(\Lambda' \cdot C) \geqslant m_p(H \cdot C) \tag{**}$$

holds for every point p *of* $H \cap C$.

There is one more remark to be made here: that Λ' as chosen above is a general point of $\Sigma'_{0,\alpha}$. To see this, note that since the fibers of the incidence correspondence

$$Z = \{(\Lambda, H) : \Lambda \subset H\} \subset \Sigma_{0,\alpha} \times \mathbf{P}^{g-1*}$$

over $\Sigma_{0,\alpha}$ are all projective spaces of dimension $g - d + r - 1$, Z is irreducible and (Λ, H), as chosen above, is a general point of Z. Then since the correspondence

$$Y = \{(\Lambda, H; \Lambda') : \Lambda' \subset H\} \subset Z \times \Sigma'_{0,\alpha}$$

surjects onto $\Sigma'_{0,\alpha}$, the inverse image in Y of any (Zariski) open set in Z dominates $\Sigma'_{0,\alpha}$; so Λ' is general in $\Sigma'_{0,\alpha}$.

This said, we have

$$\bar{d} = \sum_p m_p(\Lambda \cdot C) \qquad \bar{e} = \sum_p m_p(\Lambda' \cdot C)$$

and applying (*) and (**) to the double base points of $\Sigma_{0,\alpha}$ and the remaining points of $H \subset C_0$ respectively, we have

$$\begin{aligned} 2g - 2 &= \sum_p m_p(H \cdot C) \\ &\leqslant \sum_{p \in C} m_p(\Lambda \cdot C) + m_p(\Lambda' \cdot C) - 2\delta \\ &= \bar{d} + \bar{e} - 2\delta. \end{aligned}$$

It follows that either

$$\bar{d} \geqslant d + \delta \tag{1}$$

or

$$\bar{e} \geqslant e + \delta. \tag{2}$$

But assuming the conclusion of lemma 1′, (1) leads to

$$\begin{aligned} g-(r+1)(g-d+r)+r &\leqslant \dim \Sigma_t \\ &\leqslant \dim \Sigma_{0,\alpha} \\ &\leqslant g-(r+1)(g-d+r)+r-\delta, \end{aligned}$$

and likewise (2) gives

$$\begin{aligned} g-(r+1)(g-d+r)+y-d+r-1 &\leqslant \dim \Sigma_t' \\ &\leqslant \dim \Sigma_{0,\alpha}' \\ &\leqslant g-(r+1)(g-d+r)+g-d+r-2-\delta. \end{aligned}$$

Our conclusion, then, is that if the Castelnuovo-Severi-Kleiman conjecture holds, then $\dim W_d^r(C_t)=\rho$ for the general element of our family C_t of curves; and since $\dim W_d^r(C)=\rho$ for a single smooth curve C implies the same statement for a general curve, we have proved the

REDUCTION THEOREM. *Assuming the validity of the Castelnuovo-Severi-Kleiman conjecture, the dimension of W_d^r on a general curve of genus g is exactly the Brill-Noether number*

$$\rho = g-(r+1)(g-d+r).$$

2. Proof of the Castelnuovo-Severi-Kleiman conjecture

(a) Before attempting to establish the conjecture in general, let us consider the first non-trivial case, the dimension of the locus W_3^1 of pencils of degree 3 on a general curve of genus g. Specializing to a general Castelnuovo canonical curve, it suffices by the above to determine the dimension (and degree) of the family of lines in $\mathbf{P}^3$ meeting each of g generic chords to a twisted cubic curve $C \subset \mathbf{P}^3$. Of course, the statement that we are after for linear systems—that on a general curve

$$\begin{aligned} \dim W_3^1 &= g-2(g-3+1) \\ &= 4-g, \end{aligned}$$

and in particular that on a general curve of genus four W_3^1 consists of exactly

$$w_3^1 = \frac{\theta^4}{12} = \frac{4!}{12} = 2$$

points—is well known, but we want to discuss how the statement for lines in $\mathbf{P}^3$ may be proved.

Now, the Grassmanian $\mathrm{G}(1,3)$ of lines in $\mathbf{P}^3$ is 4-dimensional, and the Schubert cycle $\sigma_1(l_\alpha)$ of lines meeting a given line l_α has codimension one

—indeed, $\sigma_1(l_\alpha)$ is the tangent hyperplane at l_α to $G(1,3)$ under the Plücker embedding as a quadric in P^5. It then obviously is the case that for generic lines $l_1, \ldots, l_g$ in P^3 the locus $\sigma_1(l_1) \cap \cdots \cap \sigma_1(l_g)$ of lines meeting them all will have dimension $4 - g$ if $g \leqslant 4$ and will be empty if $g \geqslant 5$. The question is whether this will continue to hold if $l_1, \ldots, l_g$ are constrained to be chords to C.

The answer in this case may be obtained directly: we note that if l_1, l_2, l_3 are three skew lines in P^3, then there is a unique smooth quadric Q containing them. (There is a quadric containing them, and it must be smooth if the lines are skew.) The three lines will all be elements of one ruling of Q, and will be met by any line of the other ruling; conversely any line meeting all three has three points of intersection with Q, and so must lie in Q and be a member of the second ruling. *Thus if l_1, l_2, l_3 are any three skew chords to C, the intersection $\sigma(l_1) \cap \sigma(l_2) \cap \sigma(l_3)$ will be the other* 1*-dimensional family of lines ruling Q.* Since the chords to C fill out P^3, moreover, a generic fourth chord l_4 to C will not lie on Q. The only lines meeting $l_1, \ldots, l_4$ will thus be the lines of the second ruling of Q passing through one of the points of $l_4 \cap Q$; and so $\dim(\bigcap \sigma_1(l_\alpha)) = 0$. (Indeed, with a little more care one can show that a general l_4 will meet Q transversely in two points, so that

$$\sigma_1(l_1) \cap \cdots \cap \sigma_1(l_4) = 2;$$

which, together with Section 1 above gives the world's hardest proof of the well-known fact that a general curve of genus 4 possesses two distinct g_3^1's.) Finally, since a general fifth chord l_5 to C will not meet either of the lines meeting $l_1, \ldots, l_4$ (no line in P^3 meets all the chords of C), we see that $\sigma_1(l_1) \cap \cdots \cap \sigma_1(l_5)$ is empty, as desired.

Unfortunately, this sort of direct analysis fails in more general cases. In particular, in the above discussion the Schubert cycles $\sigma_1(l_\alpha)$ are all of codimension 1 (as will in general be the case when $r = 1$) so that it suffices to show that $\dim \sigma_1(l_1) \cap \cdots \cap \sigma_1(l_g)$ is strictly decreasing with g, i.e., that for general l_g the cycle $\sigma_1(l_g)$ contains no component of a general $\sigma_1(l_1) \cap \cdots \cap \sigma_1(l_{g-1})$. (Indeed, a nice direct proof of the conjecture in case $r = 1$ is given by Laksov in [10].) A particularly bad sign for generalizations of this argument is the fact that the statement may in fact fail even if the chords are not in "obvious" special position: by choosing a quadric Q containing the twisted cubic C and choosing lines from among the ruling meeting C twice (C will meet the lines of one ruling of Q once, and the lines of the other ruling twice) we can find an arbitrarily large set $\{l_\alpha\}$ of chords to C, all mutually skew, such that

$$\dim \bigcap_\alpha \sigma_1(l_\alpha) = 1.$$

We look, accordingly, for other approaches to the problem in P^3. One possibility that suggests itself is to parrot a popular 19th-century method of computing enumerative formulas in the Schubert calculus by specialization. To determine the intersection number σ_1^4 in $G(1,3)$, for example, one could take four lines l_α with l_1 meeting l_2 and l_3 meeting l_4; one then sees directly that there are exactly two lines meeting all four l_α, namely the line $\overline{l_1 \cap l_2, l_3 \cap l_4}$ joining the two points of intersection and the line $\overline{l_1, l_2} \cap \overline{l_3, l_4}$ of intersection of the planes spanned by the line in pairs. As a technique for computing σ_1^4 this is suspect—one has still to determine that the Schubert cycles $\sigma_1(l_\alpha)$ associated to the four lines in special position intersect transversely—but for our present purposes, and given our prior knowledge that $\sigma_1^4 = 2$ in $\mathrm{G}(1, 3)$, it is ideal: Given four general chords l_α to a rational normal curve in P^3, we may just as readily specialize to the case $l_1 \cap l_2 = p$, $l_3 \cap l_4 = q$ and, noting that under the specialization the four Schubert cycles $\sigma_1(l_\alpha)$ meet in 2 distinct points—hence transversely—conclude that the $\sigma_1(l_\alpha)$ meet transversely for general l_α.

There are, at first glance, some difficulties with this approach in general. To illustrate, suppose we wanted to show that for generic points $p, q, r, s \in C$ the two Schubert cycles $\tau(p, q) = \sigma_{d-k-1}(\overline{pq})$ and $\tau(r, s) = \sigma_{d-k-1}(\overline{rs})$ of k-planes in P^d meeting the chords $\overline{pq}$ and $\overline{rs}$ intersected transversely almost everywhere. One way to go about this would be to specialize to the case $p = r$, i.e., where the two chords share a common point of the curve. This does not appear to work: the intersection of the two Schubert cycles $\tau(p, q)$ and $\tau(p, s)$ consists of two irreducible components, the cycle $\sigma_{d-k-1, d-k-1}(p, q, s)$ of k-planes having a line in common with the 2-plane $\overline{p, q, s}$, and the cycle $\sigma_{d-k}(p)$ of planes containing p; the latter has codimension $d - k < 2 \cdot \operatorname{codim} \sigma_{d-k-1}$ when $k < d - 2$. On closer inspection, however, the argument does work: the point is, letting the point r vary in C, we want to consider not the intersection of the limiting positions of the Schubert cycles $\tau = \tau(p, q)$ and $\tau(r) = \tau(r, s)$ as r tends to p, but rather the limiting position of the intersection $\tau \cap \tau(r)$. Precisely, we denote the Grassmannian $\mathrm{G}(k, d)$ by G and let $\Sigma \subset G \times C$ be the closure

$$\Sigma = \overline{\{(\Lambda{:}r) : \Lambda \in \tau \cap \tau(r), r \neq p\}}$$

and consider the fiber Σ_p of Σ over p. Suppose $(\Lambda, p) \in \Sigma_p$ is the limit of a family $\{(\Lambda_r, r) \in \Sigma_r\}$. Then since Λ_r meets both $\overline{pq}$ and $\overline{rs}$, and has a line L_r in common with the 3-plane $\overline{pqrs}$ for each r, we see that $\Lambda = \Lambda_p$ must meet both $\overline{pq}$ and $\overline{ps}$ *and have the line* $L_p = \lim_{r \to p} L_r$ in common with the 3-plane $\overline{2p; q, s} = \lim_{r \to p} \overline{p, q, r, s}$. The fiber Σ_p is thus contained in (in fact, equal to) the union of the Schubert cycles $\sigma_{d-k-1, d-k-1}(\overline{p, q, s})$ and $\sigma_{d-k, d-k-2}(p; \overline{2p, q, s})$, each of which has codimension $2d - 2k - 2 = 2 \cdot \operatorname{codim} \tau$. Indeed, something more emerges from this argument: in view of the identity

$$(\sigma_{d-k-1})^2 = \sigma_{d-k-1, d-k-1} + \sigma_{d-k, d-k-2}$$

in the cohomology ring of the Grassmannian and the independence of the two cycles on the right, we may conclude that the general fiber $\Sigma_r = \tau(p, q) \cap \tau(r, s)$ has no multiple components: every fiber of Σ has class $\sigma_{d-k-1, d-k-1} + \sigma_{d-k, d-k-2}$; so that the central fiber Σ_p, having support contained in the union of two such cycles, must in fact be equal to these two cycles, each taken with multiplicity 1. The locus of r for which Σ_r has a multiple component being closed, we see that it is a proper subvariety of C.

This procedure may be carried a few stages further. For example, to show that the intersection of Schubert cycles $\tau(p,q)$, $\tau(r,s)$, and $\tau(u,v)$ is generically transverse, it suffices to show this for the intersection of a general $\tau(u,v)$ with each of the cycles $\sigma_{d-k-1,d-k-1}(\overline{p,q,s})$ and $\sigma_{d-k,d-k-2}(\overline{p;2p,q,s})$. This may be done similarly, by letting the point u tend to p and identifying the limiting position of the intersection in each case as a union of Schubert cycles. A technical difficulty soon arises, however: the Schubert cycles obtained in the limit become progressively more complicated, and the degeneration is hard to keep track of in general. What seems required to carry out this idea is that we find a flag $V = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset \mathbf{P}^d$ and a suitable degeneration of a chord $\overline{qr}$ to C, such that the limiting position of the intersection $\sigma_a(V) \cap \tau(qr)$ under the degeneration is identifiable as a union of Schubert cycles, again of the form $\sigma_b(V)$.

The flag which will serve this purpose is the *osculating flag* to C at p,[6] that is, the flag comprised of the osculating spaces $V_i = \overline{(i+1)p}$ to C at p, and the Schubert cycles will be general Schubert cycles defined relative to this flag: for any collection of integers $d-k \geqslant a_0 \geqslant a_1 \geqslant \cdots \geqslant a_k \geqslant 0$ (which we call a Schubert index set a), we let

$$\sigma_a(p) = \{\Lambda \in G : \dim \Lambda \cap \overline{(d-k+i-a_i+1)p} \geqslant i \text{ for all } i\}.$$

The process of degeneration to be used here involves two steps. For example, if we want to show that the cycle $\sigma_{d-k-1,d-k-2}(p)$ of k-planes meeting the tangent line $\overline{2p}$ to C at p and having a line in common with the osculating 3-plane $\overline{4p}$ intersected the cycle $\tau(q,r)$ in the right dimension for general q and r, we would first let r tend to p. Formally, this means we set

$$\Sigma(q) = \overline{\{(\Lambda; r) : \Lambda \in \sigma_{d-k-1,d-k-2}(p) \cap \tau(q,r), r \neq p\}} \subset G \times C$$

and consider the fiber $\Sigma(q)_p$ over p. As before, if any $\Lambda \in \Sigma(q)_p$ does not contain the point p, then we see it must have a line in common with the plane $\overline{2p,q}$ and a 2-plane in common with $\overline{4p,q}$, i.e.,

$$\Lambda \in \Omega_0(q) = \left\{\Lambda : \begin{matrix} \dim \Lambda \cap \overline{2p,q} \geqslant 1 \\ \dim \Lambda \cap \overline{4p,q} \geqslant 2 \end{matrix}\right\}.$$

If, on the other hand, Λ does contain p, we may list the conditions Λ satisfies as follows:

$$\begin{aligned} \Lambda &\ni p \\ \dim \Lambda \cap \overline{3p,q} &\geqslant 1 \\ \dim \Lambda \cap \overline{4p} &\geqslant 1 \\ \dim \Lambda \cap \overline{5p,q} &\geqslant 2. \end{aligned}$$

[6] We observe that these osculating spaces already appeared in the above example.

The locus of such Λ does not, unfortunately, look much like a Schubert cycle. In fact, it is not—but it is the union of two Schubert cycles. To see this, we ask one crucial question: what is the dimension of $\Lambda \cap \overline{3p}$? If it is 1 or more, then we see that Λ lies in the Schubert cycle

$$\Omega_1(q) = \left\{ \Lambda : \begin{array}{c} \Lambda \ni p \\ \dim \Lambda \cap \overline{3p} \geq 1 \\ \dim \Lambda \cap \overline{5p, q} \geq 2 \end{array} \right\}.$$

If, rather, Λ meets the osculating plane $\overline{3p}$ to C at p *only* at p, then from the condition $\dim \Lambda \cap \overline{3p, q} \geq 1$, it follows that Λ must have a point t, lying outside $3p$, in common with the 3-plane $\overline{3p, q}$. We see moreover that t must lie outside the plane $\overline{4p}$ as well, since if t lay in $\overline{4p}$, the 3-plane $\overline{3p, t}$ would lie in, hence equal $\overline{4p}$, and we would have $q \in \overline{4p}$. Thus the point t is a point of intersection of Λ with $\overline{4p, q}$ not lying in $\overline{4p}$, and so

$$\begin{aligned} \dim \Lambda \cap \overline{4p, q} &\geq \dim \Lambda \cap \overline{4p} + 1 \\ &\geq 2 \end{aligned}$$

i.e.,

$$\Lambda \in \Omega_2(q) = \left\{ \Lambda : \begin{array}{c} \Lambda \ni p \\ \dim \Lambda \cap \overline{4p, q} \geq 2 \end{array} \right\}.$$

Now we are set to complete the degeneration. Letting

$$\Sigma = \overline{\{(\Lambda, q) : \Lambda \in \Sigma(q)_p \, , q \neq p\}} \ ;$$

we see that the fiber of Σ over p is contained in the union of the three Schubert cycles

$$\begin{aligned} \Omega_0 &= \sigma_{d-k-1, d-k-1, d-k-2}(p) \\ &= \left\{ \Lambda : \begin{array}{c} \dim \Lambda \cap \overline{3p} \geq 1 \\ \dim \Lambda \cap \overline{5p} \geq 2 \end{array} \right\} \\ \Omega_1 &= \sigma_{d-k, d-k-1, d-k-3}(p) \\ &= \left\{ \Lambda : \begin{array}{c} \Lambda \ni p \\ \dim \Lambda \cap \overline{3p} \geq 1 \\ \dim \Lambda \cap \overline{6p} \geq 2 \end{array} \right\} \end{aligned}$$

and

$$\begin{aligned} \Omega_2 &= \sigma_{d-k, d-k-2, d-k-2}(p) \\ &= \left\{ \Lambda : \begin{array}{c} \Lambda \ni p \\ \dim \Lambda \cap \overline{5p} \geq 2 \end{array} \right\} \end{aligned}$$

all of which are of the right codimension

$$3(d-k)-4=\operatorname{codim}\sigma_{d-k-1,\,d-k-2}(p)+\operatorname{codim}\tau(q,r).$$

We have then for general q, r

$$\begin{aligned}\operatorname{codim}\sigma_{d-k-1,\,d-k-2}(p)\cap\tau(q,r)\\ &\geqslant \operatorname{codim}\Sigma(q)_p\\ &\geqslant \operatorname{codim}\Sigma_p\\ &=3(d-k)-4\end{aligned}$$

as desired.

(b) This, then, is how the proof goes: we specialize each of the chords $\overline{q_\alpha r_\alpha}$ into p one point at a time, exhibiting the limiting position of the intersection of the Schubert cycles $\tau(q_\alpha, r_\alpha)$ after "swallowing" each chord as the intersection of the remaining cycles $\tau(q_\alpha, r_\alpha)$ with a union of Schubert cycles of the form $\sigma_a(p)$. To formalize this we will prove the

LEMMA. *For $\{p, q_\alpha, r_\alpha\}$ a generic collection of $2g+1$ points on C, and for any Schubert index set a, the intersection*

$$\sigma_a(p)\cap\tau(q_1,r_1)\cap\cdots\cap\tau(q_g,r_g)$$

is of dimension

$$\dim G-\sum a_i-g(d-k-1)=(k+1)(d-k)-\sum a_i-g(d-k-1),$$

and has no multiple components. It is understood that the intersection is empty in case the dimension is negative.

Proof. The proof will be by induction on the number g of chords. Before beginning, we make one observation: if $a_0=d-k$—that is, if every plane $\Lambda\in\sigma_a(p)$ contains p—then projection π from p to a hyperplane P^{d-1} maps C to a rational normal curve $\tilde{C}$ in P^{d-1}, $\overline{q_\alpha r_\alpha}$ to a generic chord of $\tilde{C}$, and $\sigma_a(p)$ isomorphically to the Schubert cycle $\sigma_{a_1,\ldots,a_k}(\pi(p))\subset \mathsf{G}(k-2,d-2)$. Thus we assume the result as stated for g chords and prove it for $g+1$ chords with $a_0<d-k$.

This said, set (q,r) and (q_α, r_α) be $g+1$ generic pairs of points on C; denote by τ the intersection $\tau(q_1,r_1)\cap\cdots\cap\tau(q_g,r_g)$.

We consider first the limit of the intersection $\sigma_a(p)\cap\tau\cap\tau(q,r)$ as r tends to p; that is, we set

$$\Sigma(q)=\overline{\{(\Lambda,r):\Lambda\in\sigma_a(p)\cap\tau\cap\tau(q,r),\,r\neq p\}}\subset G\times C$$

and look at the fiber $\Sigma(q)_p$ of $\Sigma(q)$ over p. Let Λ_p be any point of $\Sigma(q)_p$, and let i_0 be the largest integer such that

$$\dim\Lambda\cap\overline{(d-k+i-a_i+2)p}\geqslant i+1 \qquad (\underset{i}{*})$$

for all $i<i_0$, adopting the convention that $a_{-1}=d-k$ in any Schubert cycle.

We can then find a point t in the intersection $\Lambda_p \cap \overline{(d-k+i_0-a_{i_0}+2)p, q}$ not contained in $\overline{(d-k+i_0-a_{i_0}+2)p}$. Moreover, t cannot lie in the osculating plane $\overline{mp}$ for any $m \leqslant d$: since $q \in \overline{(d-k+i_0-a_{i_0}+2)p, t}$, this would imply $q \in \overline{dp}$. Thus, for $i > i_0$, we have

$$\dim \Lambda_p \cap \overline{(d-k+i-a_i+1)p, q} \geqslant 1 + \dim \Lambda_p \cap \overline{(d-k+i-a_i+1)p} \geqslant i+1.$$

All in all, then, we see that $\Lambda \in \Omega_{i_0}(q) \cap \tau$, where

$$\Omega_{i_0}(q) = \left\{ \Lambda : \begin{array}{ll} \dim \Lambda \cap \overline{(d-k+i-a_i+2)p} \geqslant i+1, & i < i_0 \\ \dim \Lambda \cap \overline{(d-k+i_0-a_{i_0}+2)} \geqslant i_0+1 & \\ \dim \Lambda \cap \overline{(d-k+i-a_i+1)p, q} \geqslant i_0+1, & i > i_0 \end{array} \right\}$$

We note that in case $a_{i+1} = a_i$, we have $\Omega_i \subset \Omega_{i+1}$, thus

$$\Sigma(q)_p \subseteq \bigcup_{i\,:\,a_i > a_{i+1}} \Omega_i(q) \cap \tau.$$

We also note that if $a_k \geqslant 2$—that is, if every $\Lambda \in \sigma_a(p)$ lies in the $(d-2)$-plane $\overline{(d-1)p}$ then the intersection $\sigma_a(p) \cap \tau(q, r)$, and likewise the cycles Ω_i, are all empty. Similarly, if $a_k = 1$—that is, every $\Lambda \in \sigma_a(p)$ lies in the osculating hyperplane $\overline{dp}$ to C at p—then the cycles $\Omega_i(q)$ are all empty except for $\Omega_k(q)$.

Now we perform the second stage of our degeneration, letting q tend to p: set

$$\Sigma = \{(\Lambda, q) : \Lambda \in \Sigma(q)_p, q \neq p\}.$$

The fiber Σ_p of Σ over p is then contained in the cycle

$$\bigcup_{i\,:\,a_i > a_{i+1}} \Omega_i \cap \tau$$

where

$$\Omega_0 = \left\{ \Lambda : \begin{array}{l} \dim \Lambda \cap \overline{2p} \geqslant 0 \\ \dim \Lambda \cap \overline{(d-k+i-a_i+2)p} \geqslant i+1 \end{array} \right\}$$

$$= \sigma_{d-k-1, a_0, \ldots, a_{k-1}}(p)$$

if $a_k = 0$ and is empty if $a_k \neq 0$; and

$$\Omega_{i_0} = \left\{ \Lambda : \begin{array}{ll} \Lambda \ni p & \\ \dim \Lambda \cap \overline{(d-k+i-a_i+2)p} \geqslant i+1, & i < i_0 \\ \dim \Lambda \cap \overline{(d-k+i_0-a_{i_0}+3)p} \geqslant i_0+1 & \\ \dim \Lambda \cap \overline{(d-k+i-a_i+2)p} \geqslant i+1, & i > i_0 \end{array} \right\}$$

$$= \sigma_{d-k, a_0, \ldots, a_{i_0}-1, \ldots, a_{k-1}}(p)$$

and is empty if $a_k = 1$ and $i_0 \neq k$.

Since the cycles Ω_i are all of the form $\sigma_a(p)$ and all have codimension $\sum a_i + (d-k-1)$, we see by the induction hypothesis that

$$\begin{aligned}\dim \sigma_a(p) \cap \tau(q,r) \cap \tau &\leq \max\{-1, \dim \Omega_i(q) \cap \tau\}\\ &\leq \max\{-1, \dim \Omega_i \cap \tau\}\\ &= \max\Big\{-1, (k+1)(d-k) - \sum a_i - (g+2)(d-k-2)\Big\}.\end{aligned}$$

Finally, in view of the identity (cf. [5], page 203)

$$\sigma_a \cdot \sigma_{d-k-1} = \begin{cases} 0 & \text{if } a_k \geq 2\\ \sigma_{d-k, a_0, \ldots, a_{k-1}} & \text{if } a_k = 1\\ \sigma_{d-k-1, a_0, \ldots, a_{k-1}} + \sum\limits_{a_{i+1} < a_i} \sigma_{d-k, a_0, \ldots, a_i - 1, \ldots, a_{k-1}} & \text{if } a_k = 0\end{cases}$$

in the cohomology ring of the Grassmannian, we may conclude that the intersection has no multiple components: the class of every fibre of $\Sigma(q)$ (and of Σ) is $\sigma_a(\sigma_{d-k-1})^{g+1}$; since by induction hypothesis the intersection of τ with each component Ω_i of the fibre of Σ has no multiple components, and since—again by induction hypothesis—the pairwise intersections $\Omega_i \cap \Omega_j \cap \tau$ are of strictly smaller dimension, it follows that the intersection $(\cup \Omega_i) \cap \tau$ has no multiple components. □

Note, finally, that since more general secant flags to a rational normal curve may be specialized to the osculating flag at a point, we have immediately the following

COROLLARY. *For any collection of points $p_1, \ldots, p_m$ on a rational normal curve $C \subset \mathbf{P}^d$ and any collection of non-negative integers b_{ij}, $1 \leq i \leq d$, $1 \leq j \leq m$ satisfying*

$$b_{ij} \leq b_{i+1,j}$$

$$\sum_j b_{ij} = i$$

let $V_b(p)$ be the flag with elements

$$V_{i-1} = \overline{b_{i1}p_1 + \cdots + b_{im}p_m}\,.$$

Then for a any Schubert index, $\{p_1, \ldots, p_m, q_\alpha, r_\alpha\}$ general points on C and b any collection of integers as above, the intersection

$$\sigma_a(V_b(p)) \cap \tau(q_1, r_1) \cap \cdots \cap \tau(q_g, r_g)$$

has codimension $\sum a_i - g(d-k-1)$ in $\mathrm{G}(k,d)$, and has no multiple components.

3. Multiplicities of W_d^r

(a) We come now to the question of whether, on a general curve of genus g, the subscheme W_d^r may have multiple components. Equivalently, we now know that on a general curve W_d^r has the dimension ρ predicted by Brill-Noether, and hence from [9] that the class of the cycle $W_d^r \subset J(C)$, counting multiplicities, is $\prod_{i=0}^r [i!/(g-d+r+i)!]\theta^{g-\rho}$. We may ask whether this is in fact the class of the support of W_d^r—e.g., in case $\rho = 0$ whether W_d^r will actually consist of $g! \prod i!/(g-d+r+i)!$ *distinct* points. We observe that this is equivalent to asking the same question for $C_d^r \subset C_d$: Does the *support* of C_d^r have the class predicted by Porteous' formula? The strong form of the Castelnuovo-Severi-Kleiman conjecture asserting that for C_0 a general Castelnuovo canonical curve $W_d^r(C_0)$, realized as a subscheme of the Grassmannian $\mathrm{G}(d-r-1, d)$, has no multiple components suggests that this should be true. However, further work is required to draw the desired conclusion: While we were able in section 1c to relate coarse characteristics such as the dimension of $W_d^r(C_0)$ to that of $W_d^r(C)$, our approach does not immediately allow us to relate the more subtle question of the multiplicity of a component of $W_d^r(C)$—defined as the image scheme in the Jacobian of the subscheme $C_d^r \subset C_d$ given by the vanishing of minors of the Brill-Noether matrix—to that of $W_d^r(C_0)$—defined as the scheme-theoretic intersection of Schubert cycles on the Grassmannian.

We resolve this difficulty by solving an enumerative problem. The idea is simply that a multiple component of a subscheme $Y \subset X$ will show up when we count the points of intersection with a cycle $Z \subset X$ of complementary dimension: if we know that no component of Y is disjoint from X, then equality between the topological intersection numbers of Y and Z and the actual number of distinct points of $Y \cap Z$ will imply that Y has no multiple components. In the present situation we will intersect the subscheme C_d^r of the symmetric product C_d of C with the image of $C_{d-r-\rho}$ in C_d—that is, the locus of divisors of degree d containing $r + \rho$ fixed points. Thus we are led to the following question:

Given general points $q_1, \ldots, q_{r+\rho}$ on a general curve C of genus g, how many divisors of degree d are there on C with

$$r(D) \geqslant r$$
$$D - q_1 - \cdots - q_{r+\rho} > 0?$$

We will first give the "enumerative answer" to this problem; i.e., we will compute the intersection number

$$\mu(g, r, d) = (C_d^r \cdot C_{d-r-\rho})$$

in C_d. Then, by applying the strong form of the Castelnuovo-Severi-Kleiman conjecture and some Schubert calculus, we will give the actual answer on a generic Castelnuovo canonical curve, which turns out to be exactly $\mu(g, r, d)$. Going back to our family $\{C_t\}$ of curves degenerating to C_0, we have then the following situation: we know from the results of [9] quoted above and the computation in part (b) below, that the set of divisors on C_t satisfying the conditions above *either*

(i) is positive-dimensional; or

(ii) consists of at most $\mu(g, r, d)$ distinct points.

We know moreover by part IIa of our main theorem (with a small additional argument) that (i) *cannot be the case*. Finally, we know by the strong Castelnuovo-Severi-Kleiman conjecture and the computation in part (c) that *the set of divisors on C_0 satisfying the conditions given consists of exactly $\mu(g,r,d)$ distinct points*. From elementary considerations, then, we may conclude that for general t, *the set of divisors D satisfying the conditions consists of exactly $\mu(g,r,d)$ points*.

Checking then that no component of $C_d^r(C_t)$ may miss $C_{d-r-\rho}(C_t)$ entirely, we conclude the desired statements IIb and IIc for a general curve C_t of our family $\{C_t\}$, and hence for a general curve of genus g.

(b) To compute the intersection number $C_d^r \cdot X^{r+\rho}$ on the symmetric product we shall use the following notations and facts from [1], in which there are proofs and references to the original sources:

(1) $\pi : C_d \to J(C)$ denotes the standard map to the Jacobian (π is denoted by μ in [5]);

(2) $x \in H^2(C_d, \mathbf{Z})$ is the fundamental class of the image X of C_{d-1} in C_d; and

(3) $\tilde{\theta} = \pi^*\theta$ is the pullback to C_d of the class of the θ-divisor on $J(C)$.

The basic facts about the cohomology of C_d are:

(i) $H^*(C_d, \mathbf{Z})$ is generated over the subring $\pi^*H^*(J(C), Z)$ by the class x, that satisfies the relation

$$\sum_{i=0}^{d-g+1} (-1)^i \frac{\tilde{\theta}^i}{i!} x^{d-g+1-i} = 0.$$

(ii) The image of the class x^{d-g+k} under the push-forward (Gysin) map $\pi_* : H^*(C_d, \mathbf{Z}) \to H^*(J(C), Z)$ is

$$\pi_* x^{d-g+k} = \frac{\theta^k}{k!}\ .$$

(iii) The Chern classes of the tangent bundle to C_d are given by

$$\sum c_i(T(C_d)) = (1 + xt)^{d-g+1} e^{-\tilde{\theta}t/(1+tx)},$$

and finally, applying Porteous' formula to the vector bundle map

$$\pi_* : T(C_d) \to \pi^* T(J(C))$$

obtained from the differential of π we conclude:

(iv) The class of C_d^r is given by

$$C_d^r \sim \det \begin{pmatrix} \frac{\tilde{\theta}^{g-d+r}}{(g-d+r)!} & \frac{\tilde{\theta}^{g-d+r+1}}{(g-d+r+1)!} & \cdots & \frac{\tilde{\theta}^{g-d+2r-1}}{(g-d+2r-1)!} & x^r \\ \frac{\tilde{\theta}^{g-d+r-1}}{(g-d+r-1)!} & & & \vdots & x^{r-1} \\ \vdots & & & \vdots & \vdots \\ \vdots & & & \frac{\tilde{\theta}^{g-d+r}}{(g-d+r)!} & x \\ \frac{\tilde{\theta}^{g-d}}{(g-d)!} & \cdots & \cdots & \frac{\tilde{\theta}^{g-d+r-1}}{(g-d+r-1)!} & 1 \end{pmatrix}$$

Multiplying the right-hand column of this matrix by $x^{r+\rho}$ and applying the push-pull formula $\pi_*(\alpha \cdot \pi^*\beta) = \pi_*\alpha \cdot \beta$ gives

$$\#(C_d^r \cdot X^{r+\rho}) = \det_{C_d}\begin{bmatrix} \dfrac{\tilde{\theta}^{g-d+r}}{(g-d+r)!} & \cdots & \dfrac{\tilde{\theta}^{g+d+2r-1}}{(g-d+2r-1)!} & x^{2r+\rho} \\ \vdots & & \vdots & \vdots \\ \dfrac{\tilde{\theta}^{g-d}}{(g-d)!} & \cdots & \dfrac{\tilde{\theta}^{g-d+r-1}}{(g-d+r-1)!} & x^{r+\rho} \end{bmatrix}$$

$$= \det_{J(C)}\begin{bmatrix} \dfrac{\theta^{g-d+r}}{(g-d+r)!} & \cdots & \dfrac{\theta^{g-d+2r-1}}{(g-d+2r-1)!} & \dfrac{\theta^{g-d+2r+\rho}}{(g-d+2r+\rho)!} \\ \vdots & & \vdots & \vdots \\ \vdots & & \dfrac{\theta^{g-d+r}}{(g-d+r)!} & \vdots \\ \dfrac{\theta^{g-d}}{(g-d)!} & \cdots & \dfrac{\theta^{g-d+r-1}}{(g-d+r-1)} & \dfrac{\theta^{g-d+r+\rho}}{(g-d+r+\rho)!} \end{bmatrix}$$

$$= \theta^{(r+1)(g-d+r)+\rho} \cdot \det\begin{bmatrix} \dfrac{1}{(g-d+r)!} & \cdots & \dfrac{1}{(g-d+2r-1)!} & \dfrac{1}{(g-d+2r+\rho)!} \\ \vdots & & \vdots & \vdots \\ \dfrac{1}{(g-d)!} & \cdots & \dfrac{1}{(g-d+r+1)!} & \dfrac{1}{(g-d+r+\rho)!} \end{bmatrix}$$

It is well known that

$$\theta^{(r+1)(g-d+r)+\rho} = \theta^g = g!;$$

to evaluate the determinant, we note that in general the determinant

$$\det\begin{bmatrix} \dfrac{1}{a_0!} & \dfrac{1}{a_1!} & \cdots & \dfrac{1}{a_r!} \\ \dfrac{1}{(a_1+1)!} & & & \dfrac{1}{(a_r+1)!} \\ \vdots & & & \vdots \\ \dfrac{1}{(a_0+r)!} & \dfrac{1}{(a_1+r)!} & \cdots & \dfrac{1}{(a_r+r)!} \end{bmatrix}$$

is readily calculated: Multiplying the $(i+1)$st column by $(a_i + r)!$ we obtain the matrix

$$\begin{bmatrix} P_r(a_0) & P_r(a_1) & \cdots & P_r(a_r) \\ P_{r-1}(a_0) & \cdots & \cdots & P_{r-1}(a_r) \\ \vdots & & & \vdots \\ P_0(a_0) & \cdots & \cdots & P_0(a_r) \end{bmatrix}$$

where

$$P_i(a) = (a + r)(a + r - 1) \cdots (a + r - i + 1);$$

the polynomials $P_0, \ldots, P_r$ all being monic and forming a basis for the polynomials of degree $\leqslant r$, this matrix may be row-reduced to the Van der Monde matrix

$$\begin{bmatrix} a_0^r & \cdots & a_r^r \\ a_0^{r-1} & & \vdots \\ \vdots & & \vdots \\ a_0 & & \\ a & \cdots & 1 \end{bmatrix}$$

which has determinant

$$\Delta(a_0, \ldots, a_r) = \prod_{0 \leqslant i < j \leqslant r} (a_j - a_i).$$

Thus

$$\det \begin{bmatrix} \frac{1}{a_0!} & \cdots & \frac{1}{a_r!} \\ \vdots & & \vdots \\ \frac{1}{(a_0 + r)!} & \cdots & \frac{1}{(a_r + r)!} \end{bmatrix} = \frac{\prod_{i<j} (a_j - a_i)}{(a_0 + r)! \cdots (a_r + r)!}$$

In our present circumstance

$$\begin{aligned} a_0 &= g - d \\ a_1 &= g - d + 1 \\ &\vdots \\ a_{r-1} &= g - d + r - 1 \\ a_r &= g - d + r + \rho \end{aligned}$$

so

$$\prod_{0 \leqslant i < j \leqslant r-1} (a_j - a_i) = \prod_{i=0}^{r-1} i!,$$

while

$$\prod_{0 \leqslant i < r} (a_r - a_i) = \frac{(r + \rho)!}{\rho!}.$$

We have, then, finally

$$C_d^r \cdot X^{r+\rho} = \frac{g!\,(r+\rho)!}{\rho!\,(g-d+2r+\rho)!} \prod_{i=0}^{r-1} \frac{i!}{(g-d+r+i)!}$$

a number we shall henceforth call $\mu(g, r, d)$.

(c) Let C_0 be a general Castelnuovo canonical curve with normalization C a rational normal curve of degree d in P^d. A linear series g_d^r on C_0 is then cut out on C by hyperplanes that pass through a $(d-r-1)$-plane $\Lambda \subset \mathsf{P}^d$ meeting each of the g chords to C joining pairs of points over a node of C_0. Now let $q_1, \ldots, q_{r+\rho}$ be general points of C; then those g_d^r that contain a divisor passing through $q_1, \ldots, q_{r+\rho}$ are also readily described: They are given by those $(d-r-1)$-planes Λ that additionally lie in a hyperplane with the points $q_1, \ldots, q_{\rho+r}$—that is, meet the $(\rho+r-1)$-plane $\overline{q_1, \ldots, q_{\rho+r}}$ in a $(\rho-1)$-plane, instead of a $(\rho-2)$-plane as expected. The number of such planes Λ is then, by the Castelnuovo-Severi-Kleiman conjecture, given by the intersection number

$$((\sigma_r)^g \cdot \underbrace{\sigma_{1,\ldots,1}}_{\rho})\widehat{\mathsf{G}(d-r-1,d)};$$

we will compute this number and answer our question at the conclusion of the following digression.

Some Schubert calculus. We want to determine, in the Grassmannian $\mathsf{G}(d-r-1,d)$, the intersection number of powers $(\sigma_r)^g$ of σ_r with Schubert cycles $\sigma_{1,\ldots,1}$. To do this, we are more or less forced to solve a more general problem: the intersection number $(\sigma_k^g \cdot \tau)$ for any integer k with $1 \leqslant k \leqslant r+1$ and any Schubert cycle τ; or, dually, the intersection number

$$(\underbrace{\sigma_{1,\ldots,1}}_{k})^g \cdot \tau$$

in $\mathsf{G}(r,d)$. Inasmuch as under the inclusion $\iota: \mathsf{G}(r,d) \to \mathsf{G}(r,d')$ we have $\iota^*\sigma_{1,\ldots,1} = \sigma_{1,\ldots,1}$, it will be convenient for our present purposes to use a notation for τ that is consistent with the push-forward map i_* in homology, e.g., to set, for any collection of integers $0 \leqslant a_0 < a_1 < a_2 < \cdots < a_{r-1} < a_r$,

$$\tau_{a_0,\ldots,a_r} = \left\{ \Lambda^r : \begin{matrix} \dim \Lambda \cap \mathsf{P}^{a_0} \geqslant 0 \\ \dim \Lambda \cap \mathsf{P}^{a_1} \geqslant 1 \\ \vdots \\ \dim \Lambda \cap \mathsf{P}^{a_r} \geqslant r \end{matrix} \right\}$$

for some partial flag $\mathsf{P}^{a_0} \subset \cdots \subset \mathsf{P}^{a_r}$. Note that in $\mathsf{G}(r,d)$,

$$\tau_{a_0,\ldots,a_r} = \sigma_{d-r-a_0,\, d-r+1-a_1,\ldots,\, d-a_r}$$

and in particular this cycle has dimension $(r+1)(d-r)-\sum_{i=0}^{r}(d-r+i-a_i) = \sum a_i - r(r+1)/2$. The point of this notation is simply that the intersection numbers

$$\delta_{k;\,a_0,\dots,a_r} = \left((\sigma_{1,\dots,1})^g \cdot \tau_{a_0,\dots,a_r}\right)$$

where $0 \leqslant a_0 < a_1 < \cdots < a_r$ and $g = (\sum a_i - r(r+1)/2)/k$ are independent of d.

The information we have about the numbers $\delta_{k;\,a_0,\dots,a_r}$ is *Pieri's formula*:

$$\sigma_{1,\dots,1} \cdot \tau_{a_0,\dots,a_r} \sim \sum_{a':\ \substack{a_i - 1 \leqslant a_i' \leqslant a_i \\ \Sigma a_i' = \Sigma a_i - k}} \tau_{a_0',\dots,a_r'}$$

i.e.,

$$\delta_{k;\,a_0,\dots,a_r} = \sum_{a'} \delta_{k;\,a_0',\dots,a_r'}.$$

To use this formula, however, we have to express more clearly the indices a' over which the sum is taken. The problem is that simply subtracting 1 from k of the indices $a_0,\dots,a_r$ may give rise to an inadmissible sequence $a_0',\dots,a_r'$: the a_i' may no longer be strictly increasing, and if a_0 was 0 to start with we may wind up with $a_0' = -1$. Both of these problems are rather fortuitously taken care of when we extend the definition of $\delta_{k;\,a}$ to an alternating function on *all* $(r+1)$-tuples of integers; i.e., we set, for any permutation s of $\{0,\dots,r\}$,

$$\delta_{k;\,a_{s(0)},\dots,a_{s(r)}} = \operatorname{sgn}(s)\cdot\delta_{k;\,a_0,\dots,a_r}$$

and of course

$$\delta_{k;\,a_0,\dots,a_r} = 0 \quad \begin{array}{ll} \text{if } a_i = a_j, & \text{any } i \neq j \\ \text{or} & \\ \text{if } a_i < 0, & \text{any } i. \end{array}$$

We then have simply

$$\delta_{k;\,a} = \sum_{\# I = k} \delta_{k;\,a-I} \tag{*}$$

where

$$(a-I)_i = \begin{array}{ll} a_i - 1, & i \in I \\ a_i, & i \notin I \end{array},$$

whenever $g > 0$. Finally, in case $g = 0$—that is, $\sum a_i = r(r+1)/2$—we have by definition

$$\delta_{k;\,a_0,\dots,a_r} = \begin{cases} \operatorname{sgn}(s) & \text{if } a_0,\dots,a_r = s(0),\dots,s(r) \\ & \text{for some } s \in \Sigma_{r+1} \\ 0 & \text{if not.} \end{cases}$$

Now we may carry out the computation. Set

$$f_k(x_0, \ldots, x_r) = \sum_a \delta_{k;\, a_0, \ldots, a_r} x_0^{a_0} x_1^{a_1} \ldots x_r^{a_r}.$$

We have then, separating the terms of f_i of degree $r(r+1)/2$ and applying (*) to the remaining terms,

$$\begin{aligned} f_k &= \sum_{\Sigma a_i = r(r+1)/2} \delta_{k,a} x^a + \sum_{\substack{\Sigma a_i = r(r+1)/2 + kg, \\ g>0}} \delta_{k,a} x^a \\ &= \sum_{s \in \Sigma_{r+1}} \operatorname{sgn}(s) x_0^{s(0)} \cdots x_r^{s(r)} + \sum_{a, I} \delta_{k, a-I} x^a \\ &= \Delta(x_0, \ldots, x_r) + \sum_I x^I f_k \\ &= \Delta(x_0, \ldots, x_r) + c_k(x_0, \ldots, x_r) f_k, \end{aligned}$$

where $\Delta(x_0, \ldots, x_r)^2$ is the discriminant and $c_k(x_0, \ldots, x_r)$ the kth elementary symmetric polynomial in the x_i's. Thus

$$f_k(x_0, \ldots, x_r) = \frac{\Delta(x_0, \ldots, x_r)}{1 - c_k(x_0, \ldots, x_r)}$$

and correspondingly

$$\begin{aligned} \delta_{k;\, a_0, \ldots, a_r} &= \text{coefficient of } x_0^{a} \cdots x_r^{a_r} \text{ in} \\ &\quad \Delta(x_0, \ldots, x_r) \cdot c_k(x_0, \ldots, x_r)^q \\ &= \sum_{s \in \Sigma_{r+1}} \operatorname{sgn}(s) \cdot \text{coefficient of} \\ &\quad x_0^{a_0 - s(0)} \cdots x_r^{a_r - s(r)} \text{ in } c_k(x_0, \ldots, x_r)^g. \end{aligned}$$

Two cases that work out nicely are $k = 1$ and $k = r$. For $k = 1$, we obtain the classical formula for the degree $\tau_{a_0, \ldots, a_r} \cdot \sigma_1^{\Sigma a_i - r(r+1)/2}$ of the Schubert cycle $\tau_{a_0, \ldots, a_r}$ under the Plücker embedding

$$\deg \tau_{a_0, \ldots, a_r} = \sum_{s \in \Sigma_{r+1}} \operatorname{sgn}(s) \cdot \frac{(\dim \tau)!}{a_0 - s(0)! \cdots a_r - s(r)!}$$

$$= (\dim \tau)! \det \begin{pmatrix} \frac{1}{a_0!} & \frac{1}{a_1!} & \cdots & \frac{1}{a_r!} \\ \frac{1}{(a_0-1)!} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{(a_0-r)!} & \frac{1}{(a_1-r)!} & \cdots & \frac{1}{(a_r-r)!} \end{pmatrix}$$

which by the calculation of the previous section yields

$$\deg \tau_{a_0,\ldots,a_r} = (\dim \tau)!\,\frac{\prod_{i<j}(a_j - a_i)}{a_0!\cdots a_r!}$$

In particular, the degree of the Grassmannian $\mathsf{G}(r,d) = \tau_{d-r,d-r+1,\ldots,d}$ itself is

$$\deg \mathsf{G}(r,d) = ((r+1)(d-r))!\cdot \prod_{i=0}^{r} \frac{i!}{(d-i)!}\,.$$

Similarly, when $k = r$ the coefficient of $x_0^{b}\cdots x_r^{b_r}$ in $c_k(x_0,\ldots,x_r)^{g=\Sigma b_i/r}$ is $g!/(g-b_0)!\cdots(g-b_r)!$ and we have

$$\tau_{a_0,\ldots,a_r}\cdot(\sigma_{1,\ldots,1})^g = \sum \operatorname{sgn}(s)\cdot\frac{g!}{(g-a_0+s(0))!\cdots(g-a_r+s(r))!}$$

$$= (-1)^{(r+1)r/2} g!\cdot\det\begin{bmatrix} \frac{1}{(g-a_0)!} & \cdots & \frac{1}{(g-a_r)!} \\ \vdots & & \vdots \\ \frac{1}{(g-a_0+r)!} & & \frac{1}{(g-a_r+r)!}\end{bmatrix}$$

$$= g!\,\frac{\prod_{i<j}(a_j-a_i)}{(g-a_0+r)!\cdots(g-a_r+r)!}\,.$$

In particular, we see from this and the CSK conjecture that the number of $(d-r-1)$-planes $\Lambda \subset \mathsf{P}^d$ meeting each of g generic chords to a rational normal curve C and in addition lying in a hyperplane with $r+\rho$ general points of C is

$$\#\big(\sigma_{\underbrace{1,\ldots,1}_{\rho}}\cdot\sigma_r^g\big)\mathsf{G}(\overbrace{d-r-1,d})$$

$$= \#\big(\sigma_\rho\cdot(\sigma_{\underbrace{1,\ldots,1}_{r}})^g\big)\widehat{\mathsf{G}(r,d)}$$

$$= \tau_{d-r-\rho,d-r+1,\ldots,d}\cdot(\sigma_{1,\ldots,1})^g$$

$$= g!\,\frac{\prod_{j>0}(\rho+r+1-j)\cdot\prod_{1\leq i<j\leq r}(j-i)}{(g-d+2r+\rho)!\,(g-d+2r-1)!\cdots(g-d+r)!}$$

$$= \frac{g!\,(\rho+r)!}{\rho!\,(g-d+2r+\rho)!}\prod_{i=0}^{r-1}\frac{i!}{(g-d+r+i)!}$$

$$= \mu(g,r,d).$$

Finally, inasmuch as none of these planes Λ together with $q_1, \ldots, q_{r+\rho}$ will fail to span a hyperplane H_Λ, and that none of the hyperplanes H_Λ will meet the given chords to C at points of C—the locus of such planes satisfying Schubert conditions of strictly higher codimension and hence, by the conjecture, empty—we may conclude that

For $q_1, \ldots, q_{r+\rho}$ general points of a general Castelnuovo canonical curve C_0, there will be exactly $\mu(g, r, d)$ divisors D on C_0, disjoint from the nodes of C_0, satisfying

$$r(D) \geq r$$

$$r(D - q_1 - \cdots - q_{r+\rho}) \geq 0.$$

(d) In the previous two sections we have arrived at the same number, $\mu(g, r, d)$, by two different routes: one pertains to smooth curves and the other to Castelnuovo canonical curves. We want now to use the equality to conclude the remainder of our main theorem.

To set up, let $S \xrightarrow{\pi} \Delta$ be a surface with fibres $C_t = \pi^{-1}(t)$ a smooth curve of genus g for $t \neq 0$ and with C_0 a general Castelnuovo canonical curve. We choose arcs $q_\alpha(t) \in C_t$ lifting π such that $q_\alpha(0)$ is a general set of $r + \rho$ smooth points on C_0. Denote by $S_d \xrightarrow{\pi_d} \Delta$ the variety with fibres $\pi_d^{-1}(t) = (C_t)_d$ the dth symmetric product of the fibres of π for $t \neq 0$, and fibre $\pi_d^{-1}(0) = (C_0^*)_d$ the dth symmetric product of the smooth locus of C_0. Inside S_d we consider the closed subvariety

$$B = \{ D \in C_d(t) : r(D) \geq r \quad \text{and} \quad r(D - q_1(t) - \cdots - q_{r+\rho}(t)) \geq 0 \}.$$

That this is a subvariety follows from the interpretation of the Brill-Noether matrix (cf. §0(c) and §1(a)). Intuitively, B is the intersection of

$$\tilde{C}_d^r = \bigcup_{t \in \Delta} C_d^r(t)$$

with

$$\tilde{X}^{r+\rho} = \bigcup_{t \in \Delta} C_{d-r-\rho}(t) + q_1(t) + \cdots + q_{r+\rho}(t).$$

We note the following:

B has dimension at least 1 everywhere.

This is because $\tilde{C}_d^r$ and $\tilde{X}^{r+\rho}$ have dimension $\rho + r + 1$ and codimension $r + \rho$ respectively (the assertion about dim $\tilde{C}_d^r$ follows from our assumption that C_0 is a general Castelnuovo canonical curve together with the dimension count in §2).

B meets the general fibre $(C_t)_d$ in a finite number of points.

This is true for the following reason: For any smooth C_t we consider the incidence correspondence $I_t \subset (C_t)_d \times (C_t)_{r+\rho}$ defined by

$$I_t = \{(D, E) : r(D) \geqslant r, r(D - E) \geqslant 0\}.$$

The projection of I_t onto the first factor is $(C_t)^r_d$, which we know to have dimension $r + \rho$, and moreover the fibre of this projection over $p_1 + \cdots + p_d \in (C_t)_d$ consists of all subsets $\{p_{i_1}, \ldots, p_{i_{r+\rho}}\}$ of the p_i's. Since the $q_\alpha(t)$ are general points of C_t for all t our assertion follows.

B meets the central fibre $(C_0^*)_d$ in exactly $\mu(g, r, d)$ distinct points.

This is true by the result in the preceding section.

It follows that, by shrinking the disc Δ if necessary, that *B meets each fibre $(C_t)_d$ in at least $\mu(g, r, d)$ distinct points*. On the other hand, by the calculation in §3(b) the intersection

$$C_d^r(t) \cap \left(C_{d-r-\rho}(t) + \sum_{\alpha=1}^{r+\rho} q_\alpha(t) \right),$$

being zero-dimensional, can consist of at most $\mu(g, r, d)$ points. The only possibility is that B meets each fibre $(C_t)_d$ in exactly $\mu(g, r, d)$ distinct points, and we have proved the

THEOREM. *For C a general curve of genus g and $p_1, \ldots, p_{r+\rho} \in C$ general points, there are exactly*

$$\mu(g, r, d) = \frac{g!\,(r+\rho)!}{\rho!\,(g-d+2r+\rho)!} \prod_{i=0}^{r-1} \frac{i!}{(g-d+r+i)!}$$

divisors D on C of degree d, moving in an r-dimensional series and containing the points p_α.

Finally, as indicated in §3a, this result proves that C_d^r has no multiple components—that is, has fundamental class exactly as given above—once we check that *no component Σ of C_d^r can fail to intersect $C_{d-\rho-r} + q_1 + \cdots + q_{r+\rho}$*. To establish this it will suffice to exhibit a particular cycle $C_{d-r-\rho} + q_1 + \cdots + q_{r+\rho} \sim X^{r+\rho}$ meeting Σ transversely at one point. To find such a cycle, we first remark that, as a consequence of the dimension statement $\dim C_{d-\delta}^r < \dim C_d^r - \delta$, proved already, a general fiber of *any* component of Σ will be a linear system without base points; by Bertini's theorem it follows that *a general point $D \in \Sigma$ is a divisor $D = q_1 + \cdots + q_d$, consisting of d distinct points.*

Now, the tangent space $T_D(C_d)$ to C_d at D is a d-dimensional vector space with a natural choice of coordinate axes, corresponding to the points $q_1, \ldots, q_d$. To give a cycle in C_d representing $X^{r+\rho}$ and passing through D, we may choose any set $I = \{i_1, \ldots, i_{r+\rho}\} \subset \{1, \ldots, d\}$ and take

$$\Gamma_I = C_{d-r-\rho}^r + q_{i_1} + \cdots + q_{i_{r+\rho}}.$$

We see that the tangent space to Γ_I at D is just the coordinate $(d - r - \rho)$-plane in $T_D(C_d)$ spanned by the coordinate axes corresponding to the points $\{q_\alpha\}_{\alpha \notin I}$; and since the $(r + \rho)$-plane $T_D(\Sigma)$ must meet at least one of the coordinate $(d - r - \rho)$-planes of $T_D(C_d)$ transversely, we conclude that *D is an isolated point of intersection of Σ with Γ_I for some I.*

To finish, consider the incidence correspondence $I \subset \Sigma \times C_{r+\rho}$ given by

$$I = \{(D, E) : D \in C_{d-r-\rho} + E\}.$$

Inasmuch as the fibers of I over Σ are all finite, we see that *I has pure dimension $r + \rho$*. On the other hand, since as we have seen at least one fiber of I over $C_{r+\rho}$ contains an isolated point, at least one component of I surjects onto $C_{r+\rho}$; thus *$\Sigma \cap C_{d-r-\rho} + E \neq \emptyset$ for every E*, and so we are done.

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