THE VOLUME OF THE FUNDAMENTAL DOMAIN FOR SOME ARITHMETICAL SUBGROUPS OF CHEVALLEY GROUPS

ROBERT P. LANGLANDS

Let $\mathfrak{g}_{\mathbf{Q}}$ be a split semisimple Lie algebra of linear transformations of the finite-dimensional vector space $V_{\mathbf{Q}}$ over $\mathbf{Q}$. Let $\mathfrak{h}_{\mathbf{Q}}$ be a split Cartan subalgebra of $\mathfrak{g}_{\mathbf{Q}}$ and choose for each root α of $\mathfrak{h}_{\mathbf{Q}}$ a root vector X_α so that if $[X_\alpha, X_{-\alpha}] = H_\alpha$ then $\alpha(H_\alpha) = 2$ and so that there is an automorphism θ of $\mathfrak{g}_{\mathbf{Q}}$ with $\theta(X_\alpha) = -X_{-\alpha}$. Let L be the set of weights of $\mathfrak{h}_{\mathbf{Q}}$ and if $\lambda \in L$ let

$$V_{\mathbf{Q}}(\lambda) = \left\{ v \in V_{\mathbf{Q}} \;\middle|\; Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}_{\mathbf{Q}} \right\};$$

let $H_1, \ldots, H_p$ be a basis over $\mathbf{Z}$ of

$$\left\{ H \;\middle|\; \lambda(H) \in \mathbf{Z} \text{ if } V_{\mathbf{Q}}(\lambda) \neq 0 \right\}.$$

As usual, there is associated to $\mathfrak{g}_{\mathbf{Q}}$ a connected algebraic group $G_{\mathbf{C}}$ of linear transformations of $V_{\mathbf{C}} = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$. If H is some lattice in $V_{\mathbf{Q}}$ satisfying

(i) $M = \sum_{\lambda \in L} M \cap V(\lambda)$,

(ii) $(X_\alpha^n/n!)M \subseteq M$ for all α,

then we let $G_{\mathbf{Z}} = \{ g \in G_{\mathbf{C}} \mid gM = M \}$. Let ω be a left invariant form on $G_{\mathbf{R}}$ of highest degree which takes the value ± 1 on $\left(\bigwedge_{i=1}^p H_i\right) \wedge \left(\bigwedge_{\alpha>0} X_\alpha\right)$ and let $[dg]$ be the Haar measure associated to ω. Our purpose now is to show the following:

If $\zeta(\cdot)$ is the Riemann zeta function, $\prod_{i=1}^p (t^{2a_i - 1} + 1)$ is the Poincaré polynomial of $G_{\mathbf{C}}$, and c is the order of the fundamental group of $G_{\mathbf{C}}$ then

$$\int_{G_{\mathbf{Z}}/G_{\mathbf{R}}} [dg] = c \prod_{i=1}^p \zeta(a_i).$$

The method to be used to find the volume of $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$ is not directly applicable to $[dg]$. So it is necessary to introduce another Haar measure on the group $G_{\mathbf{R}}$. Let U be the connected subgroup of $G_{\mathbf{C}}$ whose Lie algebra is spanned over $\mathbf{R}$ by

$$\left\{ X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha}), iH_\alpha \;\middle|\; \alpha \text{ a root} \right\}$$

and let $K = G_{\mathbf{R}} \cap U$. Choose an order on the roots and let $N = N_{\mathbf{R}}$ be the set of real points on the connected algebraic subgroup of $G_{\mathbf{C}}$ with the Lie algebra $\sum_{\alpha>0} \mathbf{C} X_\alpha$. Let $A_{\mathbf{R}}$ be the normalizer of $\mathfrak{h}_{\mathbf{C}}$ in $G_{\mathbf{R}}$. Let dn be the Haar measure on N defined by a form which takes the value ± 1 on $\bigwedge_{\alpha>0} X_\alpha$ and let da be the Haar measure on $A_{\mathbf{R}}$ defined by a form which takes the value ± 1 on $\bigwedge_{i=1}^p H_i$. Let dk be the Haar measure on K such that the total volume of K is one. Let $\rho = \frac{1}{2} \sum_{\alpha>0} \alpha$ and let $\xi_{2\rho}(a)$ be the character of $A_{\mathbf{C}}$ associated to 2ρ. Finally let dg be such that

$$\int_{G_{\mathbf{R}}} \phi(g)\, dg = \int_{N \times A_{\mathbf{R}} \times K} \left|\xi_{2\rho}(a)\right|^{-1} \phi(nak)\, dn\, da\, dk.$$

Miller Fellow.

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If N^- is the set of real points on the group associated to $\sum_{\alpha<0}\mathbf{C}X_\alpha$ define dn^- in the same way as we defined dn. It is easy to see that

$$\int_G \phi(g)\,[dg] = \int_N dn \int_{A_{\mathbf{R}}} da \int_{N^-} dn^- \big|\xi_{2\rho}(a)\big|^{-1}\phi(nan^-).$$

Suppose $\phi(gk)=\phi(g)$ for all $g\in G_{\mathbf{R}}$ and all $k\in K$. Then

$$\int_G \phi(g)\,dg = \int_{N\times A_{\mathbf{R}}} dn\,da\big|\xi_{2\rho}(a)\big|^{-1}\phi(na).$$

On the other hand, if $n^-=n(n^-)a(n^-)k(n^-)$,

$$\begin{aligned}\int \phi(g)[dg] &= \int_{N^-} dn^-\Big\{\int_A da\int_N dn\big|\xi_{2\rho}(a)\big|^{-1}\phi\big(nan(n^-)a(n^-)k(n^-)\big)\Big\}\\ &= \Big\{\int_A da\int_N dn\big|\xi_{2\rho}(a)\big|^{-1}\phi(na)\Big\}\Big\{\int_{N^-}\big|\xi_{2\rho}a(n^-)\big|dn^-\Big\}.\end{aligned}$$

It follows from a formula of Gindikin and Karpelevich that the second factor equals

$$\begin{aligned}\prod_{\alpha>0}\frac{\pi^{-1/2}\Gamma\big(\rho(H_\alpha)/2\big)}{\Gamma\Big(\big(\rho(H_\alpha)+1\big)/2\Big)} &= \prod_{\alpha>0}\frac{\pi^{-\rho(H_\alpha)/2}\Gamma\big(\rho(H_\alpha)/2\big)}{\pi^{-(\rho(H_\alpha)+1)/2}\Gamma\Big(\big(\rho(H_\alpha)+1\big)/2\Big)}\\ &= \frac{\prod'_{\alpha>0}\pi^{-\rho(H_\alpha)/2}\Gamma\big(\rho(H_\alpha)/2\big)}{\prod_{\alpha>0}\pi^{-(\rho(H_\alpha)+1)/2}\Gamma\Big(\big(\rho(H_\alpha)+1\big)/2\Big)},\end{aligned}$$

since when α is simple $\rho(H_\alpha)=1$ and

$$\pi^{-1/2}\Gamma\left(\frac{1}{2}\right)=1.$$

The product in the numerator is taken over the positive roots which are not simple. By a well-known result the numbers, with multiplicities, in the set

$$\big\{\,\rho(H_\alpha)+1 \;\big|\; \alpha>0\,\big\}$$

are just the numbers $\rho(H_\alpha)$ with α positive and not simple, together with the numbers $a_1,\dots,a_p$. So if

$$\xi(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

we have to show that

$$\int_{G_{\mathbf{Z}}/G_{\mathbf{R}}} dg = \frac{c\prod_{\alpha>0}\xi\big(\rho(H_\alpha)+1\big)}{\prod'_{\alpha>0}\xi\big(\rho(H_\alpha)\big)}.$$

By the way, it is well to keep in mind that $\rho(H_\alpha)>1$ if α is not simple.

Let A be the connected component of $A_{\mathbf{R}}$ and let M be the points of finite order in $A_{\mathbf{R}}$. Certainly $A_{\mathbf{R}}=AM$. Moreover, by Iwasawa, $G=NAK$. If $g=nak$ and $a=\exp H$, we set $H=H(g)$.

If ϕ is an infinitely differentiable function with compact support on $N\backslash G$ such that $\phi(gk)=\phi(g)$ for all g in G and all k in K we can write ϕ as a Fourier integral.

$$\phi(g)=\frac{1}{(2\pi)^p}\int_{\operatorname{Re}\lambda=\lambda_0}\exp\Big(\lambda\big(H(g)\big)+\rho\big(H(g)\big)\Big)\Phi(\lambda)\,|d\lambda|;$$

λ is the symbol for an element of the dual of $\mathfrak{h}_{\mathbf{C}}$; $\Phi(\lambda)$ is an entire complex-valued function of λ; and $d\lambda = dz_1 \wedge \cdots \wedge dz_p$ with $z_i = \lambda(H_i)$. As in the lectures on Eisenstein series we can introduce

$$\widehat{\phi}(g) = \sum_{\gamma \in G_{\mathbf{Z}} \cap NM \backslash G_{\mathbf{Z}}} \phi(\gamma g).$$

Our evaluation of the volume of $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$ will be based on the simple relation

$$(\widehat{\phi}, 1)(1, \widehat{\psi}) = (1,1)(\Pi\widehat{\phi}, \Pi\widehat{\psi}).$$

The inner products are taken in $L^2(G_{\mathbf{Z}} \backslash G_{\mathbf{R}})$ with respect to dg and Π is the orthogonal projection on the space of constant functions. Since

$$(1,1) = \int_{G_{\mathbf{Z}} \backslash G_{\mathbf{R}}} dg,$$

it is enough to find an explicit formula for the other three terms. Now

$$\begin{aligned}(\widehat{\phi}, 1) &= \int_{G_{\mathbf{Z}} \cap NM \backslash G_{\mathbf{R}}} \phi(g)\, dg \\ &= \mu(G_{\mathbf{Z}} \cap NM \backslash NM) \int_A \left|\xi_{2\rho}(a)\right|^{-1} \phi(a)\, da \\ &= \Phi(\rho)\end{aligned}$$

since $\mu(G_{\mathbf{Z}} \cap NM \backslash NM) = 1$. To see the latter we have to observe that $M \subseteq G_{\mathbf{Z}}$ and that, as follows from results stated in Cartier's talk, $\mu(G_{\mathbf{Z}} \cap N \backslash N) = 1$. It is also clear that $(1, \widehat{\psi}) = \overline{\Psi}(\rho)$. The nontrivial step is to evaluate

$$(\Pi\widehat{\phi}, \Pi\widehat{\psi}).$$

From the theory of Eisenstein series we know that

$$(\widehat{\phi}, \widehat{\psi}) = \frac{1}{(2\pi)^p} \int_{\operatorname{Re}\lambda = \lambda_0} \sum_{s \in \Omega} M(s, \lambda)\Phi(\lambda)\overline{\Psi}(-s\overline{\lambda})\, |d\lambda|.$$

Ω is the Weyl group, λ_0 is any point such that $\lambda_0(H_\alpha) > 1$ for every simple root, and

$$M(s, \lambda) = \prod_{\alpha > 0} \frac{\xi\big(1 + s\lambda(H_\alpha)\big)}{\xi\big(1 + \lambda(H_\alpha)\big)} = \prod_{\substack{\alpha > 0 \\ s\alpha < 0}} \frac{\xi\big(\lambda(H_\alpha)\big)}{\xi\big(1 + \lambda(H_\alpha)\big)}.$$

In the lectures on Eisenstein series I introduced an unbounded self-adjoint operator A on the closed subspace of $L^2(G_{\mathbf{Z}} \backslash G_{\mathbf{R}})$ generated by the functions $\widehat{\phi}$ with ϕ of the form indicated above. Comparing the definition of A with the formula for $(\widehat{\phi}, 1)$ we see that

$$(A\widehat{\phi}, 1) = (\rho, \rho)(\widehat{\phi}, 1),$$

since the constant functions are in this space $A1 = (\rho, \rho) \cdot 1$. As a consequence, if $E(x)$, $-\infty < x < \infty$, is the spectral resolution of A the constant functions are in the range of $E\big((\rho,\rho)\big) - E\big((\rho,\rho) - 0\big) = E$. We show that this range consists precisely of the constant functions and compute $(E\widehat{\phi}, \widehat{\psi}) = (\Pi\widehat{\phi}, \Pi\widehat{\psi})$.

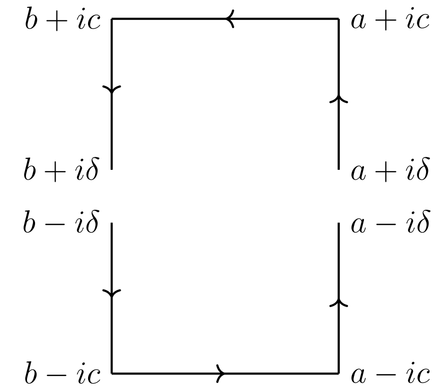
Suppose $a > (\rho, \rho) > b$ and $a - b$ is small. According to a well-known formula,

$$\frac{1}{2}\Big\{\Big(E(a)\widehat{\phi}, \widehat{\psi}\Big) + \Big(E(a-0)\widehat{\phi}, \widehat{\psi}\Big)\Big\} - \frac{1}{2}\Big\{\Big(E(b)\widehat{\phi}, \widehat{\psi}\Big) + \Big(E(b-0)\widehat{\phi}, \widehat{\psi}\Big)\Big\}$$

is equal to

$$\text{(a)} \qquad \lim_{\delta\downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} \Big(R(\mu, A)\widehat{\phi}, \widehat{\psi}\Big)\, d\mu$$

if $C(a, b, c, \delta)$ is the following contour.



Recall that, if $\operatorname{Re}\mu > (\lambda_0, \lambda_0)$,

$$\Big(R(\mu, A)\widehat{\phi}, \widehat{\psi}\Big) = \sum_{s\in\Omega} \frac{1}{(2\pi i)^p} \int_{\operatorname{Re}\lambda=\lambda_0} \frac{1}{\mu - (\lambda, \lambda)} M(s, \lambda)\Phi(\lambda)\overline{\Psi}(-s\overline{\lambda})\, d\lambda.$$

If $w = (w_1, \ldots, w_p)$ belongs to $\mathbf{C}^p$ let $\lambda(w)$ be such that $\lambda(H_{\alpha_i}) = w_i$, where $\alpha_1, \ldots, \alpha_p$ are the simple roots. Set

$$\begin{aligned} \phi_p(w, s) &= M\big(s, \lambda(w)\big)\Phi\big(\lambda(w)\big)\overline{\Psi}\big(-s\lambda(\overline{w})\big), \\ Q_p(w) &= \big(\lambda(w), \lambda(w)\big), \end{aligned}$$

then (a) is equal to

$$\frac{1}{c}\sum_{s\in\Omega} \lim_{\delta\downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{(2\pi i)^p} \int_{\operatorname{Re} w = w_0} \frac{1}{\mu - Q_p(w)} \phi_p(w, s)\, dw_1 \cdots dw_p \right\}$$

provided each of these limits exist.[1] The coordinates of w_0 must all be greater than one. We shall consider the limits individually.

Let $w^q = (w_1, \ldots, w_q)$ and define $\phi_q(w^q; s)$ inductively for $0 \leqslant q \leqslant p$ by

$$\phi_q(q_1, \ldots, w_q; s) = \operatorname*{Residue}_{w_{q+1}=1} \phi_{q+1}(w_1, \ldots, w_{q+1}; s).$$

It is easily seen that $\phi_q(w^q; s)$ has no singularities in the region defined by the inequalities $\operatorname{Re} w_i > 1$, $1 \leqslant i \leqslant q$; that $\phi_q(w^q; s)$ goes to zero very fast when the imaginary part of w^q goes to infinity and its real part remains in a compact subset of this region; and that there is a positive number ϵ such that the only singularities of $\phi_q(w^q; s)$ in

$$\Big\{\, (w_1, \ldots, w_q) \;\Big|\; |\operatorname{Re} w_i - 1| < \epsilon, 1 \leqslant i \leqslant q \,\Big\}$$

lie on the hyperplanes $w_i = 1$ and are at most simple poles. $\phi_0(s)$ is of course a constant. Set $Q_q(w^q) = Q_p(w_1, \ldots, w_q, 1, \ldots, 1)$.

[1]The inner integral is defined for $\operatorname{Re}\mu > Q_p(w_0)$. However, as can be seen from the discussion to follow, the function of μ it defines can be analytically continued to a region containing $C(a, b, c, \delta)$.

Let us show by induction that the given limit equals

$$\text{(b)} \qquad \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{(2\pi i)^q} \int_{\operatorname{Re} w^q = w_0^q} \frac{1}{\mu - Q_q(w^q)} \phi_q(w^q; s)\, dw_1 \cdots dw_q \right\}$$

if $w_0^q = (w_{0,1}, \dots, w_{0,q})$ with $w_{0,i} > 1$, $1 \leqslant i \leqslant q$. Of course, the above expression is independent of the choice of such a point w_0^q. Take $w_0^q = (1+u, \dots, 1+u, 1+v)$, with u and v positive but small and $w_0^{q-1} = (1+u, \dots, 1+u)$. If $\Lambda_1, \dots, \Lambda_q$ are such that $\Lambda_i(H_{\alpha j}) = \delta_{ij}$, then $(\Lambda_i, \Lambda_j) \geqslant 0$. As a consequence, if u is much smaller than v, then

$$Q_q(1+u, \dots, 1+u, 1-v) < (\rho, \rho).$$

Choose (b) to be larger than the number on the left. Also

$$\operatorname{Re} Q_q(w^q) = Q_q(\operatorname{Re} w^q) - Q_p(\operatorname{Im} w_1, \dots, \operatorname{Im} w_q, 0, \dots, 0).$$

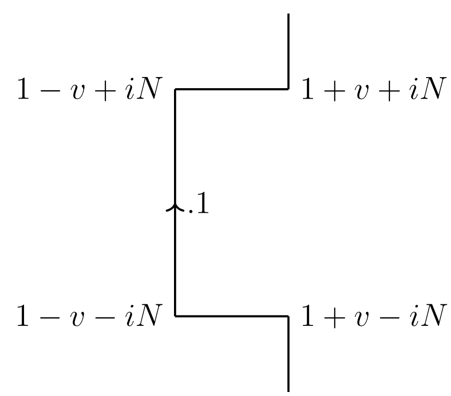
Thus there is a constant N such that if either $\operatorname{Re} w_i = 1+u$, $1 \leqslant i \leqslant q-1$ and $\operatorname{Re} w_q = 1-v$ or $\operatorname{Re} w_i = 1+u$, $1 \leqslant i \leqslant p$ and $|\operatorname{Re} w_q - 1| \leqslant v$ and $|\operatorname{Im} w_q| > N$, then

$$\operatorname{Re} Q_q(w^q) < b - 1/N.$$

In (b) we may perform the integrations in any order. Integrate first with respect to w_q. If C is the indicated contour, the result is the sum of (b) with q replaced by $q-1$ and

$$\lim_{\delta \downarrow 0} \frac{1}{(2\pi i)^q} \int_{\operatorname{Re} w^{q-1} = w_0^{q-1}} dw_1 \cdots dw_{q-1} \int_C dw_q \phi_q(w^q, s) \left\{ \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} \frac{1}{\mu - Q_q(w^q)} d\mu \right\}$$

which is obviously zero.



The contour C

Taking $q = 0$ in (b) we get

$$\lim_{\delta \downarrow 0} \frac{\phi_0(s)}{2\pi i} \int_{C(a,b,c,\delta)} \frac{1}{\mu - (\rho, \rho)} d\mu = \phi_0(s).$$

It is clear that $\phi_0(s)$ is zero unless s sends every positive root to a negative root but that for the unique element of the Weyl group which does this

$$\phi_0(s) = \frac{\prod'_{\alpha > 0} \xi\big(\rho(H_\alpha)\big) \Phi(\rho) \overline{\Psi(\rho)}}{\prod_{\alpha > 0} \xi\big(\rho(H_\alpha) + 1\big)}$$

since $s\rho = -\rho$. This is the result required.

Finally, I remark that although the method just described for computing the volume of $\Gamma\backslash G$ has obvious limitations, it can be applied to other groups. In particular it works for Chevalley groups over a number field.

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