

# THE VOLUME OF THE FUNDAMENTAL DOMAIN FOR SOME ARITHMETICAL SUBGROUPS OF CHEVALLEY GROUPS

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Let  $\mathfrak{g}_{\mathbf{Q}}$  be a split semisimple Lie algebra of linear transformations of the finite-dimensional vector space  $V_{\mathbf{Q}}$  over  $\mathbf{Q}$ . Let  $\mathfrak{h}_{\mathbf{Q}}$  be a split Cartan subalgebra of  $\mathfrak{g}_{\mathbf{Q}}$  and choose for each root  $\alpha$  of  $\mathfrak{h}_{\mathbf{Q}}$  a root vector  $X_{\alpha}$  so that if  $[X_{\alpha}, X_{-\alpha}] = H_{\alpha}$  then  $\alpha(H_{\alpha}) = 2$  and so that there is an automorphism  $\theta$  of  $\mathfrak{g}_{\mathbf{Q}}$  with  $\theta(X_{\alpha}) = -X_{-\alpha}$ . Let  $L$  be the set of weights of  $\mathfrak{h}_{\mathbf{Q}}$  and if  $\lambda \in L$  let

$$V_{\mathbf{Q}}(\lambda) = \{ v \in V_{\mathbf{Q}} \mid Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}_{\mathbf{Q}} \};$$

let  $H_1, \dots, H_p$  be a basis over  $\mathbf{Z}$  of

$$\{ H \mid \lambda(H) \in \mathbf{Z} \text{ if } V_{\mathbf{Q}}(\lambda) \neq 0 \}.$$

As usual, there is associated to  $\mathfrak{g}_{\mathbf{Q}}$  a connected algebraic group  $G_{\mathbf{C}}$  of linear transformations of  $V_{\mathbf{C}} = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$ . If  $H$  is some lattice in  $V_{\mathbf{Q}}$  satisfying

- (i)  $M = \sum_{\lambda \in L} M \cap V(\lambda)$ ,
- (ii)  $(X_{\alpha}^n/n!)M \subseteq M$  for all  $\alpha$ ,

then we let  $G_{\mathbf{Z}} = \{ g \in G_{\mathbf{C}} \mid gM = M \}$ . Let  $\omega$  be a left invariant form on  $G_{\mathbf{R}}$  of highest degree which takes the value  $\pm 1$  on  $(\bigwedge_{i=1}^p H_i) \wedge (\bigwedge_{\alpha > 0} X_{\alpha})$  and let  $[dg]$  be the Haar measure associated to  $\omega$ . Our purpose now is to show the following:

*If  $\zeta(\cdot)$  is the Riemann zeta function,  $\prod_{i=1}^p (t^{2a_i-1} + 1)$  is the Poincaré polynomial of  $G_{\mathbf{C}}$ , and  $c$  is the order of the fundamental group of  $G_{\mathbf{C}}$  then*

$$\int_{G_{\mathbf{Z}}/G_{\mathbf{R}}} [dg] = c \prod_{i=1}^p \zeta(a_i).$$

The method to be used to find the volume of  $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$  is not directly applicable to  $[dg]$ . So it is necessary to introduce another Haar measure on the group  $G_{\mathbf{R}}$ . Let  $U$  be the connected subgroup of  $G_{\mathbf{C}}$  whose Lie algebra is spanned over  $\mathbf{R}$  by

$$\{ X_{\alpha} - X_{-\alpha}, i(X_{\alpha} + X_{-\alpha}), iH_{\alpha} \mid \alpha \text{ a root} \}$$

and let  $K = G_{\mathbf{R}} \cap U$ . Choose an order on the roots and let  $N = N_{\mathbf{R}}$  be the set of real points on the connected algebraic subgroup of  $G_{\mathbf{C}}$  with the Lie algebra  $\sum_{\alpha > 0} \mathbf{C}X_{\alpha}$ . Let  $A_{\mathbf{R}}$  be the normalizer of  $\mathfrak{h}_{\mathbf{C}}$  in  $G_{\mathbf{R}}$ . Let  $dn$  be the Haar measure on  $N$  defined by a form which takes the value  $\pm 1$  on  $\bigwedge_{\alpha > 0} X_{\alpha}$  and let  $da$  be the Haar measure on  $A_{\mathbf{R}}$  defined by a form which takes the value  $\pm 1$  on  $\bigwedge_{i=1}^p H_i$ . Let  $dk$  be the Haar measure on  $K$  such that the total volume of  $K$  is one. Let  $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$  and let  $\xi_{2\rho}(a)$  be the character of  $A_{\mathbf{C}}$  associated to  $2\rho$ . Finally let  $dg$  be such that

$$\int_{G_{\mathbf{R}}} \phi(g) dg = \int_{N \times A_{\mathbf{R}} \times K} |\xi_{2\rho}(a)|^{-1} \phi(nak) dn da dk.$$

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If  $N^-$  is the set of real points on the group associated to  $\sum_{\alpha < 0} \mathbf{C}X_\alpha$  define  $dn^-$  in the same way as we defined  $dn$ . It is easy to see that

$$\int_G \phi(g) [dg] = \int_N dn \int_{A_{\mathbf{R}}} da \int_{N^-} dn^- |\xi_{2\rho}(a)|^{-1} \phi(nan^-).$$

Suppose  $\phi(gk) = \phi(g)$  for all  $g \in G_{\mathbf{R}}$  and all  $k \in K$ . Then

$$\int_G \phi(g) dg = \int_{N \times A_{\mathbf{R}}} dn da |\xi_{2\rho}(a)|^{-1} \phi(na).$$

On the other hand, if  $n^- = n(n^-)a(n^-)k(n^-)$ ,

$$\begin{aligned} \int \phi(g)[dg] &= \int_{N^-} dn^- \left\{ \int_A da \int_N dn |\xi_{2\rho}(a)|^{-1} \phi(nan(n^-)a(n^-)k(n^-)) \right\} \\ &= \left\{ \int_A da \int_N dn |\xi_{2\rho}(a)|^{-1} \phi(na) \right\} \left\{ \int_{N^-} |\xi_{2\rho}a(n^-)| dn^- \right\}. \end{aligned}$$

It follows from a formula of Gindikin and Karpelevich that the second factor equals

$$\begin{aligned} \prod_{\alpha > 0} \frac{\pi^{-1/2} \Gamma(\rho(H_\alpha)/2)}{\Gamma((\rho(H_\alpha) + 1)/2)} &= \prod_{\alpha > 0} \frac{\pi^{-\rho(H_\alpha)/2} \Gamma(\rho(H_\alpha)/2)}{\pi^{-(\rho(H_\alpha)+1)/2} \Gamma((\rho(H_\alpha) + 1)/2)} \\ &= \frac{\prod'_{\alpha > 0} \pi^{-\rho(H_\alpha)/2} \Gamma(\rho(H_\alpha)/2)}{\prod_{\alpha > 0} \pi^{-(\rho(H_\alpha)+1)/2} \Gamma((\rho(H_\alpha) + 1)/2)}, \end{aligned}$$

since when  $\alpha$  is simple  $\rho(H_\alpha) = 1$  and

$$\pi^{-1/2} \Gamma\left(\frac{1}{2}\right) = 1.$$

The product in the numerator is taken over the positive roots which are not simple. By a well-known result the numbers, with multiplicities, in the set

$$\{\rho(H_\alpha) + 1 \mid \alpha > 0\}$$

are just the numbers  $\rho(H_\alpha)$  with  $\alpha$  positive and not simple, together with the numbers  $a_1, \dots, a_p$ . So if

$$\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

we have to show that

$$\int_{G_{\mathbf{Z}}/G_{\mathbf{R}}} dg = \frac{c \prod_{\alpha > 0} \xi(\rho(H_\alpha) + 1)}{\prod'_{\alpha > 0} \xi(\rho(H_\alpha))}.$$

By the way, it is well to keep in mind that  $\rho(H_\alpha) > 1$  if  $\alpha$  is not simple.

Let  $A$  be the connected component of  $A_{\mathbf{R}}$  and let  $M$  be the points of finite order in  $A_{\mathbf{R}}$ . Certainly  $A_{\mathbf{R}} = AM$ . Moreover, by Iwasawa,  $G = NAK$ . If  $g = nak$  and  $a = \exp H$ , we set  $H = H(g)$ .

If  $\phi$  is an infinitely differentiable function with compact support on  $N \backslash G$  such that  $\phi(gk) = \phi(g)$  for all  $g$  in  $G$  and all  $k$  in  $K$  we can write  $\phi$  as a Fourier integral.

$$\phi(g) = \frac{1}{(2\pi)^p} \int_{\operatorname{Re} \lambda = \lambda_0} \exp\left(\lambda(H(g)) + \rho(H(g))\right) \Phi(\lambda) |d\lambda|;$$

$\lambda$  is the symbol for an element of the dual of  $\mathfrak{h}_{\mathbf{C}}$ ;  $\Phi(\lambda)$  is an entire complex-valued function of  $\lambda$ ; and  $d\lambda = dz_1 \wedge \cdots \wedge dz_p$  with  $z_i = \lambda(H_i)$ . As in the lectures on Eisenstein series we can introduce

$$\widehat{\phi}(g) = \sum_{\gamma \in G_{\mathbf{Z}} \cap NM \backslash G_{\mathbf{Z}}} \phi(\gamma g).$$

Our evaluation of the volume of  $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$  will be based on the simple relation

$$(\widehat{\phi}, 1)(1, \widehat{\psi}) = (1, 1)(\Pi \widehat{\phi}, \Pi \widehat{\psi}).$$

The inner products are taken in  $L^2(G_{\mathbf{Z}} \backslash G_{\mathbf{R}})$  with respect to  $dg$  and  $\Pi$  is the orthogonal projection on the space of constant functions. Since

$$(1, 1) = \int_{G_{\mathbf{Z}} \backslash G_{\mathbf{R}}} dg,$$

it is enough to find an explicit formula for the other three terms. Now

$$\begin{aligned} (\widehat{\phi}, 1) &= \int_{G_{\mathbf{Z}} \cap NM \backslash G_{\mathbf{R}}} \phi(g) dg \\ &= \mu(G_{\mathbf{Z}} \cap NM \backslash NM) \int_A |\xi_{2\rho}(a)|^{-1} \phi(a) da \\ &= \Phi(\rho) \end{aligned}$$

since  $\mu(G_{\mathbf{Z}} \cap NM \backslash NM) = 1$ . To see the latter we have to observe that  $M \subseteq G_{\mathbf{Z}}$  and that, as follows from results stated in Cartier's talk,  $\mu(G_{\mathbf{Z}} \cap N \backslash N) = 1$ . It is also clear that  $(1, \widehat{\psi}) = \overline{\Psi}(\rho)$ . The nontrivial step is to evaluate

$$(\Pi \widehat{\phi}, \Pi \widehat{\psi}).$$

From the theory of Eisenstein series we know that

$$(\widehat{\phi}, \widehat{\psi}) = \frac{1}{(2\pi)^p} \int_{\operatorname{Re} \lambda = \lambda_0} \sum_{s \in \Omega} M(s, \lambda) \Phi(\lambda) \overline{\Psi}(-s\bar{\lambda}) |d\lambda|.$$

$\Omega$  is the Weyl group,  $\lambda_0$  is any point such that  $\lambda_0(H_\alpha) > 1$  for every simple root, and

$$M(s, \lambda) = \prod_{\alpha > 0} \frac{\xi(1 + s\lambda(H_\alpha))}{\xi(1 + \lambda(H_\alpha))} = \prod_{\substack{\alpha > 0 \\ s\alpha < 0}} \frac{\xi(\lambda(H_\alpha))}{\xi(1 + \lambda(H_\alpha))}.$$

In the lectures on Eisenstein series I introduced an unbounded self-adjoint operator  $A$  on the closed subspace of  $L^2(G_{\mathbf{Z}} \backslash G_{\mathbf{R}})$  generated by the functions  $\widehat{\phi}$  with  $\phi$  of the form indicated above. Comparing the definition of  $A$  with the formula for  $(\widehat{\phi}, 1)$  we see that

$$(A\widehat{\phi}, 1) = (\rho, \rho)(\widehat{\phi}, 1),$$

since the constant functions are in this space  $A1 = (\rho, \rho) \cdot 1$ . As a consequence, if  $E(x)$ ,  $-\infty < x < \infty$ , is the spectral resolution of  $A$  the constant functions are in the range of  $E((\rho, \rho)) - E((\rho, \rho) - 0) = E$ . We show that this range consists precisely of the constant functions and compute  $(E\widehat{\phi}, \widehat{\psi}) = (\Pi \widehat{\phi}, \Pi \widehat{\psi})$ .

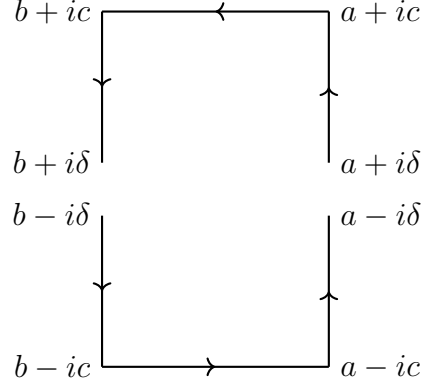
Suppose  $a > (\rho, \rho) > b$  and  $a - b$  is small. According to a well-known formula,

$$\frac{1}{2} \left\{ \left( E(a)\widehat{\phi}, \widehat{\psi} \right) + \left( E(a-0)\widehat{\phi}, \widehat{\psi} \right) \right\} - \frac{1}{2} \left\{ \left( E(b)\widehat{\phi}, \widehat{\psi} \right) + \left( E(b-0)\widehat{\phi}, \widehat{\psi} \right) \right\}$$

is equal to

$$(a) \quad \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} \left( R(\mu, A) \widehat{\phi}, \widehat{\psi} \right) d\mu$$

if  $C(a, b, c, \delta)$  is the following contour.



Recall that, if  $\operatorname{Re} \mu > (\lambda_0, \lambda_0)$ ,

$$\left( R(\mu, A) \widehat{\phi}, \widehat{\psi} \right) = \sum_{s \in \Omega} \frac{1}{(2\pi i)^p} \int_{\operatorname{Re} \lambda = \lambda_0} \frac{1}{\mu - (\lambda, \lambda)} M(s, \lambda) \Phi(\lambda) \overline{\Psi}(-s\bar{\lambda}) d\lambda.$$

If  $w = (w_1, \dots, w_p)$  belongs to  $\mathbf{C}^p$  let  $\lambda(w)$  be such that  $\lambda(H_{\alpha_i}) = w_i$ , where  $\alpha_1, \dots, \alpha_p$  are the simple roots. Set

$$\begin{aligned} \phi_p(w, s) &= M(s, \lambda(w)) \Phi(\lambda(w)) \overline{\Psi}(-s\lambda(\bar{w})), \\ Q_p(w) &= (\lambda(w), \lambda(w)), \end{aligned}$$

then (a) is equal to

$$\frac{1}{c} \sum_{s \in \Omega} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{(2\pi i)^p} \int_{\operatorname{Re} w = w_0} \frac{1}{\mu - Q_p(w)} \phi_p(w, s) dw_1 \cdots dw_p \right\}$$

provided each of these limits exist.<sup>1</sup> The coordinates of  $w_0$  must all be greater than one. We shall consider the limits individually.

Let  $w^q = (w_1, \dots, w_q)$  and define  $\phi_q(w^q; s)$  inductively for  $0 \leq q \leq p$  by

$$\phi_q(q_1, \dots, w_q; s) = \operatorname{Residue}_{w_{q+1}=1} \phi_{q+1}(w_1, \dots, w_{q+1}; s).$$

It is easily seen that  $\phi_q(w^q; s)$  has no singularities in the region defined by the inequalities  $\operatorname{Re} w_i > 1$ ,  $1 \leq i \leq q$ ; that  $\phi_q(w^q; s)$  goes to zero very fast when the imaginary part of  $w^q$  goes to infinity and its real part remains in a compact subset of this region; and that there is a positive number  $\epsilon$  such that the only singularities of  $\phi_q(w^q; s)$  in

$$\{ (w_1, \dots, w_q) \mid |\operatorname{Re} w_i - 1| < \epsilon, 1 \leq i \leq q \}$$

lie on the hyperplanes  $w_i = 1$  and are at most simple poles.  $\phi_0(s)$  is of course a constant. Set  $Q_q(w^q) = Q_p(w_1, \dots, w_q, 1, \dots, 1)$ .

<sup>1</sup>The inner integral is defined for  $\operatorname{Re} \mu > Q_p(w_0)$ . However, as can be seen from the discussion to follow, the function of  $\mu$  it defines can be analytically continued to a region containing  $C(a, b, c, \delta)$ .

Let us show by induction that the given limit equals

$$(b) \quad \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{(2\pi i)^q} \int_{\operatorname{Re} w^q = w_0^q} \frac{1}{\mu - Q_q(w^q)} \phi_q(w^q; s) dw_1 \cdots dw_q \right\}$$

if  $w_0^q = (w_{0,1}, \dots, w_{0,q})$  with  $w_{0,i} > 1$ ,  $1 \leq i \leq q$ . Of course, the above expression is independent of the choice of such a point  $w_0^q$ . Take  $w_0^q = (1+u, \dots, 1+u, 1+v)$ , with  $u$  and  $v$  positive but small and  $w_0^{q-1} = (1+u, \dots, 1+u)$ . If  $\Lambda_1, \dots, \Lambda_q$  are such that  $\Lambda_i(H_{\alpha j}) = \delta_{ij}$ , then  $(\Lambda_i, \Lambda_j) \geq 0$ . As a consequence, if  $u$  is much smaller than  $v$ , then

$$Q_q(1+u, \dots, 1+u, 1-v) < (\rho, \rho).$$

Choose (b) to be larger than the number on the left. Also

$$\operatorname{Re} Q_q(w^q) = Q_q(\operatorname{Re} w^q) - Q_p(\operatorname{Im} w_1, \dots, \operatorname{Im} w_q, 0, \dots, 0).$$

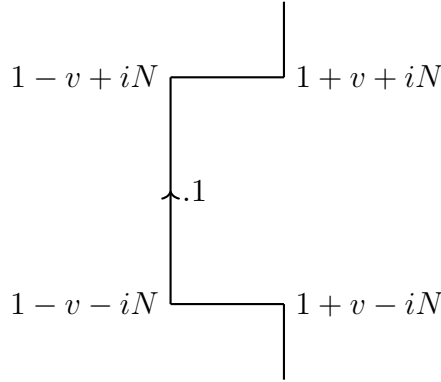
Thus there is a constant  $N$  such that if either  $\operatorname{Re} w_i = 1+u$ ,  $1 \leq i \leq q-1$  and  $\operatorname{Re} w_q = 1-v$  or  $\operatorname{Re} w_i = 1+u$ ,  $1 \leq i \leq p$  and  $|\operatorname{Re} w_q - 1| \leq v$  and  $|\operatorname{Im} w_q| > N$ , then

$$\operatorname{Re} Q_q(w^q) < b - 1/N.$$

In (b) we may perform the integrations in any order. Integrate first with respect to  $w_q$ . If  $C$  is the indicated contour, the result is the sum of (b) with  $q$  replaced by  $q-1$  and

$$\lim_{\delta \downarrow 0} \frac{1}{(2\pi i)^q} \int_{\operatorname{Re} w^{q-1} = w_0^{q-1}} dw_1 \cdots dw_{q-1} \int_C dw_q \phi_q(w^q, s) \left\{ \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} \frac{1}{\mu - Q_q(w^q)} d\mu \right\}$$

which is obviously zero.



The contour  $C$

Taking  $q = 0$  in (b) we get

$$\lim_{\delta \downarrow 0} \frac{\phi_0(s)}{2\pi i} \int_{C(a,b,c,\delta)} \frac{1}{\mu - (\rho, \rho)} d\mu = \phi_0(s).$$

It is clear that  $\phi_0(s)$  is zero unless  $s$  sends every positive root to a negative root but that for the unique element of the Weyl group which does this

$$\phi_0(s) = \frac{\prod'_{\alpha > 0} \xi(\rho(H_\alpha)) \Phi(\rho) \overline{\Psi(\rho)}}{\prod_{\alpha > 0} \xi(\rho(H_\alpha) + 1)}$$

since  $s\rho = -\rho$ . This is the result required.

Finally, I remark that although the method just described for computing the volume of  $\Gamma \backslash G$  has obvious limitations, it can be applied to other groups. In particular it works for Chevalley groups over a number field.

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