(0)

# RESTRICTED CHEBOTAREV THEOREMS FOR $SL_2(\mathbb{Z})$ AND RELATED GROUPS

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# N. CHEBOTAREV'S THEOREM (1922) (SEE ACCOUNT STEVENHAGEN/LENSTR 1996)

$K/\mathbb{Q}$ GALOIS EXTENSION OF $\mathbb{Q}$ WITH GALOIS GROUP $H$.

- FOR $p$ A (RATIONAL) PRIME WHICH IS UNRAMIFIED IN $K$ LET

$$\mathrm{FROB}_{K/\mathbb{Q}}(p)$$

BE THE CORRESPONDING CONJUGACY CLASS OF $p$ IN $H$ CORRESPONDING TO FROBENIUS. THEN FOR $C$ A CONJUGACY CLASS IN $H$

$$\sum_{\substack{p \leq x \\ \mathrm{FROB}(p) = C}} 1 \sim \frac{|C|}{|H|} \sum_{p \leq x} 1 \sim \frac{|C|}{|H|} \mathrm{Li}(x), \text{ AS } x \to \infty$$

$$\mathrm{Li}(v) = \int_2^v \frac{dt}{\log t}$$

- ACTUALLY HE PROVED A WEAKER DENSITY VERSION WHICH IMPLIES THAT THERE ARE INFINITELY MANY $p$ WITH $\mathrm{FROB}(p) = C$.

- IT IS A VAST GENERALIZATION OF DIRICHLET'S THEOREM ON PRIMES IN PROGRESSIONS AND ONE CANNOT OVERSTATE ITS IMPACT.

## LOCAL OR RESTRICTED VERSIONS:

FOR "NATURALLY" DEFINED GROWING SETS OF PRIMES $B_x$ IS IT STILL TRUE THAT

$$\sum_{\substack{p \in B_x \\ \mathrm{FROB}(p) = C}} 1 \sim \frac{|C|}{|H|} \sum_{p \in B_x} 1 \quad \text{AS } x \to \infty ?$$

- THIS SHOULD HOLD IF $B_x$ DOES NOT KNOW ABOUT $K/\mathbb{Q}$ AND IF THERE ARE CORRELATIONS THEN CORRECTIONS TO THE PROPORTION $\frac{|C|}{|H|}$ ARE NEEDED.

IN HIS 2012 MINERVA PRINCETON LECTURES SERRE JOKES

- MOTIVATED SUCH $B$'s "MOTIVIC"
  EG PRIMES IN PROGRESSIONS.

- UNMOTIVATED, BUT PERHAPS AS INTERESTING

  EG PROGRESSIONS IN PRIMES.

MOTIVATED EXAMPLE:

$$B_{x,h} = \{ \text{PRIMES } p : x-h \leq p \leq x \}$$

$$x^{\varepsilon} \leq h \leq x \qquad (\varepsilon > 0).$$

- UNDER THE RIEMANN HYPOTHESIS IF $h \geqslant x^{1/2+\varepsilon}$ THEN

$$|B_{x,h}| \sim \frac{h}{\log x} \quad \text{AS} \quad x \to \infty.$$

- HOHEISEL (1930) SHOWED THAT THIS HOLDS UNCONDITIONALLY FOR $h \geqslant x^{1-\delta}$ FOR SOME SMALL $\delta > 0$.

- WORLD RECORD (2024) GUTH-MAYNARD HOLDS WITH $h \geqslant x^{17/30}$.

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ONE CAN EXTEND HOHEISEL TO THE CHEBOTAREV SETTING, BALOG-ONO (2021):
GIVEN $K/\mathbb{Q}$ AND $C$ AS ABOVE THERE IS $\delta(K) > 0$ SUCH THAT FOR $h \geqslant x^{1-\delta}$

$$\sum_{\substack{p \in B_{x,h} \\ \mathrm{FROB}(p) \in C}} 1 \sim \frac{|C|}{|H|} \frac{h}{\log x} \quad \text{AS} \quad x \to \infty$$

## "UNMOTIVATED" EXAMPLE:

ORDER THE PRIME NUMBERS

$$\underline{P} = \{ 2 = p_1 < p_2 < p_3 < p_4 \cdots \}$$

ONE EXPECTS THAT THE ARITHMETIC OF THE INDEX AND THAT OF THE CORRESPONDING PRIME P ARE UNCORRELATED. SO IF WE CHOOSE $B_2$ TO BE EVERY OTHER PRIME

$$B_2 = \{ p_{2n} : n \geq 1 \}$$

(OR MORE GENERALLY $B_{a,b} = \{ p_{an+b} , n \geq 1 \}$ )

THEN THE CORRESPONDING CHEBOTAREV SHOULD HOLD

$$\sum_{\substack{p \in B_2 ; p \leq x \\ \mathrm{FROB}(p) = C}} 1 \sim \frac{|C|}{|H|} \sum_{p \in B_2, p \leq x} 1 \quad \text{AS } x \to \infty. \qquad \text{———} (*)$$

- WHILE (*) APPEARS TO BE WELL OUT OF THE REACH OF PRESENT TECHNIQUES, AT LEAST ONE CAN SHOW THAT THE (LHS) OF (*) GOES TO INFINITY WITH $x$.

- THIS FOLLOWS FROM

$$K/\mathbb{Q}\ ,\ H = \mathrm{GAL}(K/\mathbb{Q}),\ c \in H^{\#}$$

(CONJ CLASSES OF $H$)

$\mathcal{P}_c = \{p : \mathrm{FROB}_{K/\mathbb{Q}}(p) = C\}$ AND $m \geq 0$ ALL GIVEN, THERE ARE INFINITLY MANY $n$ SUCH THAT THE $m$-<u>CONSECUTIVE</u> PRIMES $p_{n+1}, p_{n+2}, \ldots, p_{n+m}$ ARE ALL IN $\mathcal{P}_c$.

- THIS CAN BE PROVED USING THE ADVANCES IN BOUNDED GAPS IN PRIMES [GOLDSTON-PINTZ-YILDRIM], [ZHANG] AND ESPECIALLY [MAYNARD], AND ADAPTING THE ARGUMENTS BANK-FREIBERG AND TURNAGE-BUTTERBAUCH ON ARITHMETIC PROGRESSIONS.

# AN ARITHMETIC GROUP ANALOGUE

LET $X = \Gamma \backslash G / K$ BE A FINITE VOLUME LOCALLY SYMMETRIC SPACE OF (REAL) RANK 1; (OF NEGATIVE CURVATURE).

- THE PRIMITIVE CLOSED GEODESICS $P$ ("PRIMES") ON $X$ CORRESPOND TO PRIMITIVE CONJUGACY CLASSES $P \longleftrightarrow \{\gamma\}_\Gamma$ IN $\Gamma$.

- $\ell(P)$ DENOTES THE LENGTH OF $P$.

- HOLONOMY OF PARALLEL TRANSPORT OF VECTORS AROUND $P$ GIVES A CONJUGACY CLASS $m(P)$ IN A CLOSED SUBGROUP $M$ OF $K$.

- LET $\rho : \Gamma \twoheadrightarrow H$ BE A DENSE MORPHISM INTO A COMPACT $H$.

THEN $\rho(P) = \rho(\{\gamma\}_\Gamma)$ IS WELL DEFINED IN $H^\#$ WHERE $H^\#$ AND $M^\#$ ARE THE CONJUGACY CLASSES OF $H$ AND $M$.

- SET $\mu_H^\#$ (RESP $\mu_M^\#$) TO BE THE PUSH FORWARDS OF THE CORRESPONDING NORMALIZED HAAR MEASURES.

THE HOLONOMY CHEBOTAREV THEOREM CONCERNS THE DISTRIBUTION OF

$$\underline{P} \longmapsto \left(\ell(\underline{P}), m(\underline{P}), \rho(\underline{P})\right) \in (0,\infty) \times M^{\#} \times H^{\#}$$

<u>THEOREM</u> (S - WAKAYAMA)

FOR $J \subset M^{\#}$, $A \subset H^{\#}$

$$\sum_{\substack{\ell(p) \le x \\ m(p) \in J \\ \rho(p) \in A}}{}' 1 \sim \mu_M^{\#}(J)\, \mu_H^{\#}(A) \sum_{\ell(p) \le x}{}' 1 \quad \text{AS } x \to \infty.$$

MOREOVER USING PROPERTIES OF THE UNITARY DUAL OF G ONE HAS UNIFORM EXPONENTIAL DECAY RATES (IN M ASPECT) WHICH ALLOWS FOR LOCAL VERSIONS (MOTIVATED).

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$X = \Gamma \backslash SL_2(\mathbb{C}) / SU(2)$ HYPERBOLIC 3-MANIFOLD

THEN $M = SO(2) = [0, 2\pi)$ ABELIAN $d\theta$, $= M^{\#}$

$N(\underline{P}) := e^{\ell(p)}$ ; THEN

$$\sum_{N(p) \le x} 1 \sim \frac{e^{2x}}{2x} \text{ AS } x \to \infty$$

(SELBERG PRIME GEOD. THEOREM)

THEOREM (DEVER–MILICEVIC 2023) LOCAL CHEBOTAREV

$$\sum_{\substack{0 \leq x-h \leq N(\gamma) \leq x \\ m(\gamma) \in J}} 1 \sim h|J| \cdot \frac{x}{\log x} \quad \text{AS } x \to \infty$$

UNIFORMLY FOR $h|J| \gg x^{1/3+\epsilon}$.

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FOR THE REST OF THE LECTURE WE FOCUS ON

$$\Gamma = SL_2(\mathbb{Z})$$

OR THE SUBGROUP OF INDEX 6

$$\Lambda = \langle A, B \rangle, \quad A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

WHICH IS FREE ON A AND B AND IS THE FUNDAMENTAL GROUP OF THE ONCE PUNCTURED TORUS $\mathbb{H}/\Lambda$.

LET $\rho : \Gamma \twoheadrightarrow H$ BE AN EPIMORPHISM WITH $H$ FINITE OR COMPACT.

- FOR $H$ FINITE ONE HAS THE LOCAL CHEBOTAREV THEOREM:

$$\sum_{\substack{0 \le x-h \le N(\mathfrak{p}) \le x \\ \rho(\mathfrak{p}) = C}} 1 \sim \frac{|C|}{|H|} \frac{h}{\log x} \qquad x \to \infty \quad \text{FOR } h \geq x^{3/4+\varepsilon}$$

THIS FOLLOWS FROM THE TRACE FORMULA COUNTING OF PRIMES WITH REMAINDER $O(x^{3/4+\varepsilon})$.

ONE EXPECTS THAT THIS WILL CONTINUE TO HOLD FOR $h \geq x^{1/2+\varepsilon}$ BUT FOR SMALLER $h$ IT CAN FAIL AT LEAST IF

$$\ker \rho : \Gamma \to H \quad \text{IS CONGRUENCE.}$$

TO SEE THIS

$\gamma \in SL_2(\mathbb{Z})$ PRIMITIVE $\qquad t(\{\gamma\}_\Gamma) = \mathrm{trace}(\gamma) \in \mathbb{Z}$

$$N(\{\gamma\}_\Gamma)^{1/2} + N(\{\gamma\}_\Gamma)^{-1/2} = t$$

SO WHEN $h = x^{1/2}$ BASICALLY WE HAVE ONE PACKET OF $\mathfrak{p}$'S WITH trace $= t$

THERE ARE APROXIMATELY $t$ SUCH CLOSED GEODESICS $\{\gamma\}_p$ AND WE MIGHT EXPECT THAT $\rho(\gamma)$ ~~THEY~~ EQUIDISTRIBUTE IN $H^\#$.

HOWEVER IF SAY

$$\rho : SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/p\mathbb{Z}) \cong SL_2(\mathbb{F}_p) = H$$

IS REDUCTION mod $p$

THEN THE RESULTING $\rho(\{\gamma\}_p)$ HAS

$$\text{TRACE}(\rho(\{\gamma\}_p)) = t \in \mathbb{F}_p$$

AS CALCULATED BY FROBENIUS THE CONJUGACY CLASSES IN $SL_2(\mathbb{F}_p)$ FOR $t \neq \pm 2$ ARE DETERMINED BY THE TRACE (AND FOR $t = \pm 2$ THERE ARE TWO OR THREE CLASSES)

$\Longrightarrow$ $\rho(\{\gamma\}_p)$ IS NOT EQUIDISTRIBUTED IN $H^\#$

• IF $\ker \rho$ IS NOT CONGRUENCE IT MAY WELL BE EQUIDISTRIBUTED.

IF $H$ IS COMPACT AND INFINITE THEN THE QUESTION OF AN EXPONENTIAL RATE OF CONVERGENCE MAY DEPEND ON $\rho$ AND IS SUBTLE.

FOR EXAMPLE IF $H = SU(2)$ AND

$$\rho(\Gamma) \subset H(\overline{\mathbb{Q}})$$

THEN THE SPECTRAL GAP THEOREM OF BOURGAIN-GAMBURD GIVES AN EXPONENTIAL RATE IN THE REMAINDER FOR THE CHEBOTAREV THEOREM.

NOTE: FOR $H = SU(2)$

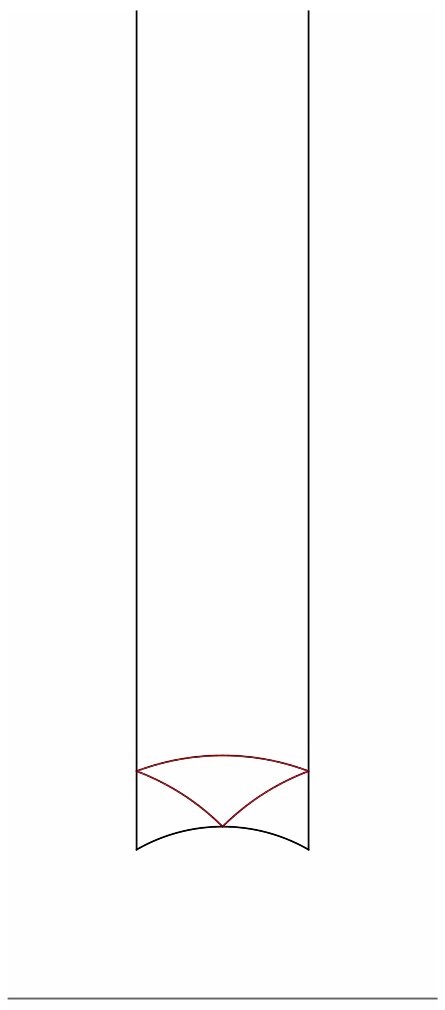
$$\mu_H^{\#} = \frac{1}{\pi} \sin^2\theta \, d\theta \quad \text{ON } [0,\pi].$$

THE FOLLOWING PICTURE OF PRIMITIVE CLOSED GEODESICS ON $\mathbb{H}/SL_2(\mathbb{Z})$ ARE BY CONSTANTIN KOLGER.

# Geodesics on the Modular Surface

We show closed geodesics on the modular surface, which correspond for varying $d$ to the matrix:

$$x_d = \frac{1}{d^{\frac{1}{4}}} \begin{pmatrix} \frac{d+\sqrt{d}}{2} & \frac{d-\sqrt{d}}{2} \\ 1 & 1 \end{pmatrix}$$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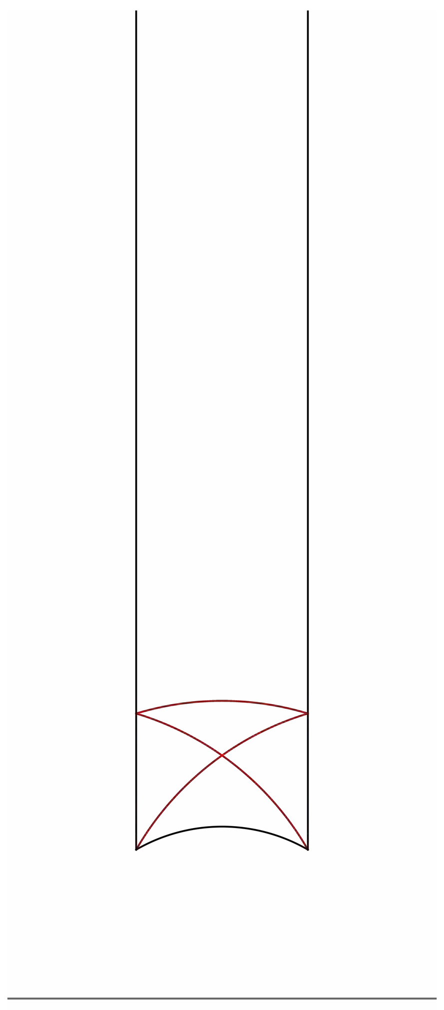
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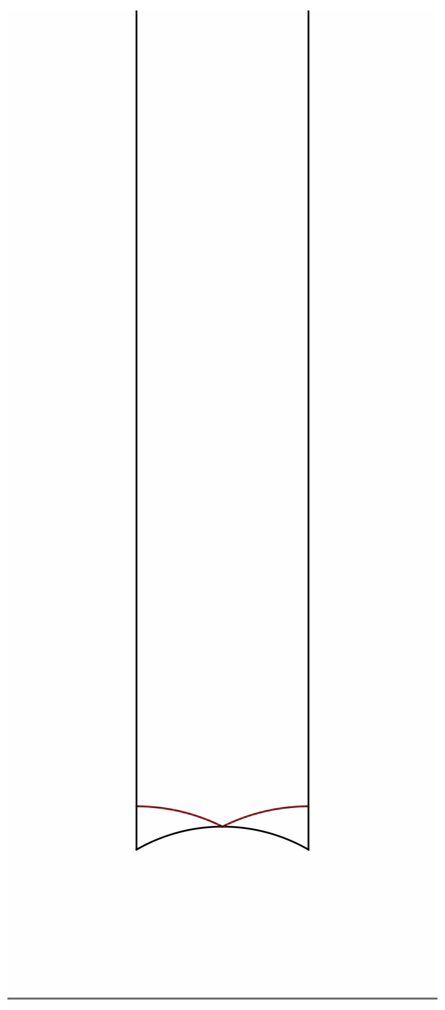
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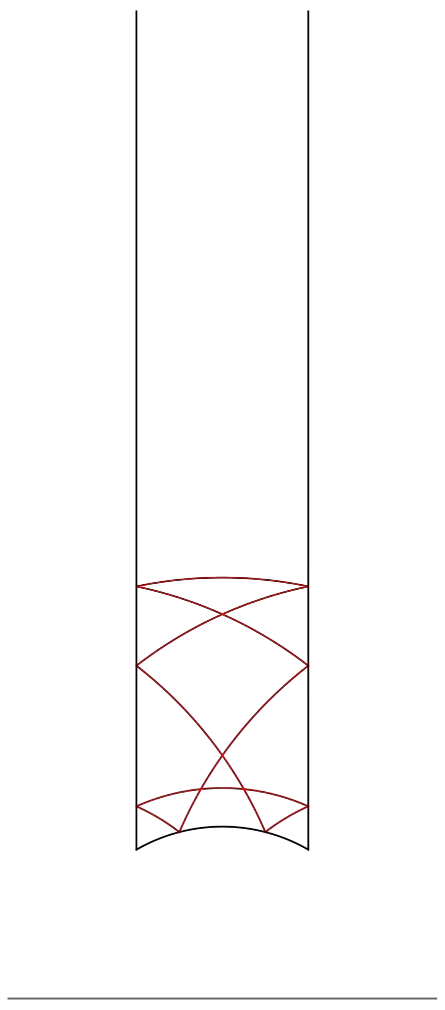
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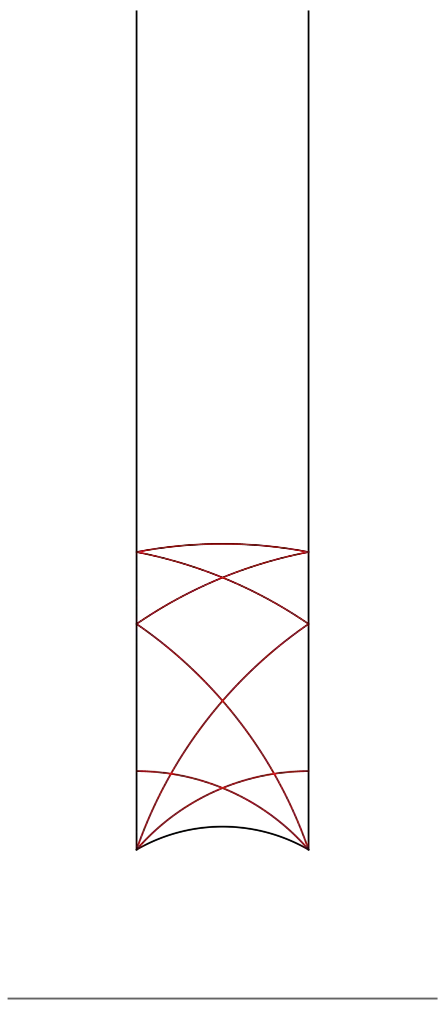
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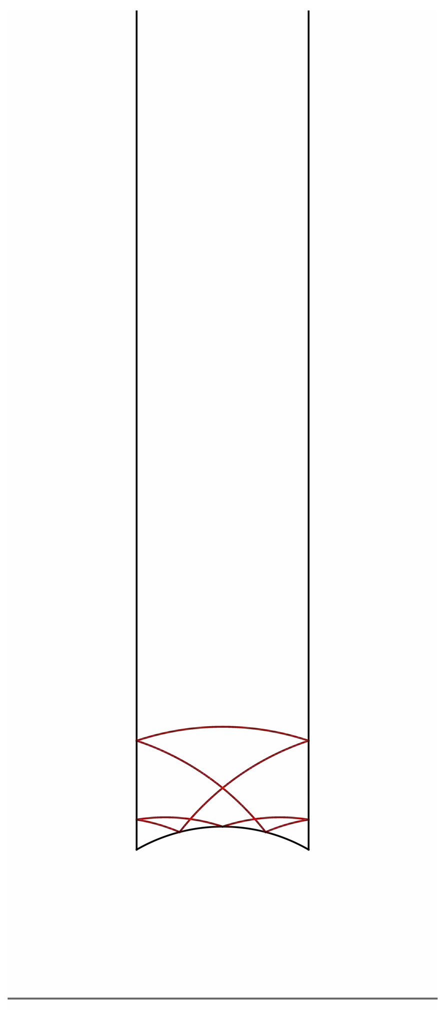
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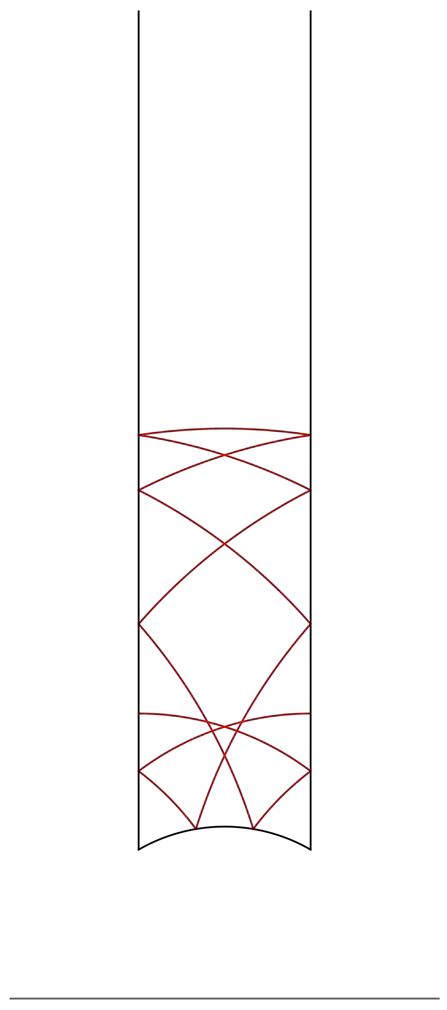
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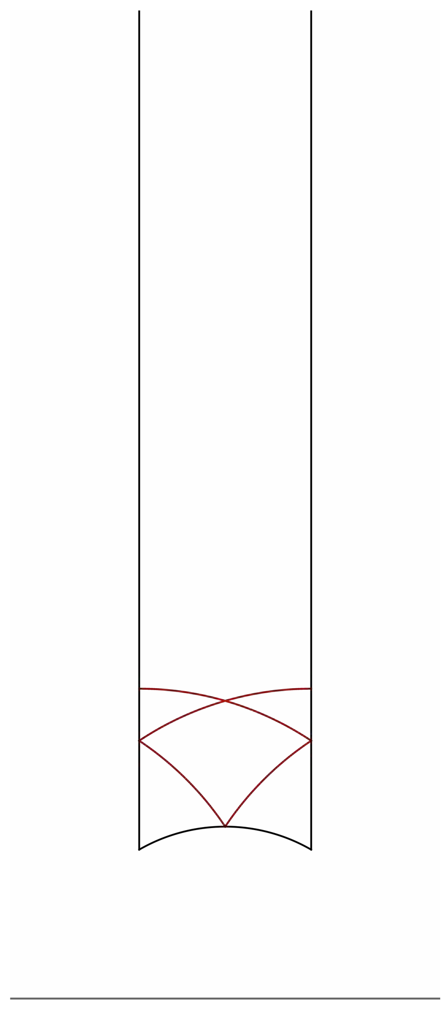
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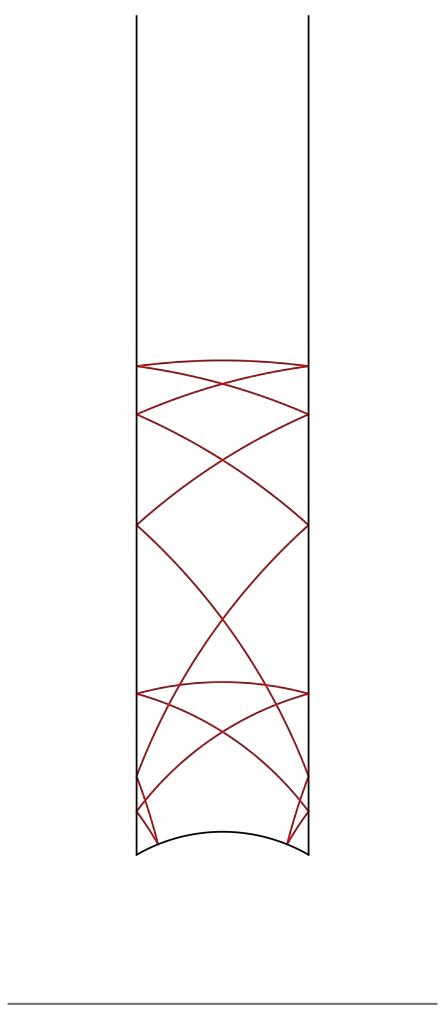
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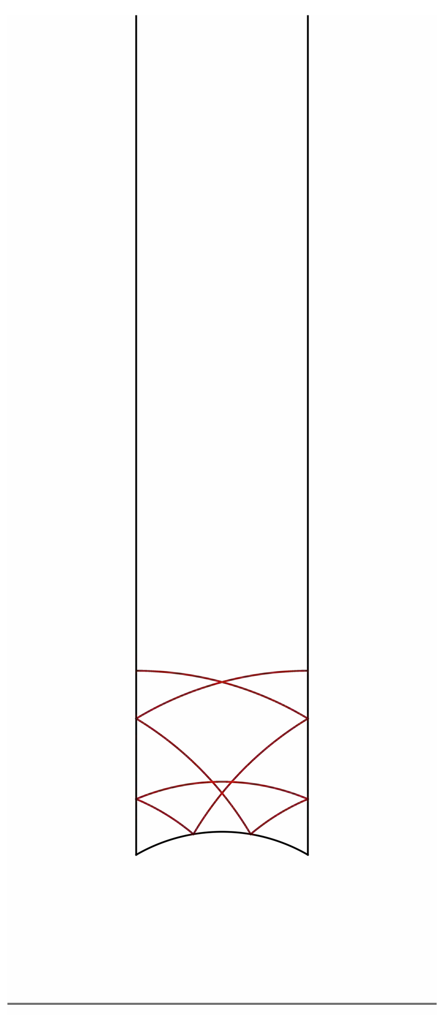
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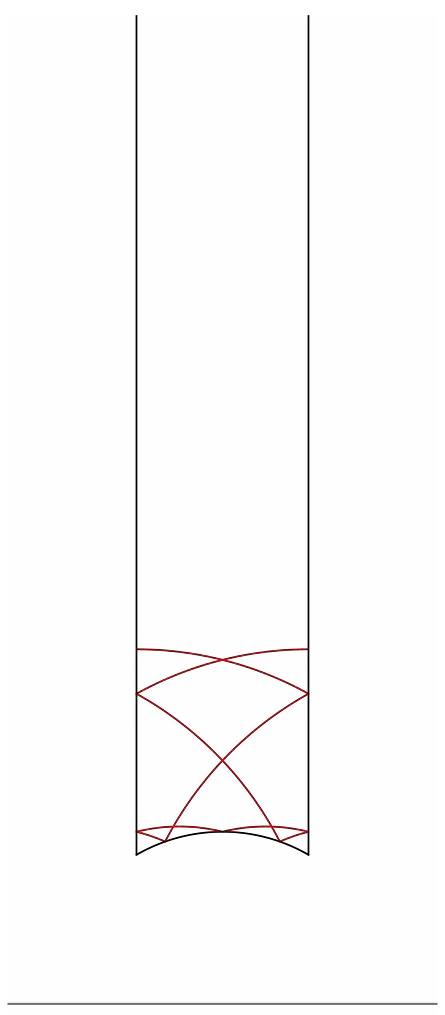
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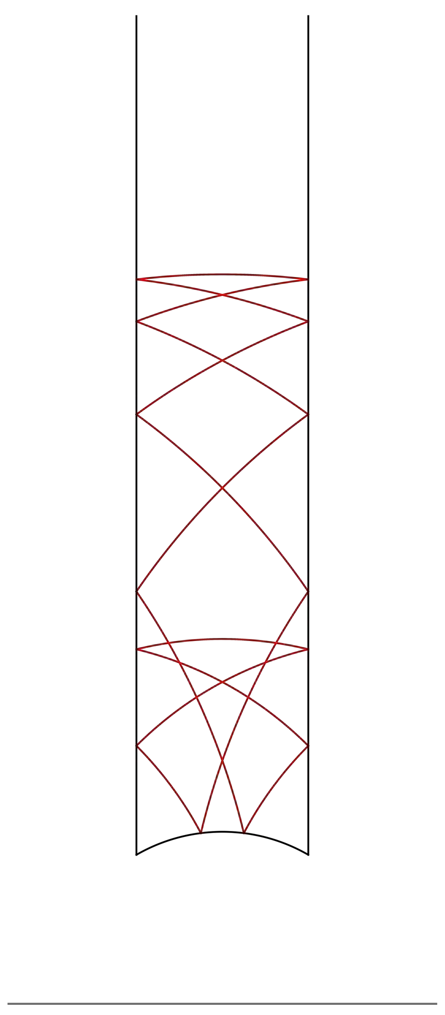
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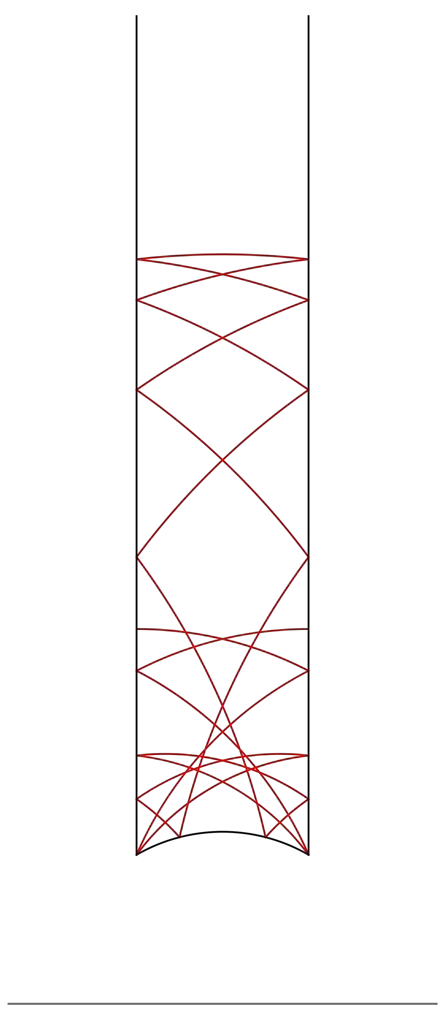
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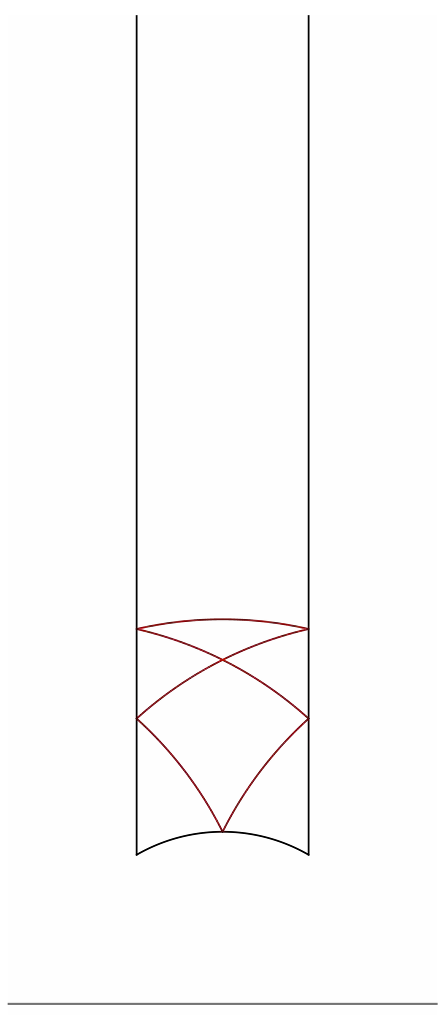
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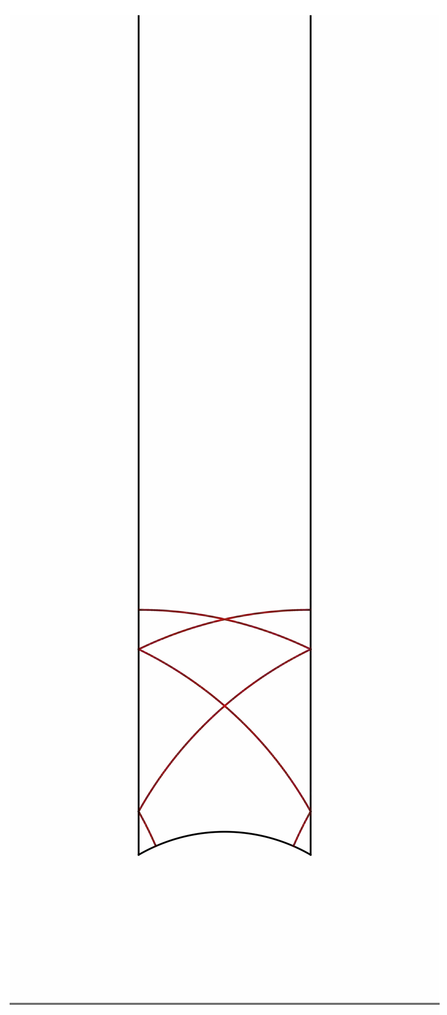
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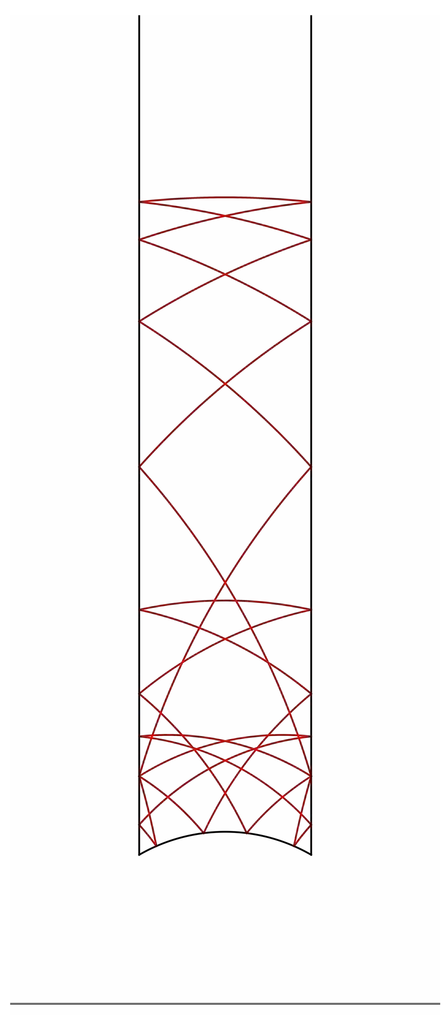
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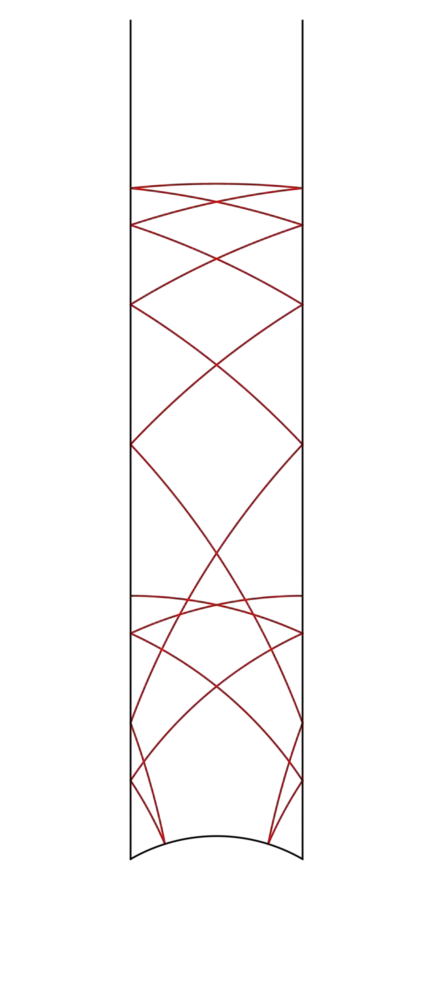
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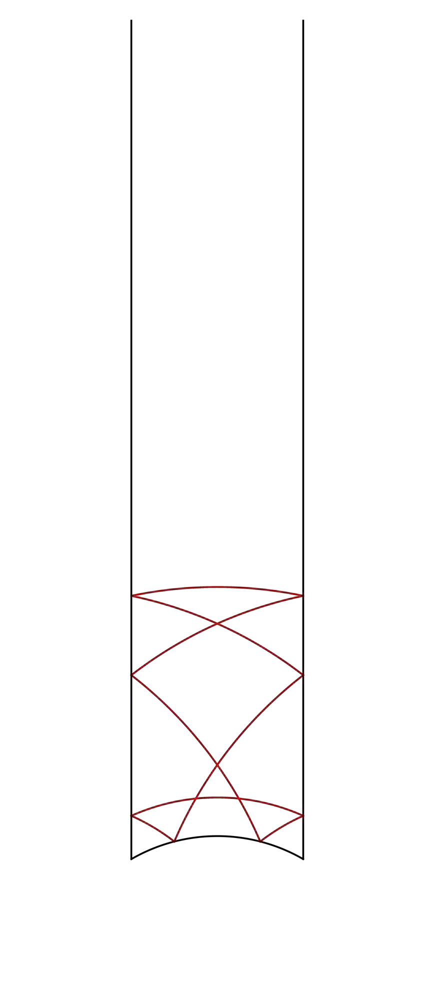
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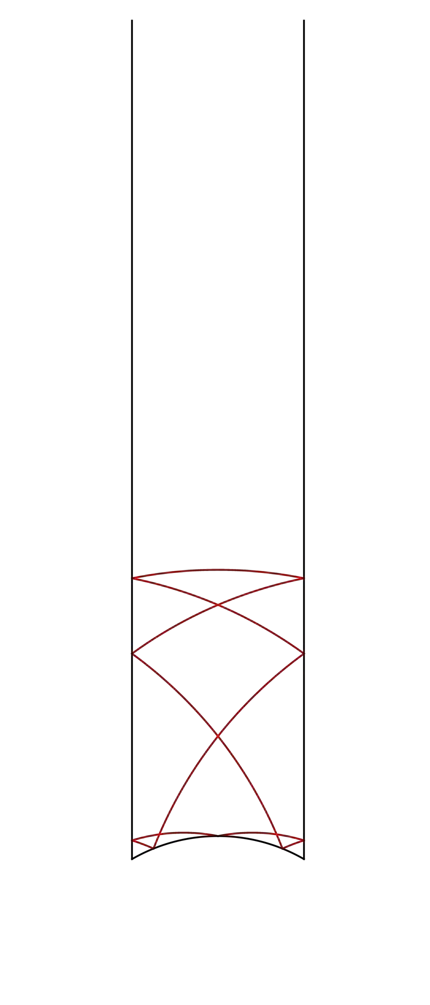
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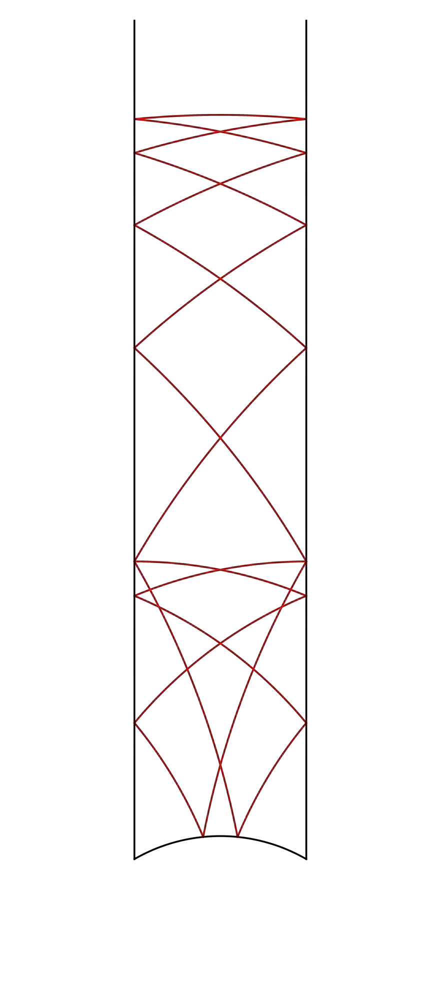
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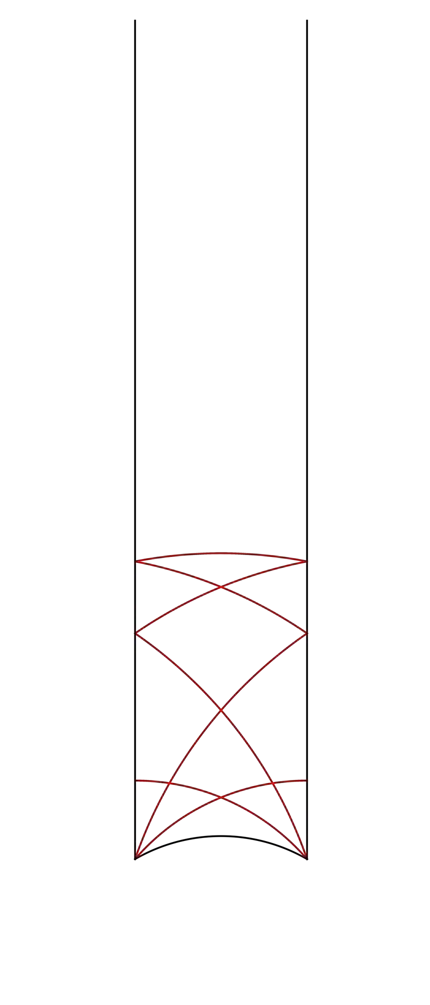
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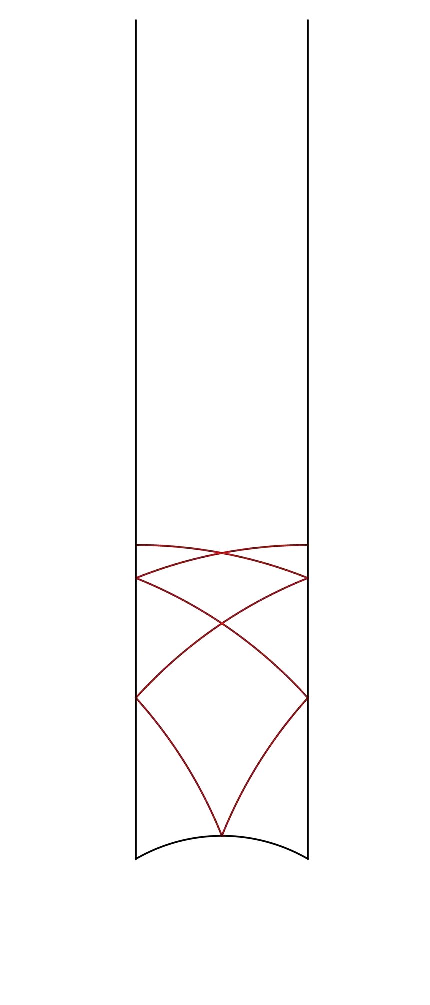
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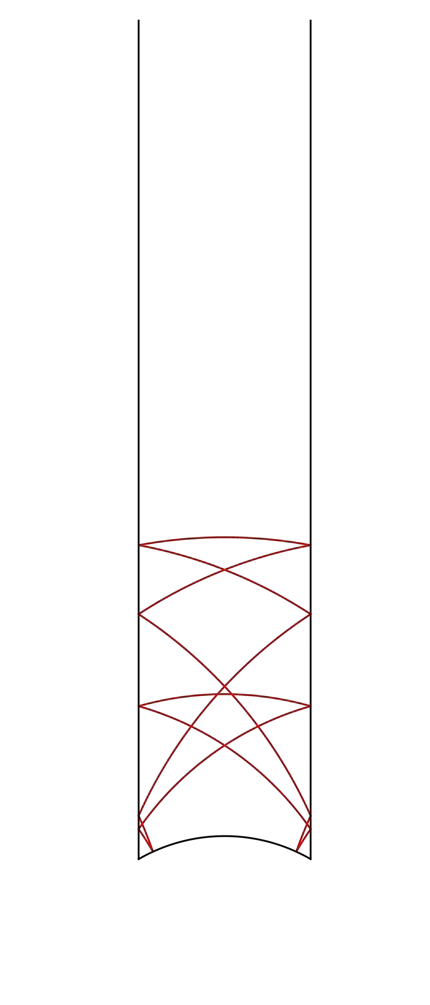
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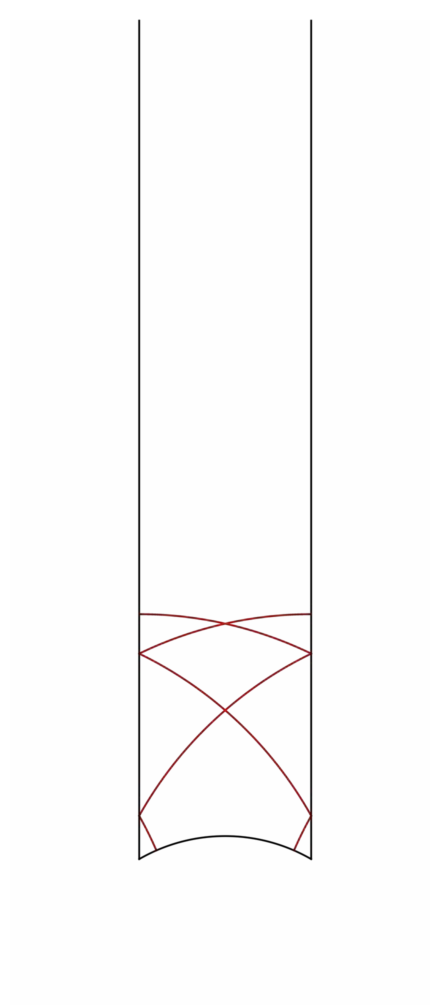
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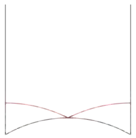
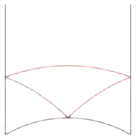
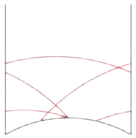


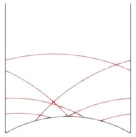
$M_1 = 1$
$y_{\max} = 1.1180$



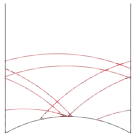
$M_2 = 2$
$y_{\max} = 1.4142$



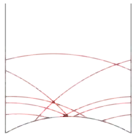
$M_3 = 5$
$y_{\max} = 1.4866$



$M_4 = 13$
$y_{\max} = 1.4980$



$M_5 = 29$
$y_{\max} = 1.4996$



$M_6 = 34$
$y_{\max} = 1.4997$

# A MOTIVATED VARIANT OF PRIME GEODESICS

THERE IS A VERY MOTIVATED VARIANT DUE TO GAUSS. EACH PRIME GEODESIC $p$ COMES WITH TWO INVARIANTS $p = \{\gamma\}_\Gamma$ HAS ITS TRACE $t(p)$ AND ITS DISCRIMINANT $d(p)$. THESE ARE RELATED BY PELL'S EQ$^{\underline{N}}$

$$t^2 - du^2 = 4 \qquad \text{(LEAST SOLUTION)}$$

WHOSE BEHAVIOR IS A NOTORIOUSLY DIFFICULT PROBLEM.

- DOES THE CHEBOTAREV THEOREM HOLD WHEN THE $p$'s ARE ORDERED BY $d(p)$? FOR $\rho \in Epi(\Gamma, H)$ IS

$$\sum_{\substack{d(p) \leq x \\ \rho_H(p) = c}} 1 \sim \frac{|C|}{|H|} \sum_{d(p) \leq x} 1 \quad ? \tag{xx}$$

A CONJECTURE OF HOOLEY GIVES A PROPOSED ASYMPTOTIC FOR THE RHS OF (xx) AND HAS BEEN CHECKED NUMERICALLY IN THE SENIOR PRINCETON THESES OF KWON AND PETROW.

## NON-MOTIVATED EXAMPLES

THERE ARE MANY SUCH 'THIN' SETS OF CLOSED GEODESICS.

(1) RECIPROCAL PRIME GEODESICS ARE ONES THAT PASS THROUGH $i$ OR EQUIVALENTLY $\gamma$'s THAT ARE CONJUGATE TO $\gamma^{-1}$ IN $SL_2(\mathbb{Z})$.

$$\sum_{\substack{N(p)\le x \\ p \text{ RECIPROCAL}}} 1 \sim \frac{3}{8}\sqrt{x}$$

AND CHEBOTAREV THEOREMS FOR CONGRUENCE KERNELS.

(2) MUCH MORE CHALLENGING ARE MARKOFF GEODESICS WHOSE MAXIMAL HEIGHT IN $\mathbb{H}/\Gamma$ IS AT MOST $3/2$. THERE ARE INFINITELY MANY OF THEM AND MARKOFF PARAMETRIZED THEM BY THE POSITIVE INTEGER SOLUTIONS TO:

$$M: \quad x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0$$

WE ORDER THE COORDINATES $x_1 \geq x_2 \geq x_3$

THE NUMBERS $x_1$ (WITH MULTIPICITY) ARE THE MARKOFF NUMBERS $M_n$;

$1, 2, 5, 13, 29, 34, 89, \ldots$

EACH SUCH SOLUTION GIVES A BINARY QUADRATIC FORM $Q$

$$\mathrm{disc}(Q) = 9M_n^2 - 4$$

AND THE CONJUGACY CLASS $\{\gamma\}_\Gamma$ CORRESPONDING THE GENERATOR OF $AUT_Q(\mathbb{Z})$ GIVES A PRIMITIVE CLASS WITH

$$t(\{\gamma\}) = 3M_n \, . \qquad \text{———} (\times\times\times)$$

THE $M_n$'s GROW RAPIDLY (GURWOOD, ZAGIER; McSHANE-RIVIN, MIRZAKHANI)

$$\sum_{t(Q) \le x} 1 \sim c (\log x)^2$$

IN PARTICULAR FOR

$$\rho : \Gamma \twoheadrightarrow SL_2(\mathbb{Z}/p\mathbb{Z}) = H_p, \quad \text{CONGRUENCE REDUCTION}$$

FROM $(\times\times\times)$ $\rho(Q)$ IS DETERMINED BY $M_n \bmod p$

AT LEAST IF

$$M_n \not\equiv \pm 2 \pmod p$$

IF $p \equiv 3(4)$ THEN FROM THE MARKOFF EQUATION IT FOLLOWS THAT $(p \neq 3)$

$$M_n \not\equiv 0, \pm \tfrac{2}{3} \pmod p$$

AS WAS OBSERVED BY FROBENIUS.

SO THE IMAGE $\rho(Q_n)$ IS $C_{3M_n}$ IN $H_p^{\#}$ WHERE $C_t$ IS THE CLASS CORRESPONDING TO TRACE $t$ IN $\mathbb{F}_p$.

THE CHEBOTAREV THEOREM FOR $\rho : \Gamma \twoheadrightarrow H_p$ FOR $p \equiv 3(4)$ LARGE AND MARKOFF'S GEODESICS THEN FOLLOWS FROM:

$$\sum_{\substack{M_n \le x \\ \rho(Q_n) = C_{t/3}}} 1 \sim \frac{\#\{t^2 + x_2^2 + x_3^2 = 3t x_2 x_3 \text{ IN } \mathbb{F}_p\}}{\#\{x_1^2 + x_2^2 + x_3^2 = 3x_1 x_2 x_3 \text{ IN } \mathbb{F}_p\}} \sum_{M_n \le x} 1 \qquad \text{(v)}.$$

AND IS 0 IF $t = 0, \pm\frac{2}{3} (\bmod p)$

THUS THE WEIGHTS IN THE DISTRIBUTION OF MARKOFF CHEBOTAREV ON $H_p^{\#}$ ARE NOT $\mu_{H_p}^{\#}$ AND VARIOUS CLASSES ARE OMITTED.

THERE IS A SIMILAR RESULT FOR $p \equiv 1(4)$ AND LARGE.

# KEY INGREDIENTS IN THE PROOF OF (V)

THAT FOR $p$ LARGE ( $> 10^{393}$ ) THE VIETA (OR MARKOFF) GROUP MOD OF POLYNOMIAL MORPHISMS OF $\mathbb{A}^3$ GENERATED BY THE INVOLUTIONS

$R_1, R_2, R_3$ $\quad R_1(x_1,x_2,x_3) = (3x_2x_3 - x_1, x_2, x_3)$

$R_2, R_3$ SIMILARLY

ACTS TRANSITIVELY ON $M^*(\mathbb{Z}/p\mathbb{Z})$

(IN FACT IT ACTS MINIMALLY AND UNIQUELY ERGODICALLY ON THE COMPACT SPACE $M^*(\mathbb{Z}_p)$ )

- BOURGAIN-GAMBURD-S (2016, 2025 JAMS) SHOW THAT THERE IS A GIANT COMPONENT TO THE CONNECTIVITY GRAPH AND THAT ANY COMPONENT OF SIZE $p^\varepsilon$ IS CONNECTED TO THE GIANT COMPONENT.

- W. CHEN (2024 ANNALS) THROUGH HIS STUDY OF CONNECTIVITY PROPERTIES OF MODULI SPACES OF ELLIPTIC CURVES WITH H-STRUCTURES SHOWS THAT EVERY COMPONENT OF THE ACTION OF MOD ON $M^*(\mathbb{Z}/p\mathbb{Z})$ HAS SIZE DIVISIBLE BY $p$.

- D. MARTIN (2025 INVENT.) HAS GIVEN AN ELEMENTARY PROOF OF THIS DIVISIBILITY.

# ANALYTIC MACHINERY:

McSHANE-RIVIN : FOR COUNTING SIMPLE CLOSED GEODESICS ON HYPERBOLIC ONCE PUNCTURED TORI, $\Sigma_{1,1}$.

M. MIRZAKHANI : COUNTING ORBITS OF THE MAPPING CLASS GROUP MOD.

$$\pi_1(\Sigma_{1,1}) = F_2 = \langle A, B \rangle \quad \text{FREE ON } A, B.$$

H A FINITE GROUP

$\mathrm{Epi}(F_2, H)$ : THE EPIMORPHISMS $\rho$ OF $F_2$ TO H. THEY ARE PARAMETRIZED BY PAIRS $(C, D)$ IN $H \times H$ WHICH GENERATE H, $\rho(A) = C$ $\rho(B) = D$

$$Y = \mathrm{Epi}(F_2, H)/H$$ THE CHARACTER CLASSES

$(C, D) \sim (C', D')$ IF $(hCh^{-1}, hDh^{-1}) = (C', D')$ FOR SOME $h \in H$.

- THE MAPPING CLASS GROUP $\mathrm{MOD} \cong \mathrm{OUT}(F_2) \cong PGL_2(\mathbb{Z})$ ACTS AS PERMUTATIONS OF $Y$ BY NIELSEN MOVES.

- IT PRESERVES THE HIGMAN INVARIANT $t_\rho \in H^\#$ OF $\rho$ IN $Y$, NAMELY THE CONJUGACY CLASS OF $\rho(A)\rho(B)\rho(A^{-1})\rho(B^{-1})$ IN $H^\#$.

- LET $Y_1 \sqcup Y_2 \cdots \sqcup Y_\nu = Y$ BE THE DISTINCT ORBITS OF MOD ON $Y$. EACH $Y_j$ HAS COMMON INVARIANT $t_j$ AND MOD ACTS TRANSITIVELY ON THESE "$t$-SYSTEMS" (OF H AND B NEUMAN).

- THE PROJECTION

$$\pi : Y \longrightarrow H^\# \quad \text{ON THE FIRST COORD}$$

$$\widetilde{(C, D)} \rightarrow \{C\}_H \quad \text{ie}$$

IS WELL DEFINED AND IS $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ INVARIANT WITH MOD $= PGL_2(\mathbb{Z})$; CORRESPONDING DEHN TWIST.

FOR EACH $j = 1, \dots, \nu$

$$\frac{\left|\pi^{-1}(\{h\}_H) \cap Y_j\right|}{|Y_j|} \quad \text{IS A PROBABILITY MEASURE ON } H^\#$$

## CHEBOTAREV FOR SIMPLE CLOSED GEODESICS:

LET $\rho \in \mathrm{Epi}(\pi_1(\Sigma_{1,1}), H)$ WHOSE IMAGE LIES IN $Y_j$ IN $Y$, THEN FOR $C \in H^\#$

$$\sum_{\substack{P \text{ simple} \\ \ell(P) \leq x \\ \rho(P) = C}} 1 \sim \frac{|\pi^{-1}(C) \cap Y_j|}{|Y_j|} \sum_{\substack{P \text{ simple} \\ \ell(P) \leq x}} 1$$

AS $x \to \infty$.

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FOR THE PROOF WE USE THE DESCRIPTION OF THE SIMPLE CLOSED GEODESICS BY McSHANE-RIVIN AS PRIMITIVE INTEGRAL LATTICE POINTS IN $\mathbb{R}^2$ W.R.T. THE THURSTON NORM AND THE JOINT ACTION OF $PGL_2(\mathbb{Z}) \cong \mathrm{MOD}$ ON THE PLANE (LINEAR) AND ON THE CHARACTER VARIETY NON-LINEAR.

- FOR $\rho : \Gamma \twoheadrightarrow H$ CONGRUENCE ; B-G-S GIVES THAT ESSENTIALLY MOD ACTS TRANSITIVELY ON EACH HIGMAN COMPONENT.

## H COMPACT:

THE SET UP IS SIMILAR BUT THE DETERMINATION OF LIMIT MEASURE MORE PROBLEMATIC.

$$H = SU(2)$$

$\rho \in Epi(F_2, SU(2))$ ITS HIGMAN INVARIANT

$$t_\rho = \mathrm{TRACE}\left(\rho(A)\rho(B)\rho(A^{-1})\rho(B^{-1})\right) \in [-2,2].$$

THE CORRESPONDING (REAL) CHARACTER VARIETY IS

$$Y_{k}: \quad x^2 + y^2 + z^2 + xyz := k = t_\rho + 2 \in [0,4].$$

$$|x|, |y|, |z| \leq 2.$$

THE MEASURE $\nu_k$ ON $Y_k$

$$\nu_k = \frac{dx\,dy}{2z + xy}$$ IS MOD INVARIANT AND ERGODIC (GOLDMAN)

$\pi^{\#}(\nu_k)$ IS SUPPORTED ON

$$|\cos\theta| \leq \frac{\sqrt{k}}{2}, \quad 0 \leq \theta \leq \pi.$$

AND IS A MULTIPLE OF $d\theta$ ON THIS INTERVAL.

PROGRESS ON THE CLASSIFICATION OF MOD INVARIANT MEASURES ON $Y_k$

(GOLDMAN-PREVITE-XIA, BROWN-ESKIN-FILIP-RODRIGUEZ HERTZ, CANTAT-DU PONT-MARIN BALLON ..) SUGGEST VERY STRONGLY THAT THE CHEBOTAREV THEOREM SHOULD BE:

FOR $J \subset [0, 2\pi)$

$$\sum_{\substack{P \text{ SIMPLE} \\ \ell(P) \le x \\ \rho(P) \in J}} 1 \sim \nu_k(J) \sum_{\substack{\ell(P) \le x \\ P \text{ SIMPLE}}} 1$$

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