Questions. In $\text{SL}(2, \mathbb{R})$

in general questions are best asked in terms of structure groups $G$ and discrete subgroup $\Gamma$, assume $\Gamma \backslash G$ finite volume.

max. compact subgroup. A max. unimodular subgroup.

$G = KAK$, \quad \Gamma G = K\Gamma A K$.

chap. prob. in $\text{SL}(2, \mathbb{R})$, $x = e$, $\begin{pmatrix} \phi & \theta \\ \theta & \phi \end{pmatrix}$.

how are angles distributed.

$: f \left( \begin{pmatrix} \frac{a}{c} \\ \frac{b}{c} \end{pmatrix} \right) = \left( \begin{pmatrix} \phi & \theta \\ \theta & \phi \end{pmatrix} \right) \left( \begin{pmatrix} 0 & -\frac{c}{b} \\ \frac{c}{b} & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & \frac{a}{c} \\ \frac{a}{c} & 1 \end{pmatrix} \right)$

$|c| \leq x$ how many are $\frac{a}{c}$ and $\frac{b}{c}$ distincts modulo 1.

We fix several analyses; best carried out directly on group, using functions of two arguments $q_1$ and $q_2$ defined by series

$$
\sum_{(q_1, q_2)} f(q_1, q_2)
$$

where $\xi_1$ fixed instead here.

Dedicate with two points $\xi$, $\xi'$ in $H$, and certain Jacobi transfer that represent automorph forms (not necessarily analytic) in both variables.
quickly recall

\[ ds^2 = \frac{dx^2}{y^2} \] \quad \text{in operator} \quad y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)

invariant of two points.

\[ \mathcal{U}(z, \xi) = \frac{|z - \xi|^2}{4y^2} = \frac{e^{p/2} - e^{-p/2}}{y} \]

where \( p = d(z, \xi) \) invariant distance.

Area of circle \( A(z, \xi) = \frac{\pi}{4} (\mathcal{U}(z, \xi) - 1) \)

\[ q^2 = \frac{a^2 + d}{c^2 + d} \] \quad \text{define } \Sigma_q(z) = e^{|\arg(cz+d)} \]

\[ \frac{y}{q^2} = \frac{y}{(c^2 + d)^2} \quad \text{here} \]

scalar equation

\[ \frac{d^k}{dq^{2k}} = (c^2 + d)^{k+1} \frac{d^k}{dz^{2k}} (c^2 + d)^{-k-1} \]

We say \( f(z) \) is a form \( f \) of \( \Gamma \) of \( \text{index } (p, q) \)

\[ f(z) = \Sigma_q(z)^k f(z) \quad \text{in } \Gamma \]

for \( k \leq h \) put

\[ D_{\nu, k} = (z - \xi)^{h-k} \frac{d^{h-k}}{dz^{h-k}} y^{h-1} \]

for \( k \geq h \)

\[ D_{\nu, h} = (-z - \xi)^{h-k} \frac{d^{h-k}}{dz^{h-k}} y^{h-1} \]
(P, k) form

so that \[ D_{m+k} = \overline{D}^{-h_1-k} \]

then

\[ D_{m+k} (q^2) = \overline{D}^{-h_1} D_{m+k} (q^2) \overline{D}^{-h_1-k} \]

then if \( f \) is form of index \( k \) in \( P \)
\[ D_{m+k} f \] is form of index \( h \).

The operator

\[ \Delta_k = \frac{q^2}{8} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial q^2} \right) - z i k q \frac{\partial}{\partial q} \]

carries a form of index \( k \) to another. Can show that if \( f \) is
eigenfunction of \( \Delta_k \) then \( D_{m+k} f \)
is eigenfunction of \( \Delta_k \) with same
eigenvalue (but \( D_{m+k} f \) may of course
identically 0 for some \( f \).

Look at spectrum of eigenfunctions
of \( \Delta_k \) which are forms of \((P, k)\) forms
and \( f \) for which \[ \int f^2 \frac{dx dq}{q^2} < \infty \]
with eigenvalue in form \( \frac{1}{4} + r^2 \) (within \( r \))

The analysis of the spectrum for general \( k \) is essentially the same as for \( k = 0 \), only one eigenfunction is annihilated when passing from one level to another (apart from \( k = 4 \)).

In \( k < 0 \) and \( k > 0 \) we have a finite set of eigenvalues of form \( (\frac{2}{4} - k + \epsilon)(1 + \epsilon - k) \) where \( 0 \leq \epsilon \leq k \), multiplicity of each equals sum of analytic \( 2 \) of weight \( \frac{1}{4} \) in \( P \), similarly for \( k < 0 \).

For \( k = 0 \)

Hankel functions appearing here are orthonormally given

\[
\sum f(x, y, z) = \sum h(x) \, \mu_{n+1}^+(z) \mu_{n+1}^-(z)
\]

with

\[
h(x) = \int_0^x \sin(t) \, f(t, z) \, \frac{dt}{y^2}
\]

For the other levels similar theorems occur. we will refer to \( \mu_{n+1}^+ (z) \) as orthonormal system of level \( k \).
the analytic center. constant \( \frac{1}{\sqrt{\lambda(\theta)}} \) present only for \( \lambda = 0 \) level \( 0 \).

\[ h > k \quad \text{and} \quad D_{h,k} = \mu_n(\zeta) \text{ and } D_{h,k} = \mu_n(\zeta) \]

with \( h < k \).

\[ \lambda^2 = \frac{\prod (k + \frac{1}{2} + iv) \prod (k + \frac{1}{2} - iv)}{\prod (h + \frac{1}{2} + iv) \prod (h + \frac{1}{2} - iv)} \]

In cases where the eigenfunction disappears in level \( \lambda \), \( \lambda = 0 \) by this formula.

Experience shows that it is most convenient to work with dimensionless series.

The general analytic vehicle we shall choose is given by the

\[ \sum \left( \theta(\xi, \eta) \right) e^{-i \theta(\xi, \eta)} \]

which converges for \( R > 1 \) and

\( \lambda \) of level \( k \) in \( \xi \) and \( -h \) in \( \eta \).

However, in general (for \( h \neq k \)) this function is singular at \( \zeta = 0 \) and so not very usable. Therefore we choose instead
for \( \ell \geq k \)

\[ K_{\ell, h}(z, \xi; \Delta) = \sum_{\gamma \in \Gamma} \left( \frac{n(x, \zeta)}{n(x, \xi)} \right)^{-\Delta} \frac{1}{\gamma z - \frac{\xi}{\gamma}} \]

\[ = \sum_{\gamma \in \Gamma} \frac{\zeta^{2 - \frac{\xi}{\gamma}} + i \kappa \arg \frac{\xi - \gamma^{-1} \xi}{\zeta - \gamma^{-1} \xi}}{\gamma z - \frac{\xi}{\gamma}} \]

which has no finite singularity

and for \( \ell < k \), we define

\[ K_{\ell, h}(z, \xi; \Delta) = \sum_{\gamma \in \Gamma} \frac{1}{\gamma z - \frac{\xi}{\gamma}} \]

For compact \( \Gamma \subset H \) can show

for \( \ell \geq k \)

\[ K_{\ell, h}(z, \xi; \Delta) = \]

\[ = \sum_{\gamma \in \Gamma} \frac{\zeta^{2 - \frac{\xi}{\gamma}}}{\gamma (z - \frac{\xi}{\gamma})} \frac{\zeta^{2 - \frac{\xi}{\gamma}}}{\gamma (z - \frac{\xi}{\gamma})} \frac{1}{\gamma z - \frac{\xi}{\gamma}} \]

\[ \sum_{\gamma \in \Gamma} \frac{\zeta^{2 - \frac{\xi}{\gamma}}}{\gamma (z - \frac{\xi}{\gamma})} \frac{\zeta^{2 - \frac{\xi}{\gamma}}}{\gamma (z - \frac{\xi}{\gamma})} \frac{1}{\gamma z - \frac{\xi}{\gamma}} \]

For constant eigenfunction \( \frac{1}{\sqrt{\Delta(z)}} \) occurs only for

\[ k = h = 0 \]
When it gives a term \( \frac{4\pi^2}{\sigma(D)} \frac{1}{\varepsilon - 1} \)

which is only pole in region \( \sigma > 1 \)

Results: (assume first our eigenvalues between \( 0 < \varepsilon < \frac{1}{2} \))

\[
\sum \frac{h \log \frac{\beta - \varepsilon}{\beta - \varepsilon^*} + i k \log \frac{\beta - \varepsilon^*}{\beta - \varepsilon^*} \frac{2 - \varepsilon^*}{2 - \varepsilon}}{\alpha(2, q)} \leq \chi
\]

\[
= \frac{4\pi}{\alpha(D)} \chi + O\left( \chi^{\frac{2}{3}} \right)
\]

for \( h = k = 0 \)

\[
= O \left( \chi^{\frac{2}{3}} \right)
\]

\[
= O \left( \chi^{\frac{2}{3}} \right)
\]

for \( h = k \neq 0 \)

\[
= O \left( \chi^{\frac{2}{3}} \right)
\]

for \( h \neq k \).

If eigenvalues between 0 and \( \frac{1}{2} \)

we get some remainder terms

but have some leading term \( \frac{1}{\beta^2} \cdot \log(R) \)

\[
\sum \left| \mu_n^{(2)} \right|^2 \leq \frac{\rho(2\varepsilon - 1)}{\beta(1 - \varepsilon)(1 - \varepsilon)} \mu_n^{(2)} \mu_n^{(2)} \leq R \frac{\rho(2\varepsilon - 1)}{\beta(1 - \varepsilon)(1 - \varepsilon)} \mu_n^{(2)} \mu_n^{(2)}
\]

\( n \leq R\)
auto meromorphic for an $n > 0$ square integer $n$ can be expanded in eigenfunctions, 

there exist $\varepsilon$-Eisenstein series. 

find relation 

then look at Fourier transform expansion of $L^m (2)$ in terms of $e^{i n x}$. coeff of $e^{i n x}$ in involves series. 

$$\sum_{c \neq 0} \frac{e^{i n x} \frac{a+c}{c}}{|c|^{2 \varepsilon}} = \sum_{c \neq 0} c^{\frac{1}{2}} x^{2 \varepsilon} + o(x^2)$$ 

meromorphic for $\sigma > \frac{1}{2}$ has poles at most in points $\sigma = \frac{1}{2} + i n$, but for $m, n \neq 0, 0$ we have poles at $\sigma = 1$. In $m = n = 0$ we have poles at $\sigma = \frac{1}{2}$.

Eisenstein series. 

$$\sum_{|c| < \lambda} (x-\lambda c) e^{i x \frac{a+c}{c}} = \sum_{1 > p > \frac{1}{2}} c^{i \frac{1}{2}} x^{2 \varepsilon} + o(x^2)$$ 

thus if $\text{ord } p > \frac{1}{2}$, 

$$\sum_{1 > \lambda c < x} (x-\lambda c) e^{i x \frac{a+c}{c}} = o(x^2)$$

in most cases giving $m = 1$. This is 

$$\sum_{1 > \lambda c < x} (x-\lambda c) \mu(\lambda) = o(x^2).$$

which simplifies $\sum_{n \leq x} \mu(n) = O(x)$.
5.7. Problem 11. Evaluate \( U(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \) for real \( x \in \mathbb{R} \) using \( e^x \).

Analytic vehicle: \( e^x \).

Proven in uniform plan.

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x}
\]

\[
\frac{d^2}{dx^2} e^{-x} = -e^{-x}
\]

\[
\int e^{-x} dx = -e^{-x} + C
\]

and:

\[
YAC = 2B = D
\]

Final answer:

1. \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x} \)

2. \( e^{-x} = \frac{d^2}{dx^2} \left( \frac{1}{2} \right) \)

3. \( -e^{-x} = \int \left( \frac{d^2}{dx^2} \right) dx \)

4. \( -e^{-x} + C = \int \left( \frac{d^2}{dx^2} \right) dx \)

5. \( e^{-x} = C \)

6. \( e^{-x} = \frac{d^2}{dx^2} \left( \frac{1}{2} \right) \)

7. \( e^{-x} = \) constant for \( x \) in \( \mathbb{R} \)