Euler:
(1) \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \),

(2) \( B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 (1-t)^{y-1} t^{x-1} dt \),

generalizations:

Dirichlet:
(3) \( \int_{t_1}^{x_1} \cdots \int_{t_n}^{x_n} (1-t_1 \cdots t_n) dt_1 \cdots dt_n \),
\( 0 < t_i \)
\( \sum t_i < 1 \)

or, more general:
(4) \( \int_{t_1}^{x_1} \cdots \int_{t_n}^{x_n} (1-t_1 \cdots t_n)^{y_1-1} \cdots (1-t_1 \cdots t_n)^{y_n-1} dt_1 \cdots dt_n \),
\( 0 < t_i \)
\( \sum t_i < 1 \)

\( p \) prime; \( \varepsilon = e^{2\pi i/p} \), \( \chi \neq \chi_0 \), Jacoby ca. 1837

(1') \( \sum_{h \in \mathbb{Z}/p} \chi(h) \varepsilon^{\frac{h}{p}} \); \( |\Gamma\chi|^2 = p \),
(2') \( \sum_{h \in \mathbb{Z}/p} \chi_1(h) \chi_2(-h) = \frac{\overline{\chi_1} \overline{\chi_2}}{\chi_1 \chi_2} \),

if \( \chi_1 \neq \chi_0 \), \( \chi_2 \neq \chi_0 \), \( \chi_1 \chi_2 \neq \chi_0 \).

Similar analogies for (3) and (4).
\[ \Delta(t) = \prod_{i<j}^2 (t_j - t_i), \]

\[ \frac{1}{\prod_{i=1}^n} \int_0^\infty \frac{\Gamma(x+y) \Gamma(x+(v-1)y) \Gamma(y+(v-1)x)}{\Gamma(x+y+(n+v-2)y)} \]

valid for real parts of \( 1+vy \), \( x+y-vz \) and \( y+v-xz \) are positive for \( 1 \leq v \leq n \).

\[ \frac{\Gamma(x+y) \Gamma(x+(v-1)y) \Gamma(y+(v-1)x)}{\Gamma(x+y+(n+v-2)y)} \]

valid for real parts of \( 1+vy \), \( x+y-1-(n+v-2)y \) are positive for \( 1 \leq v \leq m \).

\[ \sum_{i<j}^2 (t_j - t_i) \]

valid for \( \gamma, 1+vy \) and \( x+y-1 \) positive when \( 1 \leq v \leq n \).
Limiting cases of (5)

\[ \int \cdots \int (t_1 \cdots t_n)^{x-1} e^{-t_1 - \cdots - t_n} |\Delta(t)|^{2z} dt_1 \cdots dt_n \]

\[ = \prod_{\nu=1}^{n} \frac{\Gamma((\nu+2)z)}{\Gamma(\nu+2) \Gamma(z+\nu)} \]

(9)

\[ \int \cdots \int e^{-t_1 - \cdots - t_n} |\Delta(t)|^{2z} dt_1 \cdots dt_n = \]

\[ = (2\pi)^{n} \prod_{\nu=1}^{n} \frac{\Gamma((\nu+2)z)}{\Gamma(\nu+2) \Gamma(z+\nu)} \]

Both valid when \( \Gamma \)-functions in numerator have arguments with positive real parts.

Also versions for unit-circle (or interval \( -\pi, \pi \)) involving \( |\Delta(e^{i\theta})|^{2z} \), e.g.,

\[ \prod_{\nu=1}^{n} \int_{-\pi}^{\pi} \left( \frac{a}{2} \right)^{x+y-2} (1+e^{i\theta})^y |\Delta(e^{i\theta})|^{2z} d\theta_1 \cdots d\theta_n \]

\[ = (2\pi)^{n} \prod_{\nu=1}^{n} \frac{\Gamma((\nu+2)z)}{\Gamma(\nu+2) \Gamma(z+\nu)} \]

which is in reality only a modified form of (6) from analogous form of (5) can also be given.
For \( z > 0 \) and integral, there is a version of (5) involving complex integration path and with no restrictions on \( x \) and \( y \).

\[
\int \ldots \int (t_1 \ldots t_n)^{-1} \prod_{i=1}^{n} \Delta(t_i) \frac{dz}{z} = \left(\frac{\sin \pi x \sin \pi iy}{\pi} \right) \prod_{v=1}^{n} \frac{\Gamma(1+v \pi) \Gamma(x+v-1\pi) \Gamma(y+v-1\pi) \Gamma(x+y+(m+v-2)\pi)}{\Gamma(1+2\pi) \Gamma(x+y+(m+v-2)\pi)}.
\]

(1941)

\[
\sum_{t_1, t_2 (\text{mod } p)} \chi_1(t_1, t_2) \chi_2((1-t_1)(1-t_2)) \chi_3(t_1, t_2) = \\
= \frac{\mathcal{T} \chi_3^2 \mathcal{T} \chi_1 \mathcal{T} \chi_1 \mathcal{T} \chi_2 \mathcal{T} \chi_1 \mathcal{T} \chi_2}{\mathcal{T} \chi_3 \mathcal{T} \chi_1 \mathcal{T} \chi_3 \mathcal{T} \chi_1 \mathcal{T} \chi_2 \mathcal{T} \chi_3^2} + (\chi_3 \Rightarrow \chi_3 \psi)
\]

where \( \psi \) is a quadratic character, except when any of the \( \mathcal{T} \) have principal character \( \chi_0 \) as subscript (some of these exceptions can be avoided if we interpret \( \overline{\chi_0} \) as \(-1\)).

Proof of (12) indicated that correct analogue of (5) should be:
Conjecture: Let \( \mathcal{P}_n \) run over all polynomials
\[ x^n + a_1 x^{n-1} + \ldots + a_n \mod p \] of degree \( n \) and write \( D(\mathcal{P}_n) \) for the discriminant of \( \mathcal{P}_n \), then
\[
\sum_{\mathcal{P}_n} \chi_1 (\mathcal{P}_n(0)) \chi_2 (\mathcal{P}_n(1)) \chi_3 (D(\mathcal{P}_n))
\]
\[
= \frac{1}{T} \frac{T x_3^n - T x_1 x_3^{n-1} - T x_2 x_3^{n-1}}{T x_3 - T x_1 x_2 x_3}
\]

Analogues of say, (8), (9), (7) can similarly be written down (the analogue of (6) turns out to be essentially identical with (13)). (13) proved by myself for \( n = 2 \) in 1941, as well as analogue of (8). Analogue of (9) proved independently by myself and Ronald Evans ca 1980 for \( n = 3 \), about same time Evans also independently found proof of (13) and analogue of (8) for \( n = 2 \).

Finally, general form of (13) proved 1990 by Greg Anderson.
Some other Beta-type integrals:

Let \( X_1, \ldots, X_m \) be \( m \) dimensional vectors and \( e \) vector of unit length \( |e| = 1 \), then:

\[
\int_{\ldots} \frac{dX_1 \ldots dX_m}{(X_1^{\alpha_1} \ldots X_m^{\alpha_m}) |e - X_1 - X_2 - \cdots - X_m|^{\alpha_{m+1}}}
\]

\[= \pi^{\frac{m(m+1)}{2}} \frac{\Gamma\left(\frac{\sum_{i=1}^{m} \alpha_i - mn}{2}\right)}{\Gamma\left(\frac{m(m+1) - \sum_{i=1}^{m} \alpha_i}{2}\right)} \prod_{i=1}^{m+1} \frac{\Gamma\left(\frac{\alpha_i}{2}\right)}{\Gamma\left(\frac{\alpha_i}{2} - n + 1\right)}
\]

valid when all arguments of \( \Gamma \)'s in numerator have positive real part.

A character analogue is fairly easy to state and prove.

---

Let the \( Y \) denote symmetric or by real matrices and write

\[dY = |Y|^{-\alpha_{m+1}} \prod_{i \neq j} dY_{i,j}.
\]

Let \( E \) denote the unit matrix, then:

\[
\int_{\ldots} |Y_1^{\alpha_1} \ldots Y_m^{\alpha_m} |E - Y_1|^{\alpha_{m+1}} \ldots |E - Y_m|^{\alpha_{m+1}} dx_1 \ldots dx_m
\]

\(Y_1 > 0\)

\(Y_1 + \cdots + Y_m < E\)

is again expressible as a quotient of products of \( \Gamma \)-functions whose arguments are linear expressions in the \( x_i \) and \( z_i \).
If we write
\[ Y(\Delta_1, \ldots, \Delta_m) = \prod_{\nu=1}^{\infty} \left| Y^{(\nu)} \right|^{A_{\nu}}, \]
where
\[ Y^{(\nu)} = \left( \eta_{i,j}^{(\nu)} \right)_{i,j<\nu}, \]
then
\[ \int \ldots \int Y_1(\Delta_1, \ldots, \Delta_{i-1}) \cdots Y_m(\Delta_{m+1}, \ldots, \Delta_{m+m}) \, dy_1 \cdots dy_m \]
where
\[ Y^* = (I + Y_1 + \cdots + Y_m)^{-1}, \]
is again expressible as a quotient of products of \( \Gamma \)-functions whose arguments are linear expressions in the \( \Delta_{i,j} \).
Integrals exist if the \( \Delta_{i,j} \) with \( 1 \leq i \leq m \) are negative real parts and the \( \Delta_{m+1,j} \) have sufficiently large real parts.
Character analogues undoubtedly exist of these last two Beta-type integrals, but I have not tried to establish them.
q-analogue of (5) conjectured by Richard 
Askey for general n, proved for n = 2.

While
\[(x; q)_m = (1-x)(1-xq) \cdots (1-xq^{m-1}),\]
and define for \(|q| < 1\)
\[\int_0^1 f(t) d_q t = (1-q) \sum_{m=0}^{\infty} \frac{f(q^m)}{q^m} t^m,
\]
and
\[\Gamma_q(x) = \frac{(q_1; q_1)^{\infty}}{(q^x; q_1)^{\infty}} (1-q)^{1-x}.
\]

Then:
\[\int_0^1 \frac{\prod_{i=1}^{n} t_i}{q \prod_{i=1}^{n} t_i} \left( \frac{t_i t_j q^{-1}}{t_i + t_j - q^{-1}} \right)^{x-1} \prod_{i=1}^{2k} \Gamma_q\left( \frac{x-1}{2k} \right) \prod_{i=1}^{n} d_q t_i = \prod_{i=1}^{n} \frac{\Gamma_q(1+nv) \Gamma_q(x+q-1) \Gamma_q(x+y+q-1)}{\Gamma_q(1+qv) \Gamma_q(x+y+(m+n-2)v)}.
\]

If we let q \rightarrow 1 we recover formula (5).

The formula was proved for general n in 
1988 published independently by
K.W.J. Kadell and Laurent Habsieger.
It is not clear whether Greg Anderson's
ideas can be modified so as to work for
this case also.
Own Proof: Consider first case that $e$ is a positive integer, then

$$|\Delta_t|^2 = \sum e_{n_1, \ldots, n_m} t_1^{a_1} \cdots t_m^{a_m}$$

with integer coefficients $e$. Thus the integral in (5) is a linear combination of terms

$$\prod_{n=1}^m \frac{\Gamma(x + \alpha_n)}{\Gamma(x + \alpha_n + \alpha_n)}$$

which without loss of generality we may assume $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$.

Since obviously

$$\sum \alpha_n = m(m-1)e,$$

we have

$$\alpha_m \geq (m-1)e.$$ 

In the same way, since $\Delta(t, \ldots, t_m)$ is divisible by $\Delta(t, \ldots, t_n)$ for each

$$1 \leq n \leq m,$$

we have generally

$$\alpha_n \geq (n-1)e.$$ 

Also, since

$$|\Delta(t)|^2v = (t_1 \cdots t_m)$$

we find also

$$\alpha_v \leq 2(m-1)e - (m-v)2 = (m+v-2)e.$$
This means that
\[
\frac{P(x+y_v)}{P(x+y+\alpha_v)} = \frac{P(x+(y-1)V)}{P(x+y+(m+V-2)\alpha)} \phi_v(x,y),
\]
where \( \phi_v(x,y) \) is a polynomial in \( x \) and \( y \) of degree \((m+V-2)\alpha \) in \( y \), and
\[
\frac{\prod_{v=1}^{n} P(x+y_v)}{\prod_{v=1}^{n} P(x+y+(m+V-2)v)} = Q_0(x,y) \prod_{v=1}^{n} \frac{P(x+y+(m+V-2)v)}{P(x+y_v)},
\]
where \( Q_0(x,y) \) is a polynomial in \( x \) and \( y \) of degree \( \frac{1}{2} m (m-1) \alpha \) in \( y \).

Since \( I \), the integrand in (5), is a linear combination of residues, we have
\[
I = Q(x,y) \prod_{v=1}^{n} \frac{P(x+(y-1)V)}{P(x+y+(m+V-2)v)} = \frac{Q(x,y)}{P(y)} \prod_{v=1}^{n} \frac{P(x+y+(m+V-2)v)}{P(x+y_v)},
\]
where \( Q(x,y) \) is a polynomial of degree at most \( \frac{1}{2} m (m-1) \alpha \) in \( y \), and
\[
P(y) = \prod_{v=1}^{n} \frac{P(y+(m+V-2)v)}{P(y_v)},
\]
a polynomial of degree \( \frac{1}{2} m (m-1) \alpha \) in \( y \).

Since \( I \) is symmetric in \( x \) and \( y \), we must have
\[
\frac{Q(x,y)}{P(y)} = \frac{Q(y,x)}{P(x)}.
\]
3.

It follows that this quotient is independent of $x$ and $y$, so that

$$I = \frac{C_m(z)}{\prod_{y=1} \frac{\Gamma(x+2-yz) \Gamma(y+2)}{\Gamma(x+y+(m+y-2)z)}}.$$ 

To determine $C_m(z)$, we take $x=y=1$ in $I = \Im I_m(x,y;z)$. A simple transformation of variables gives

$$\Im I_m(1,1;2) = \frac{1}{\Gamma(2-z+1)} \Im I_{m-1}(1,2z+1;2),$$

which reduces to

$$C_m(z) = \frac{\Gamma(1+2z)}{\Gamma(1+2)} C_{m-1}(z),$$

or

$$C_m(z) = \prod_{y=1}^{z} \frac{\Gamma(1+yz)}{\Gamma(1+z)}.$$

since $C_1(z)=1$.

This proves (5) for $z$ a positive integer. A standard interpolation argument extends this to all $z \in \mathbb{R}$ with positive real part, and

thus by analytic continuation to all complex $x, y, z$ for which $I$ is well defined.
Proof: Let

\[ S_m(x_1, y_1; z) = \frac{1}{m!} \int S_m(x_1, y_1; z) \cdot \frac{2\pi}{\Delta(t)} \cdot dt_1 \ldots dt_m, \]

where

\[ S_m(x_1, y_1; z) = \int |F(0)| \cdot |F(1)| \cdot |D(F)| \cdot df_1 \ldots df_m, \]

and

\[ \Phi(t) = (t-\theta_1) \ldots (t-\theta_m) = \sum_{i=0}^{m} F_i \cdot t^i, \]

and \( \theta_1 < \theta_2 < \ldots < \theta_m < 1 \), \( D(F) \) is the discriminant of \( F \), and \( C(F) \) denotes the integration domain in \( F_0, F_1, \ldots, F_{m-1} \), implied by the conditions on the \( \theta_i \).

**Lem.** Let \( \tau_0 < \theta_1 < \tau_1 < \theta_2 < \ldots < \theta_m < \tau_m \),

\[ F(t) = \prod_{i=0}^{m} (t-\theta_i), \quad T(t) = \prod_{i=0}^{m} (t-\tau_i); \]

then

\[ \int_{C(F)} \frac{1}{T'(\tau_i)} \prod_{i=0}^{m} dF_0 \ldots dF_{m-1} = \]

\[ = \frac{\prod_{i=0}^{m} \Phi(\tau_i)}{\prod_{i=0}^{m} \Phi(\theta_i)} \cdot \prod_{i=0}^{m} \left| T'(\tau_i) \right| \cdot A_{m-1}. \]

This is proved by writing \( \frac{F(t)}{T(t)} = \sum_{i=0}^{m} \frac{\phi_i}{t-\tau_i} \),

then \( \phi_i = \frac{F(\tau_i)}{T'(\tau_i)} \), and we have \( \phi_i > 0 \),

\[ \sum_{i=0}^{m} \phi_i = 1. \] Also, for every set of \( \phi_i \) fulfilling the last two conditions, then
corresponds to a polynomial $F$ satisfying the
conditions of the lemma. If we write
the integral of the lemma with the
new variables $\varphi_1, \ldots, \varphi_m$ it becomes:
\[
\prod_{i=0}^{m} |T(T_i)|^{\beta_i - \frac{1}{2}} \int \cdots \int \prod_{\varphi_i > 0} \frac{p_i^{\lambda_i - 1} \cdots \varphi_i^{\lambda_i - 1}}{\varphi_i > 0} \prod_{\varphi_i > 0} (1 - \varphi_i \cdots - \varphi_m \varphi_1 \cdots \varphi_i) \prod_{\varphi_i > 0} \varphi_i^{\beta_i} \prod_{\varphi_i > 0} \varphi_i^{\beta_i - 1} \prod_{\varphi_i > 0} \sum_{\varphi_i < 1}
\]
which proves the lemma.

Now let $0 < \varphi_1 < \varphi_2 < \cdots < \varphi_{n-1} < \varphi_n < 1,$
write $F(t) = \sum_{i=0}^{n-1} F_i t^i = G(t) = \sum_{i=0}^{n} G_i t^i.$

Conditions define integration domain $(F, G)$ in
the integral
\[
J = \int \left[ \frac{1}{G(\varphi)} \right]^{x} \left[ \frac{1}{G(t)} \right]^{y-1} |R(F, G)| dF_0 \cdots dF_1 \cdots dF_n \cdots dG_n
\]
$(F, G)$
where $|R(F, G)|$ is the abs. value of the resultant
of $F$ and $G.$
\[
|R| = \prod_{i=1}^{\frac{n}{2}} |F(\varphi_i)| = \prod_{i=1}^{\frac{n}{2}} |G(\varphi_i)|.
\]

If we integrate first over $F$ (and let
$G$ play the role of $T$ in the lemma) we
get
\[
J = S_m(x, y, z) \left( \frac{\Gamma(z)}{\Gamma(mz)} \right)^{\frac{m}{z}}.
\]

If we integrate first over $G$ (and let
$T(z-1)F$ play the role of $T$ in the lemma),
we get
\[ y = \sum_{m=1}^{\infty} \frac{P(x+z, x+y+z; 2m) \frac{P(x) P(y)}{P(x+y+z; 2m)}}{P(x+y+z; 2m)} \cdot \]
Thus
\[ S_{m+1}(x, y; z) = \sum_{m=1}^{\infty} \frac{P(mz) P(x) P(y)}{P(z) P(x+y+z; 2m)}, \]
which at once gives the evaluation of \( S_{m+1}(x, y; z) \) since \( S_1(x, y; z) = \frac{P(x) P(y)}{P(x+y)} \),
and so proves (5).

One can modify Greg Anderson's lemma to get direct proofs of the two formulas (8) and (9) which were earlier only obtained as limiting cases of (5). This is important since in the character case the limiting process has no analogue.