Selberg Paper. Norsk. Mathematisk Tidsskrift, 26, 71-78 (1944)

Write

$$\Delta(u) = \Delta(u_1, \ldots, u_p) = \prod_{i<j}^{p}(u_j - u_i).$$

**Theorem:** For integer $p$ and complex $x, y, z$ with

$$\text{Re}(x) > 0, \text{Re}(y) > 0, \text{Re}(z) > -\min \left[ \frac{1}{p}, \frac{\text{Re}(x)}{p-1}, \frac{\text{Re}(y)}{p-1} \right],$$

we have

$$I = \int_0^1 \cdots \int_0^1 \left( \prod_{i=1}^{p} u_j \right)^{x-1} \left( \prod_{i=1}^{p} (1-u_j) \right)^{y-1} |\Delta(u)|^{2z} \, du_1 \cdots du_p$$

$$= \prod_{\nu=1}^{p} \left[ \frac{\Gamma(1+\nu z)\Gamma(x+(\nu-1)z)\Gamma(y+(\nu-1)z)}{\Gamma(1+z)\Gamma(x+y+(p+\nu-2)z)} \right].$$

**Proof:** For $p = 1$ this reduces to the well-known Euler integral

$$\int_0^1 u^{x-1}(1-u)^{y-1} \, du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

so we assume $p > 1$.

Consider first the case when $z$ is a positive integer. Then

$$|\Delta(u)|^{2z} = \sum C_{\alpha_1, \alpha_2, \ldots, \alpha_p} u_1^{\alpha_1} \cdots u_p^{\alpha_p}$$

with integer coefficients $c$. Therefore the integral $I$ is a linear combination of terms

$$\prod_{\nu=1}^{p} \left[ \frac{\Gamma(x+\alpha_\nu)\Gamma(y)}{\Gamma(x+y+\alpha_\nu)} \right],$$

where, without loss of generality, we may suppose that $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p$. Since $\Delta(u)$ is homogeneous of degree $\frac{3}{2}p(p-1)$, we have

$$\sum_{\nu=1}^{p} \alpha_\nu = p(p-1)z,$$

$$\alpha_p \geq (p-1)z.$$

In the same way, since $\Delta(u_1, \ldots, u_p)$ is divisible by $\Delta(u_1, \ldots, u_\nu)$ for each $\nu$, we have

$$\alpha_\nu \geq (\nu-1)z.$$
Now
\[ |\Delta(u)|^{2z} = \left( \prod_{j=1}^{p} u_j \right)^{2(p-1)z} |\Delta(1/u)|^{2z}, \]
and therefore the exponents \( \alpha'_\nu = 2(p-1)z - \alpha_{p+1-\nu} \) satisfy the same inequalities
\[ \alpha'_\nu \geq (\nu - 1)z. \]

Therefore
\[ \alpha_\nu \leq 2(p-1)z - (p-\nu)z = (p+\nu-2)z. \]

This means that
\[ \frac{\Gamma(x + \alpha_\nu)}{\Gamma(x + y + \alpha_\nu)} = \frac{\Gamma(x + (\nu - 1)z)}{\Gamma(x + y + (p+\nu-2)z)} q_{\alpha_\nu}(x, y) \]
where \( q_{\alpha_\nu}(x, y) \) is a polynomial in \( x \) and \( y \) with degree \( [(p+\nu-2)z - \alpha_\nu] \) in \( y \). Thus
\[ \prod_{\nu=1}^{p} \frac{\Gamma(x + \alpha_\nu)\Gamma(y)}{\Gamma(x + y + \alpha_\nu)} = Q_\alpha(x, y) \prod_{\nu=1}^{p} \left[ \frac{\Gamma(x + (\nu - 1)z)\Gamma(y)}{\Gamma(x + y + (p+\nu-2)z)} \right] \]
where \( Q_\alpha(x, y) \) is a polynomial in \( x \) and \( y \) with degree in \( y \)
\[ \sum_{\nu=1}^{p} [(p+\nu-2)z - \alpha_\nu] = \frac{1}{2} p(p-1)z. \]

Since \( I \) is a linear combination of such terms,
\[ I = Q(x, y) \prod_{\nu=1}^{p} \left[ \frac{\Gamma(x + (\nu - 1)z)\Gamma(y)}{\Gamma(x + y + (p+\nu-2)z)} \right] \]
\[ = \frac{Q(x, y)}{R(y)} \prod_{\nu=1}^{p} \left[ \frac{\Gamma(x + (\nu - 1)z)\Gamma(y + (\nu - 1)z)}{\Gamma(x + y + (p+\nu-2)z)} \right] \]
where
\[ R(y) = \prod_{\nu=1}^{p} [y(y+1) \cdots (y + (\nu - 1)z - 1)], \]

and \( Q(x, y) \) is a polynomial in \( x, y \) of degree at most \( \frac{1}{2} p(p-1)z \) in \( y \). It follows from \( \Delta(u) = \pm \Delta(1-u) \) that \( I \) is symmetric in \( x \) and \( y \). Therefore
\[ \frac{Q(x, y)}{R(y)} = \frac{Q(y, x)}{R(x)}. \]

But the right side of this identity is a polynomial in \( y \), and therefore \( Q(x, y) \) must be divisible by \( R(y) \). Since the degree of \( Q(x, y) \) in \( y \) is equal to the degree of \( R(y) \), the
quotient must be independent of $y$. By symmetry, the quotient is also independent of $x$. That is to say

$$ I = c_p(z) \prod_{\nu=1}^{p} \left[ \frac{\Gamma(x + (\nu - 1)z)\Gamma(y + (\nu - 1)z)}{\Gamma(x + y + (p + \nu - 2)z)} \right] $$

To determine $c_p(z)$, we take $x = y = 1$. Then

$$ J_p = \int_0^1 \cdots \int_0^1 |\Delta(u)|^{2z} \, du_1 \cdots du_p = c_p(z) \prod_{\nu=1}^{p} \left[ \frac{(\Gamma(1 + (\nu - 1)z))^2}{\Gamma(2 + (p + \nu - 2)z)} \right]. $$

Now we let $w$ be the largest of the $u_j$ and take for the other $u_j$

$$ u_j = w v_j, \quad 0 \leq v_j \leq 1. $$

Then

$$ J_p = p \int_0^1 w^{p-1} \int_0^1 \cdots \int_0^1 |\Delta(u)|^{2z} \, dv_1 \cdots dv_{p-1} \, dw $$

$$ = p \int_0^1 w^{p-1+z(p-1)} \int_0^1 \cdots \int_0^1 \left[ \prod_{\nu=1}^{p-1} (1 - v_\nu) \Delta(v) \right]^{2z} \, dv_1 \cdots dv_{p-1} \, dw $$

$$ = \frac{1}{(p-1)z + 1} I' $$

where $I'$ is the integral $I$ with $x = 1, y = 2z + 1$ and $p - 1$ for $p$. That is to say,

$$ c_p(z) \prod_{\nu=1}^{p} \left[ \frac{(\Gamma(1 + (\nu - 1)z))^2}{\Gamma(2 + (p + \nu - 2)z)} \right] = \frac{c_{p-1}(z)}{(p-1)z + 1} \prod_{\nu=1}^{p-1} \left[ \frac{\Gamma(1 + (\nu - 1)z)\Gamma(1 + (\nu + 1)z)}{\Gamma(2 + (p + \nu - 2)z)} \right]. $$

This reduces to

$$ \frac{c_p(z)}{c_{p-1}(z)} = \frac{\Gamma(2 + (p-1)z)\Gamma(1 + pz)}{((p-1)z + 1)\Gamma(1 + (p-1)z)\Gamma(1 + z)} = \frac{\Gamma(1 + pz)}{\Gamma(1 + z)}. $$

Since $c_1(z) = 1$, we have

$$ c_p(z) = \prod_{\nu=1}^{p} \left[ \frac{\Gamma(1 + \nu z)}{\Gamma(1 + z)} \right], $$

which completes the proof for integer $x$.

The proof extends to complex $z$ with $\text{Re} \, z > 0$ by a standard argument using Carlson's theorem. Finally, by analytic continuation, it extends to all complex $x, y, z$ for which the integral $I$ is well-defined.
\[
S_m(x,y,z) = \int \cdots \int \left( \sum_{i=1}^{x-1} \cdots \sum_{i=1}^{y-1} (1-t_i) \right) \frac{\partial^2 \mathcal{F}}{\partial \theta_i^2} \, dt_1 \cdots dt_y = \frac{1}{x!} \frac{\prod_1 \mathcal{P}(v_2) \mathcal{P}(x+v_2) \mathcal{P}(y+v_2)}{\prod_1 \mathcal{P}(v_2) \mathcal{P}(x+y+v_2) \mathcal{P}(y+x+v_2)}
\]

\[
S_0(x,y,z) = \int \left( \sum_{i=0}^{x-1} \cdots \sum_{i=0}^{y-1} \Delta \mathcal{F} \right) \, d\mathcal{F}_0 \cdots d\mathcal{F}_{k-1}
\]

\[
F(t) = \frac{\partial^2 \mathcal{F}_{i}^2}{\partial \theta_i^2} \quad \Delta \mathcal{F} = \sum_{i=0}^{x} F(t) \frac{\partial^2 \mathcal{F}_{i}^2}{\partial \theta_i^2} = \prod_1 \theta_i - \theta_i
\]

**Lemma:** Let \( 0 < \theta_1 < \theta_2 < \cdots < \theta_n < \tau_m \)

\[
F(t) = \prod_{i=1}^{n} (t-\theta_i) \quad T(t) = \prod_{i=0}^{n} (t-\theta_i)
\]

Define \( D \mathcal{F} \):

\[
D \mathcal{F} = \prod_1 \mathcal{P}(\theta_0) \cdots \mathcal{P}(\theta_n) \prod_1 \left( T(t) \right)^{\lambda_i-\frac{1}{2}}
\]

Set \( \frac{F(t)}{T(t)} = \sum_{i=0}^{n} \frac{p_i}{t-\theta_i} \), where \( p_i = \frac{F(\theta_i)}{T(\theta_i)} \)

\[ \sum_{i=0}^{n} p_i = 1, \quad \text{and} \quad p_i > 0 \]

Set \( p_i \) to avoid the polynomial \( F \) : \( D \mathcal{F} \). Show that the integral is

\[ \int \mathcal{P}(\theta_0) \cdots \mathcal{P}(\theta_n) \prod_1 \left( T(t) \right)^{\lambda_i-\frac{1}{2}} \]

**Integral with respect to variable \( \theta_i \):**

\[ \mathcal{V} \mathcal{H} \mathcal{A} \mathcal{R} \quad \frac{\partial \mathcal{F}}{\partial \theta_i} \quad \Delta \mathcal{F} = \prod_1 \left( T(t) \right)^{-\frac{1}{2}} = \left| D \mathcal{F} \right|^{-1} \]

**Integral of**
\[ \frac{\eta}{1!} \left[ T \left( \sum \rho_i \right) \right] \sum_{i=0}^{\eta} \rho_i \left( \prod_{i=0}^{\eta} x_i \right) \]