General Problem

Class (Euler product 89)

\[ L(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}; a_1 = 1, \quad a_m = O(m^\delta) \\text{ for any } \delta > 0. \]

"Euler product"

\[ \log L(s) = \sum_{m=1}^{\infty} \frac{b_m}{m^s}, \quad \text{where } b_m = 0 \]

unless \( m = p^\alpha; \alpha > 0. \) \( (s-1)^m L(s) \) integral

function of finite order for some integer \( m > 0. \)

\( b_m = O(m^\theta) \) with \( \theta < \frac{1}{2}. \)

Functional equation: Let

\[ \phi(s) = \prod_{j=1}^{n} \Gamma(\lambda_j s + \mu_j), \quad L(s), \quad \text{with} \]

constants \( 1 \leq i = 1, \quad Q = 0, \quad \lambda_j > 0, \quad R / \mu_j \geq 0. \)

and

\[ \phi(s) = \overline{\phi(1-s)} \cdot \phi(\frac{1}{2} + \xi) \text{real for real } \xi. \]

Suppose we have \( n \) distinct \( \lambda_j \) with

the same \( \Gamma \) factors and form with

real constants \( c_j \neq 0 \)

\[ F(s) = \sum_{j=1}^{n} c_j \Re \Gamma(\lambda_j s + \mu_j), \]

then

\[ \frac{1}{n} \Gamma(\lambda_j s + \mu_j) \quad F(s) \text{ is real for } \]

\[ s = \frac{1}{2} + it; \quad t \text{ real.} \]
Conjecture: Almost all "non-trivial" zeros of $F(s)$ are on the line $\sigma = \frac{1}{2}$, $\sigma = \sigma + it$. The non-trivial zeros lie in a strip $-A < \sigma < A$ and the number with imaginary part in $(0,T)$ is
\[ N(T,F) = \frac{A}{\pi} T (\log T + B) + O(\log T), \]
where $\Lambda = \frac{A}{\pi}$. If we denote the number of zeros of $F(s)$ with real part $\frac{1}{2}$ in $0 < t < T$ by $N_0(T,F)$, the conjecture $N_0(T,F) \sim N(T,F)$ can be proved if certain plausible conjectures are assumed (some of which can today be proved for even a single function in this class).

What can be proved without using any hypotheses?

It is clear that we could only hope to prove anything of significance for the linear combination $F(s)$, where we can prove something significant for the single $L(s)$. This has been done only in the case $\Lambda = \frac{1}{2}$, and for some cases with $\Lambda = 1$. 
case \( \Lambda = \frac{1}{2} \).

Dirichlet L-functions, \( \chi \) primitive character mod \( q \) (incl. case \( q = 1 \) and \( \chi(m) \equiv 1 \)).

\[
L(s, \chi) = \sum_{m} \frac{\chi(m)}{m^s}
\]

integral function for \( q \neq 1 \), \((s-1)\) \( L(s) \) integral for \( q = 1 \).

Write \( a = \frac{1 - \chi(-1)}{2} \), and

\[
\phi(s, \chi) = \sum_{q \frac{1}{2}} q^{\frac{s}{2}} \pi^{-s} \frac{1}{\psi(s)} L(s, \chi),
\]

then \( \phi(s, \chi) = \phi(1 - s, \chi) \).

For simplicity we look at the case \( \chi \) even \( (a = 0) \), the odd case can be handled in the same way.

Let \( \chi_j, j = 1, 2, \ldots, m \), be \( m \) distinct primitive even characters, the \( \xi_j \) real and \( a_j \neq 0 \), and form

\[
F(a) = \sum_{j=1}^{m} c_j \xi_j q_j^{\frac{s}{2}} L(s, \chi_j)
\]

(Alternatively, we might consider

\[
F^*(a) = \sum_{j=1}^{m} c_j \xi_j (1 + q_j^{a - \frac{1}{2}}) L(s, \chi_j)
\]

Then \( \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) F(a) \sim \pi^{-\frac{s}{2}} \Gamma(\frac{a}{2}) F^*(a) \).
are real for \( s = \frac{1}{2} + it \), \( t \) real.

We have in this case

\[
N(T, F) = \frac{T}{2\pi} (\log T + B) + O(\log T).
\]

For the single \( L(\sigma, \chi) \) it has been proved that a positive proportion of the zeros have real part \( \frac{1}{2} \), more precisely:

\[
N_0(T, L) > cT \log T \quad \text{for} \quad T > Aq^2,
\]

where \( c \) and \( A \) are absolute constants.

For the general linear combination some results are implied in the literature. From Hardy-Littlewood work follows

\[
N_0(T, F) > cT \quad \text{for} \quad T > T_0(f).
\]

Recently A.A. Karatsuba has considered combinations

\[
f(s) = \sum \chi L(\sigma, \chi) + \sum \overline{\chi} L(\sigma, \overline{\chi}),
\]

where \( \chi \) is a complex character and obtained the result (1994)

\[
N_0(T, f) > T (\log T)^{\frac{1}{2}} e^{-\sqrt{\log T}} , \quad T > T_0;
\]

with some positive constant \( c \) (for more general combinations he has a much weaker and more complicated result.)
I shall sketch a proof that for the general combination $F(x)$ we have

1. $N_0(T, F) > C(m) T \log T$ for $T > T_0(F)$,
   where $C(m)$ depends on $m$ only. Also:

2. If $\omega(t) \to \infty$ as $t \to \infty$ then
   $F(\frac{1}{2} + it)$ has a zero in the interval
   $(t, t + \frac{\omega(t)}{\log t})$ for almost all $t$.

First let us see how these results are proved for the single $L$-function.

Let $s = \frac{1}{2} + it, \ t > 0$, and

$$\omega(t) = \theta \pi^{-\frac{1}{2}} \Gamma(\frac{1}{4})$$

and write

$$\chi(t, \chi) = E(t, \chi) \sum_{n \leq \frac{x}{2}} \chi(n) n^{-s} +$$

$$\frac{1}{x} \sum_{n \leq \frac{x}{2}} \chi(n) n^{-s-1} + O\left(\frac{x}{T}\right)$$

Write

$$\left(\xi(s)\right)^{-\frac{1}{2}} = \sum_m \frac{\alpha_m}{m^s}; \ \alpha_1 = 1; \ \left(\xi(s)\right)^{-\frac{1}{2}} = \sum_m \chi(n) \frac{\alpha_m}{m^s}$$
and for $1 \leq t \leq 2T$; $\xi = T^{1/10}$; $T > 16 q^3$ write
\[
\gamma(t, x) = \sum_{m = 1}^{\infty} \frac{x(m)}{m^2} (1 - \frac{\log m}{\log \xi}).
\]
(we shall often write $\gamma(t, x)$ for $\gamma(\xi + it, x)$.)

For $\frac{1}{\log T} \leq H \leq \frac{\log \log T}{2 \log q}$, consider the three expressions:

\[I_x(t, H) = \int_t^{t+H} |X(u, x)|^2 du,
\]

\[M_x(t, H) = \int_t^{t+H} L(\frac{1}{2} + it, x) \gamma^2(u, x) du - H,
\]

and

\[J_x(t, H) = \int_t^{t+H} |X(u, x)| \gamma(u, x) du.
\]

Then $J_x(t, H) > |I_x(t, H)|$,

$X(t, x)$ changes sign in $(t, t+H)$ and so has at least one zero there.

Also $J_x(t, H) \geq H - |M_x(t, H)|$,

so if

$|M_x(t, H)| + |I_x(t, H)| < H$,

there is a zero in $(t, t+H)$. 

Using the approximate functional equation for $L(s, X)$ or $X(t, X)$, we can show

1. \[ \sum_{T}^{2T} |I_X(t, H)|^2 dt = O\left(T \frac{H^{3/2}}{\sqrt{\log T}}\right), \]

2. \[ \sum_{T}^{2T} |M_X(t, H)|^2 dt = O\left(T \frac{H^{3/2}}{\sqrt{\log T}}\right), \]

and (as we shall use much later)

3. \[ \sum_{T}^{2T} \left| X(t, X) \right|^2 dt = O\left(T\right). \]

The constants implied by the $O$ are absolute.

We see now that \( |I_X(t, H)| \leq \frac{T}{\sqrt{3}} \) and \( |M_X(t, H)| \leq \frac{T}{3} \) holds except in a subset of \((T, 2T)\) of measure \( O\left(\frac{T}{\sqrt{H \log T}}\right) \); choosing now \( H = \frac{\lambda}{\sqrt{\log T}} \) with \( \lambda \) a large enough constant, we get statement (I) for \( N_0(T, L_X) \), and by choosing \( \lambda = (\omega(T))^2 \) we get statement (II).
To adapt this idea to the linear combination \( F(x) \), we need some results about the value distribution of 
\[
\log |L(\frac{1}{2}+it, X)| \text{ or } \log |X(t, X)|.
\]
For \( T > \log T^3 \), \( k \) a positive integer and \( T^{\frac{1}{2k}} \leq x \leq T^{\frac{1}{k}} \), we can show
\[
\frac{1}{T} \int_{1}^{2T} \left| \log |X(t, X)| - R \sum_{p \leq x} \chi(p) p^{-\frac{1}{2}} e^{-it \log x} \right|^2 dt = \frac{1}{T} \sum_{p \leq x} \chi(p) p^{-\frac{1}{2}} e^{-it \log x} \leq T^{-k} e^{A T^k},
\]
The constants implied by the \( O \) are again absolute.
From this we can prove that
\[
\frac{\log |X(t, X)|}{\sqrt{\pi \log \log t}}
\]
has a normal Gaussian distribution.
More precisely: let \( X_{a,b} \) denote the characteristic function of the interval \( (a, b) \), then
\[
\frac{1}{T} \int_{a}^{b} \left( \frac{\log |X(t, X)|}{\sqrt{\pi \log \log t}} \right) dt = T \int_{a}^{b} e^{-\frac{u^2}{2}} du + \frac{1}{T} \left( \frac{\log \log \log T}{\sqrt{\log \log T}} \right)^2 + O \left( T^{-\frac{1}{2}} \right).
\]
Also for two distinct characters \( x \) and \( x' \), similar results hold for the difference

\[
\log |X(t, x)| - \log |X(t, x')|,
\]
only here we must divide by 
\[
\sqrt{2\pi \log \log t}
\]
to get the normal

\[
\text{gaussian distribution. Thus if } 0 < \delta < \frac{1}{2}, \text{ the set in } (T, 2T) \text{ where }
\]

\[
|\log |X(t, x)| - \log |X(t, x')|| \leq (\log \log T)^5
\]
has measure

\[
O(T (\log \log T)^{-\frac{1}{2} + \delta}).
\]

Thus most of the time one \( X(t, x_j) \) dominates all the other decisively. This dominance is somewhat persistent and stretches long compared to \( \frac{1}{\log T} \). Define

\[
\Delta X(t, x) = \frac{1}{H} \int_{-H}^{H} \log |X(t, x)| \, dx,
\]

for \( 0 \leq h \leq H \), we can show for any positive integer \( k \) that
\[ \int_0^{2T} \left( \Delta x(t, H) - \log_1 x(t+h, x_1) \right)^2 dt = \]
\[ = O \left( T e^{\lambda k} \left( e^{k} \log^k (H \log T) + k^k \right) \right). \]

Integrating over \( h \) we get
\[ \int_0^{2T} \int_0^{2T} \left( \Delta x(t, H) - \log_1 x(t+h, x_1) \right) dh dt = \]
\[ = O \left( THE^{\lambda k} \left( e^{k} \log^k (H \log T) + k^k \right) \right). \]

If we denote by \( W(t, x) \) the subset of \( h \) for which
\[ |\Delta x(t, H) - \log_1 x(t+h, x_1)| > (\log_y T)^{\frac{1}{2}}, \]

we find choosing \( k \) so large that
\[ k \delta > 2N + 1, \]

\[ W(t, x) \leq \frac{H}{(\log_y T)^N}, \]

except for a subset of \( t \) in \( (T, 2T) \) of measure \( O \left( \frac{1}{(\log_y T)^N} \right) \).

We also can show that for \( x \neq x'_0 \)
\[ |\Delta x(t, H) - \Delta x'(t, H)| > (\log_y \log T)^{\frac{1}{2}}. \]
except for a subset of \((T, 2T)\) of measure \(O(T \log \log T)^{-\frac{1}{2} + \delta}\).

For \(x, \ldots, x_m\) we now define \(j \neq k\), \(S_{j,k}\) as the subset of \((T, 2T)\) where

\[
|\Delta x_j(t, H) - \Delta x_k(t, H)| \leq (\log \log T)^{\frac{5}{2}}
\]

we have \(\omega(S_{j,k}) = O(T (\log \log T)^{-\frac{5}{2} + \delta})\).

If we exclude all of these subsets from \((T, 2T)\), the rest consists of \(m\) sets \(S_j\) such that in \(S_j\) for \(k \neq j\)

\[
\Delta x_j(t, H) \geq \Delta x_k(t, H) + (\log \log T)^{\frac{5}{2}}
\]

If we also from each \(S_j\) exclude all \(t\) for which for any \(k\)

\[
W(t, x_k) > \frac{H}{(\log T)^N},
\]

we get that \((T, 2T)\) except for a subset of measure \(O(T (\log \log T)^{-\frac{5}{2} + \delta})\) is divided into \(m\) subsets \(S_j\) such
that \[ \sum_{m} \mathcal{C}(S^*_j) = T - O(T (\log T)^{-\frac{1}{2} + \delta}) \]

and for each \( t \) in \( S^*_j \) we have for \( k \neq j \) that

\[
\log |X(t+h, x_j)| - \log |X(t+h, x_k)| > (\log \log T)^{\delta} - 2 (\log \log T)^{\frac{\delta}{2}} > \frac{1}{2} (\log \log T),
\]

except for a set of \( t \) in \((0, H)\) of measure \( O\left( \frac{H}{(\log \log T)^{\nu}} \right) \).

From

\[
\int_{\frac{T}{2}}^{2T} \frac{1}{2} X(t, x) \eta(t, x) \, dt = O(T)
\]

we see that

\[
\int_{\frac{T}{2}}^{2T} \left| X(u, x_j) \eta^2(u, x_j) \right| \, du \leq H \log \log T
\]

except for a subset of \( t \) of measure

\( O\left( \frac{H}{(\log \log T)^{\nu}} \right) \), we exclude also these \( t \)

from the \( S^*_j \) (without renaming them).

Now look at

\[ I_{x_j}^*(t, H), M_{x_j}^*(t, H) \text{ and } J_{x_j}^*(t, H) \]
which are for $t$ in $S^*_j$ the integrals $I_{x_j}, H_{x_j}$ and $J_{x_j}$ but with the bad subset removed. They differ from these at most by

$$O\left( \sqrt{\frac{H}{(\log \log T)^N}} \cdot \sqrt{H \log T} \right)$$

$$= O\left( \frac{H}{\log T} \right),$$

taking $N = 3$. We see now that we get a sign change of $X(u, x_j)$ in $(t, t + H)$ for $t$ in $S^*_j$ and

$$J_{x_j}^{(e)} > |I_{x_j}^{(e)}(t, H)|$$

which is equivalent to

$$H > |I_{x_j}(t, H)| + |M_{x_j}(t, H)| + O\left( \frac{H}{\log T} \right)$$

For $T$ large enough this holds outside a measure $O\left( \frac{1}{H \log T} \right)$ and so in most of $S^*_j$ if $H = \frac{\lambda m^2}{\log T}$ with $\lambda$ a large enough constant. Probes more than $\frac{c}{c m^2 T \log T}$ sign changes of
\[ \pi^{-\frac{1}{2}} \psi \left( \frac{1}{2} \right) F(\sigma); \; \sigma = \frac{1}{2} + it \quad \text{in} \; \mathbb{S}^n \]

Adding up over \( j \) we get more than \( \frac{c}{m^2} T \log T \) or

\[ N_0(T, F) > \frac{c}{m^2} T \log T \quad \text{for} \; T>T_0(F) \]

can be improved to

\[ N_0(T, F) > \frac{c \log \log T}{m^2} T \log T \quad \text{for any} \; x > 1. \]

Case \( \lambda = 1 \). Some cases of such \( \lambda(x) \) have been handled and

\[ N_0(T, \lambda) > c T \log T \quad \text{for} \; T>T_0(\lambda) \]

proved. Essentially what is required is that one can estimate expressions like

\[ \frac{2}{T} \int_{-T}^{T} |L(\frac{1}{2} + it)|^2 \psi(\frac{1}{2} + it) P(\frac{1}{2} + it) \, dt \]

where \( P \) is a Dirichlet polynomial

\[ P(s) = \sum_{n \leq \xi} \frac{\lambda(n)}{n^s} \]

where \( \xi \) is like

some small power of \( T \). In these cases the \( L(\frac{1}{2}, \xi) \) have been handled but not the \( M_\lambda(\xi, \xi) \) which were
avoided using a simpler device which does not work for the linear combination. By slightly modifying the way one defines the analog of \( \eta(s) \) for \( L(s) \), we put

\[
L(s)^{-\frac{1}{2}} = \sum_{n} \frac{\alpha_n}{n^s} ; \alpha_n = 1
\]

and

\[
\eta(s) = \sum_{0 < n < \frac{1}{4}} \frac{\alpha_n}{n^s} + \sum_{\frac{1}{4} \leq n \leq N} \frac{\alpha_n}{n^s} \left(1 - 2 \frac{\beta_n}{\xi_n} \right)
\]

This new \( \eta(s) \) works equally well for \( I(t, H) \), and much better for

\[
M(t, H) = \int_{t}^{t+H} L\left(\frac{1}{2} + it\right) \eta^{2}\left(\frac{1}{2} + it\right) dt - H,
\]

We can actually get

\[
\int_{1}^{T} |M(t, H)|^2 dt = O\left( \frac{T}{\log^2 T} \right)
\]

uniformly for \( 0 < H < T \). Everything else works as before.
Briefly sketched: We write
\[ |N(t,H)| = \left| \int_{1}^{t+iH} \left( c(t+iu) \theta(t+iu)-1 \right) du \right| \leq \]
\[ \leq \int_{1}^{t+iH} \left| c(t+iu) \theta(t+iu)-1 \right| du + \left| c(t+iu)-c(t+iH) \right| \]
\[ + O \left( \frac{1}{\log^{1/3} \frac{t}{\lambda}} \right). \]

Also,
\[ \left( \int_{1}^{t+iH} (c(t) \theta(t)-1) dt \right)^2 \leq \]
\[ \int_{1}^{t+iH} \frac{1}{t+iH} \left( \int_{1}^{t+iH} (c(t) \theta(t)-1) dt \right)^2 dt. \]

If we write
\[ J_0 = \int_{1}^{t+iH} \left( c(t) \theta(t)-1 \right)^2 dt, \]
we have
\[ J_0 = O(T) \quad \text{and} \quad J_2 = O \left( T 3^{-\epsilon} \right). \]
A continuity argument now gives
\[ J_0 = O \left( T 3^{-\epsilon} (T-\frac{1}{2}) \right), \]
and the estimation
\[ \int_{1}^{t+iH} |N(t,H)|^2 dt = O \left( \frac{T}{\log^{2} \frac{t}{\lambda}} \right) = O \left( \frac{T}{\log^{2} \frac{1}{\lambda}} \right) \]
now follows easily.
The drawback is that with this \( \gamma(s) \), we have to go through the estimation of
\[
\frac{2^T}{T} \int |I(t, h)|^2 \, dt
\]
new, since it does not follow from our earlier result. The same approach can however be adapted to our old \( \gamma(s) \) defined as
\[
\gamma(s) = \sum_{m < \xi} \frac{\alpha_m}{m^s} \left( 1 - \frac{\log m}{\log \xi} \right),
\]
by observing that the Dirichlet series for \( \zeta(s) \gamma^2(s) \) is identical with that of
\[
\left( 1 - \frac{1}{2 \log \xi} \frac{\zeta'(s)}{\zeta(s)} \right)^2 \text{ for } m \leq \xi.
\]
Subtracting from the expression of \( \zeta(s) \gamma^2(s) - 1 \) the terms with
\[1 < m \leq \xi \], and handling...
This part separately on the line \( \sigma = \frac{1}{2} \), the remainder with the method just outlined one get easily for \( \sigma > \frac{2}{\log T} \)

\[
\int_0^2 T \left| M(t, H) \right|^2 \, dt = O \left( T \frac{\log (H \log T)}{\log^2 T} \right)
\]

a result only slightly worse (and much better than we actually need).