Elementary methods and the distribution of "generalized primes".

The problem I am going to speak of is the following:

- We suppose that we have an infinite set of real numbers \( r > 1 \) tending to infinity

\[ 1 < r_1 \leq r_2 \leq r_3 \leq \ldots \]

these numbers we call "primes". From these "primes" we build up a sequence of integers in the way that we form all possible products of the form \( r_1, r_2, \ldots, r_n \) with

\[ x_1 > r_1, x_2 > r_2, \ldots, x_n > r_n \]. The number that we get we order in a nondecreasing sequence

\[ \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots \]

The problem we denote by \( N(x) \) the number of \( \alpha_i \leq x \) and by \( \Pi(x) \) the number of \( \alpha_i \leq x \); the problem is now to find an asymptotic behaviour of the \( N(x) \) as \( x \to \infty \).
This problem has been treated first by A. Beurling in: *Sur la loi asymptotique de la distribution des nombres premiers*, *Journal de Math.* (1933), by analytical means. We shall treat this problem under the assumption that

\[(1) \quad N(x) = tx \log x + O\left(\frac{x}{(\log x)^\alpha}\right),\]

as \(x \to \infty\), \(t\) and \(\alpha\) being positive constants, and \(2 < \alpha < 3\), by means of a method which is completely elementary in the technical sense of this word, and prove that if \(x > 2\), (1) implies that

\[(2) \quad \Pi(x) \sim \frac{x}{\log x},\]

as \(x \to \infty\).

First, let us remark that obviously all properties of the ordinary natural numbers, which depend only upon multiplicative properties, also hold true for our sequence of "integers". That means that concepts as divisor, divisibility, and also numerical functions as the Möbius function, can be extended with all the properties "conversed to" our sequence of "integers".
In particular we define as usual the function
\[ l(m_i) = \begin{cases} \log p_i & \text{if } m_i \text{ is a prime of height} \\ 0 & \text{otherwise} \end{cases} \]

And as usual in the case of the ordinary we define
\[ \psi(x) = \sum_{m_i \leq x} l(m_i) \]
and (2) is then equivalent to

\[ \psi(x) \sim x \quad \text{as } x \to \infty \]

To prove (3) we first have to establish some preliminary formulae.

\[ \psi(x) = o(x) \]

\[ \sum_{m_i \leq x} \frac{l(m_i)}{m_i} = \log x + O(1) \]

(4) is proved by considering the relation
\[ \sum_{m_i \leq x} \log m_i = \sum_{m_i \leq x} \sum_{d_i | m_i} \frac{l(d_i)}{d_i} \]
\[ = \sum_{d_i \leq x} l(d_i) N\left(\frac{x}{d_i}\right) \]

From (4) follows \[ \psi(x) = o(x) \]
Besides (4) we need a deeper asymptotic formula, which can be written
in one of the two forms

\[ \log x, \psi(x) + \sum_{m : \leq x} \lambda(m) \psi\left(\frac{x}{m}\right) = 2x \log x + O(x) \quad (5) \]

or

\[ \sum_{\nu : \leq x} \lambda(\nu) \log \nu + \sum_{\nu : \nu' \leq x} \lambda(\nu) \lambda(\nu') = 2x \log x + O(x) \quad (5') \]

To prove (5') we start with the expression

\[ \log^2 x \sum_{\nu : \leq x} \mu(\nu) \log^2 \frac{x}{\nu} \]

or

\[ \log^2 x \sum_{\nu : \nu' \leq x} \mu(\nu) \log^2 \frac{x}{\nu} \]

Thus

\[ \sum_{\nu : \leq x} \left\{ \sum_{\nu : \nu' \leq x} \mu(\nu) \log^2 \frac{x}{\nu} \right\} = \log^2 x \]

and

\[ \sum_{\nu : \leq x} \left( \log^2 x - \frac{\log^2 x}{\nu} \right) + \sum_{\nu : \nu' \leq x} \log \nu \log \frac{x}{\nu} \]

\[ = \log x \sum_{\nu : \leq x} \lambda(\nu) + \sum_{\nu : \nu' \leq x} \lambda(\nu) \lambda(\nu') + O(x) \]

Secondly we have
\[
\int_{|t| > \mu} \frac{e^{i \frac{1}{\sqrt{\mu}}} \exp \left[ \frac{t^2}{m^2} - \frac{t^2 \log \Lambda^2}{(t^2 + (\Delta \omega))^{\frac{1}{2}}} \right]}{t^\theta} \, dt
\]

\[
\sum_{n \geq 0} \frac{e^{i \pi n}}{n!} \int_{|t| > \mu} e^{i \frac{\pi}{4} t (\alpha + \lambda \mu)} \, dt
\]

\[
\sum_{|t| > \mu} \frac{1}{t^{2\theta}} = O(n) \leq \sum_{|t| > \mu} \frac{1}{t^\theta} \]

\[
\sum_{\alpha \geq 0} \sum_{n \geq 0} \left( \frac{(\Delta \omega)}{t^2 + \lambda t} \right)^{\frac{3}{2} + \frac{1}{2} \theta}
\]

\[
\int_{|t| > \mu} \left( \sum_{n \geq 0} \frac{e^{i \pi n}}{n!} \right) \left( \sum_{\alpha \geq 0} \frac{e^{i \frac{\pi}{4} t (\alpha + \lambda \mu)}}{t^\theta} \right) \, dt
\]
\begin{align*}
\sum_{a: x} \{ \sum_{d_i / m_i} \mu(d_i) \log \frac{x}{d_i} \} &= \sum_{d_i \leq x} \mu(d_i) \log \frac{x}{d_i} N \left( \frac{x}{d_i} \right) \\
\sum_{d_i / m_i} \mu(d_i) \log \frac{x}{d_i} &= \sum_{d_i \leq x} \mu(d_i) \log \frac{x}{d_i} + \\
&= h_x \sum_{d_i \leq x} \frac{\mu(d_i)}{d_i} \log \frac{x}{d_i} + \Theta \left( x \left( \log \frac{x}{d_i} \right)^{3-\alpha} \right) \\
= h_x \sum_{d_i \leq x} \frac{\mu(d_i)}{d_i} \log \frac{x}{d_i} + \Theta \left( x \left( \log \frac{x}{d_i} \right)^{3-\alpha} \right) \text{.}
\end{align*}

Then we have to determine
\[ \sum_{d_i \leq x} \frac{\mu(d_i)}{d_i} \log \frac{x}{d_i} \] for this end we need

the formulas

(1) \[ \sum_{a: x} \frac{1}{m_i} = \frac{1}{\log z} + c_1 + \Theta \left( \frac{1}{(\log z)^{\alpha-1}} \right) \]

and

(2) \[ \sum_{a: x} \frac{\mu(a: i)}{n_i} = \frac{1}{2} \log \frac{z}{2} + c_2 \log z + c_3 + \Theta \left( \frac{1}{(\log z)^{\alpha-2}} \right) \]

further (3) \[ \sum_{d_i \leq x} \frac{\mu(d_i)}{d_i} = \Theta(1) \]
From (1) and (2) we get

\[ \log^2 2 = \frac{2}{\pi} \sum_{n \leq x} \frac{\log n}{n} + e^x \sum_{n \leq x} \frac{1}{n^2} + c_5 + 6\left(\frac{1}{(\log x)^{3/2}}\right) \]

From this we get pushing \( x = \frac{d}{2} \),

\[ \frac{\mu(x)}{x} \cdot \frac{\log^2 \frac{x}{d}}{\log \frac{x}{d}} = \frac{2}{\pi} \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{n \leq x} \frac{1}{n^2} + c_5 \sum_{d \leq x} \frac{\mu(d)}{d} + \left(\frac{1}{\log \frac{x}{d}}\right) \]

\[ = \frac{2}{\pi} \log x + O\left(\frac{1}{(\log x)^{3/2}}\right) \]

Inserting this we get

\[ (\beta) \quad \frac{x}{x} \cdot \psi(x) + \sum_{n \leq x} \Lambda(n) \frac{\psi(\frac{x}{n})}{\psi(x)} = 2x \log x + O\left(x(\log x)^{1 - \epsilon}\right) \]

for \( \epsilon > 0 \).

From this we get

\[ (\beta') \quad \int_{0}^{x} \Lambda(n) \frac{\psi(\frac{x}{n})}{\psi(x)} \, dn = 2x + O\left(x(\log x)^{1 - \epsilon}\right) \]

and further applying \( \beta' \) to \( \beta \) we get

\[ (\beta'') \quad \int_{0}^{x} \psi(x) = \sum_{n \leq x} \Lambda(n) \frac{\psi(\frac{x}{n})}{\psi(x)} \frac{x}{\psi(\frac{x}{n})} + O\left(x(\log x)^{1 - \epsilon}\right) \]
$$(7) \quad \log x \cdot R(x) = \sum_{\alpha_i \leq x} \Lambda(n_i) \log \frac{x}{n_i} + O(x \log x \log \log x).$$

Now we put

$$\phi(x) = x + R(x)$$

and get from (5) and (7) that

$$\log x \cdot R(x) = -\sum_{\alpha_i \leq x} \Lambda(n_i) R\left(\frac{x}{\alpha_i}\right) + O(x \log x \log \log x).$$

and for

$$\log x \cdot R(x) = \sum_{\alpha_i \leq x} \Lambda(n_i) \Lambda(n_i') R\left(\frac{x}{\alpha_i \alpha_i'}\right) + O(x \log x \log \log x)$$

or by combining

$$2 \log x |R(x)| \leq \sum_{\alpha_i \leq x} \Lambda(n_i) \left|R\left(\frac{x}{\alpha_i}\right)\right| + \sum_{\alpha_i \leq x} \frac{\Lambda(n_i) \Lambda(n_i')}{\log \frac{x}{\alpha_i \alpha_i'}} R\left(\frac{x}{\alpha_i \alpha_i'}\right)
\quad + O(x \log x \log \log x) = \int \left|R\left(\frac{x}{t}\right)\right| \phi(t) + O\left(x \log x \log \log x\right)$$

since by (6)

$$\phi(t) = 2t + \phi(t) \quad \text{where} \quad \phi(t) = O\left(t \log t \log \log t\right)$$

we get

$$2 \log x |R(x)| \leq 2 \int \left|R\left(\frac{x}{t}\right)\right| dt + \int \left|R\left(\frac{x}{t}\right)\right| \phi(t) + O\left(\_\_\_\_\_\right)$$
\[
\int_X |R(\frac{x}{t})| \: dq(t) = q(x)|R(x)| - q(x)|R(x)|
\]

\[
- \int_X q(t) \: d|R(\frac{x}{t})| = 0(x) + o\left(\frac{t}{(\ln t)^{\alpha-2}}\right) - \int_1^x \frac{t}{(\ln t)^{\alpha-2}} \: d\left\{ \frac{q(\frac{x}{t}) + \frac{x}{t}}{l(\ln t)} \right\} dt
\]

\[
= o(x) + o\left(\frac{t}{(\ln t)^{\alpha-2}} \right) - \int_1^x \frac{t}{(\ln t)^{\alpha-2}} \: d\left\{ \frac{q(\frac{x}{t}) + \frac{x}{t}}{l(\ln t)} \right\} dt
\]

\[
= o(x) + o\left(\frac{t}{(\ln t)^{\alpha-2}} \right)
\]

Hence

\[
(\overline{2}) \quad \frac{1}{x} \int_X |R(x)| \leq \frac{1}{l(\ln x)} \int_1^x \frac{1}{l(\ln t)^{\alpha-2}} \: dt + o\left(\frac{x}{(\ln x)^{2-\alpha}}\right)
\]

From (4) we get

\[
\int_X \frac{R(t)}{t^2} \: dt = \int_1^x \frac{q(t) - t}{t^2} \: dt = \frac{q(x)}{x} + \int_1^x \frac{1}{t} \: d\psi(t)
\]

\[
- \log x = \sum_{m \leq x} \frac{\Lambda(m)}{m} - \log x + o(1) = o(1)
\]

and from (6) if \( t < t' \)

\[
0 \leq \psi(t') - \psi(t) \leq 2(t' - t) + o(t'(\log t')^{2-\alpha})
\]
From which

\[(C'') \quad |R(t') - R(t)| \leq |t'-t| + O\left( \frac{t'^{1+\epsilon}}{(q(t+t')^{*})^{\epsilon-2}} \right)\]

The proof that \(R(x) = r(x)\) now goes as follows: We know that \(1R(x) < Kx\) for \(x > 1\). Now suppose that for some \(x > 0\), we have \(R(x) < 0\) for \(x > x_0\).

From (C') in the form

\[\left| \int_{t_1}^{t_2} \frac{R(t)}{t^2} \, dt \right| = O(1)\]

it follows that if \(t\) is in the interval \((x_1, x_2)\) the \(R(t)\) changes into sign at most once. Then

\[\int_{x_1}^{x_2} \frac{|R(t)|}{t^2} \, dt < K^2\]

On the other hand if \(R(t)\) changes the sign more than once, there is a point in the interval where \(R(t)\) vanishes.

Now divide the interval \((1, x)\) into intervals

\[(p_v, p_{v+1}) \quad \text{for} \quad v = 0, 1, \ldots \left[ \frac{x}{q^2} \right], \text{and discus} \]
\[ J_\nu = \int_{p_\nu}^{p_{\nu+1}} \frac{|R(t)|}{t^2} \, dt \]

for \( p_\nu > x_0 \), then we have \( \exists \, \tau^* \) if \( R(t) \) changes sign at most once in \( (p_\nu, p_{\nu+1}) \) that

\[ |J_\nu| \leq K_2 = \frac{\alpha}{2} \ln \eta \quad (\text{if we take} \ p = e^{x_0}) \]

if \( R(t) \) changes at least twice then there is a point \( \tau^* \leq p_{\nu+1} \) when \( R(\tau_0) = 0 \), and so

\[ |R(t)| \leq |t - t_0| + O\left( \frac{t + t_0}{(\eta + t_0)^{\alpha-2}} \right). \]

\[ \left| \frac{R(t)}{t} \right| \leq 1 - \frac{t_0}{t} + O\left( \frac{1}{(\eta + t)^{\alpha-2}} \right) \]

If we form this we easily see that there is an interval \((e^{-\delta} t_0, e^{\delta} t_0)\) which

\[ \delta = 1 + \frac{5}{3} \; ; \; \delta = K_{\eta}. \]

Starting sufficiently large, a part \((t_1, e^{\delta} t_1)\) of this is contained in \((p_\nu, p_{\nu+1})\).