1. Hejhal: On the distribution of \( \log |L'(\frac{1}{2} + it)| \).


In [1], Hejhal considers the expression

\[ (1) \quad \nu'(t) = \frac{\log |L'(\frac{1}{2} + it)|}{\sqrt{\pi \log \log t}}, \quad t > 9 \]

and proves that if \( m_{a,b}(T) \) denotes the measure of the set for which \( a < \nu'(t) < b \) in \( 9 < t < T \), then

\[ (2) \quad m_{a,b}(T) \sim T \int_a^b e^{-\pi u^2} du \]

as \( T \to \infty \). This proof, which assumes R.H., is based on [2] (where the relevant part dealt with \( \arg \xi(\frac{1}{2} + it) \)) as well as on old, but at the time unpublished results concerning the expression

\[ (3) \quad \nu(t) = \frac{\log |L'(\frac{1}{2} + it)|}{\sqrt{\pi \log \log t}}, \quad \text{for which if we define } m_{a,b}(T) \text{ in a corresponding way, we have} \]
\[ m_{a, b}(T) = T \sum_{a} e^{-\pi a^2} \quad \text{with} \quad \Theta \left( \frac{T}{\log \log T} \right), \]
which was proved without any hypothesis.

In a form concerned with more general

dirichlet series (4) occurs in [3], where

\( T \) is proved assuming a certain density

hypothesis, which can be proved for

some fairly large groups of dirichlet series

with „interproduct and functional equation,

roughly when \( \Lambda \leq 1 \) in the class con-

sidered in [3].

I shall sketch a proof of (3) in the

stronger form:
\[ m_{a, b}^1(T) = T \sum_{a} e^{-\pi a^2} \quad \text{with} \quad \Theta \left( \frac{T}{\log \log T} \right). \]

The proof is considerably shorter than

Hejhal’s and uses no hypothesis.

We begin by quoting some definitions

and formulas from [2].

Let \( 0 < \delta, \epsilon < x \leq t^2 \), \( t > 0 \) and

define:
\[ \delta_{x, t} = \frac{1}{\delta} + 2 \max \left( \beta - \frac{1}{2}, \frac{2}{\log x} \right), \]

where \( \delta = \beta + i \gamma \) runs over the

zeros of \( \xi(x) \) for which
(7) \[ |t - \Re| \leq \frac{x^3(\beta - \frac{1}{2})}{\log x}. \]

Also let

\[
\Lambda_x(n) = \begin{cases} 
\Lambda(n) & \text{for } 1 \leq n \leq x, \\
\log_x \frac{x^3}{m} - 2 \log_2 \frac{x^2}{m} & \text{for } x \leq n \leq x^2, \\
\log \frac{x^3}{m} - 2 \log^2 x & \text{for } x^2 \leq n \leq x^3.
\end{cases}
\]

Put \( s = \frac{1}{2} + it \) and \( L_x = \sigma_x e^{-it} \), then

\[
(8) \quad \sum_{\sigma} \frac{\sigma_x e^{-\frac{1}{2}}}{(s_x - \sigma)^2} = \Theta \left( \frac{1}{\theta} \sum_{\sigma < r_x} \frac{\Lambda_x(n)}{n \theta^n} + \log r \right).
\]

and

\[
(9) \quad \frac{\xi_1}{\xi} (L_x) = \Theta \left( \frac{1}{\theta} \sum_{\sigma < r_x} \frac{\Lambda_x(n)}{n \theta^n} + \log r \right),
\]

\[
\left( \frac{\xi_1}{\xi} \right)^{(1)} (L_x) = \sum_{\sigma} \frac{1}{(s_x - \sigma)^2} + \Theta \left( \frac{1}{\theta} \right) \quad (11)
\]

Finally we have

\[
\frac{\xi_1}{\xi} (L_x) = \frac{\xi_1}{\xi} (L_x) + \Theta \left( \frac{1}{\sigma_x e^{-\frac{1}{2}}} \left( \frac{1}{\theta} \sum_{\sigma < r_x} \frac{\Lambda_x(n)}{n \theta^n} + \log r \right) \right).
\]
\[
\begin{align*}
(1) \quad \frac{\xi_1'}{\xi} (\Delta) &= \frac{\xi_1'}{\xi} (\Delta X) + (\Delta - \Delta X) (\frac{\xi_1'}{\xi})'(\Delta X) + \\
&+ \sum_{\xi} \frac{(\Delta - \Delta X)^2}{(\Delta X - \eta)^2(\Delta - \eta)} + O\left( \frac{1}{t^2} \right) = \\
&= O\left( \sum_{x < X^3} \frac{L_X(m)}{M^{\frac{1}{2}}} + \log t \right) \\
&+ \sum_{\xi} \frac{(\Delta - \Delta X)^2}{(\Delta X - \eta)^2(\Delta - \eta)}.
\end{align*}
\]

From the expression \( \sum_{\xi} \) we subtract for \( t < T \) the expression \( \sum_{\xi} \frac{1}{\Delta - \eta} \) when \( |t - \eta| < \frac{1}{\log T} \).

And obtain after a little manipulation, using (9), that

\[
(13) \quad \sum_{\xi} \frac{(\Delta - \Delta X)^2}{(\Delta X - \eta)^2(\Delta - \eta)} = \sum_{|t - \eta| < \frac{1}{\log T}} \frac{1}{\Delta - \eta} + \\
+ O\left( \sigma_X \frac{1}{\log T} \left( \sum_{x < X^3} \frac{L_X(m)}{M^{\frac{1}{2}}} + \log t \right) \right),
\]

or inserting this in (12)

\[
(14) \quad \frac{\xi_1'}{\xi} (\Delta) = \Sigma_1(t) + \Sigma_2(t),
\]
where
\[ (15) \quad \sum_1(t) = \sum_{|t-t_k| < \frac{1}{\log t}} \frac{1}{\sqrt{t-t_k}} \]

and
\[ (16) \quad \sum_2(t) = \Theta((\log t)^{-\frac{1}{2}}) \log T \left( \sum_{a < \sqrt{x}} \frac{\lambda(x)}{\sqrt{a}} \right) \log t) \]

Using Lemma 12 of [27] we can now easily show by choosing \( x \) as a sufficiently small power of \( T \) (with exponent depending on \( k \)), that for any integer \( k \)

\[ (17) \quad \int_{\frac{1}{\log T}}^T \frac{\sum_2(t)}{t} |t^k dt = \Theta_k(T), \]

while for \( \Sigma_2 \) we easily obtain that for \( 0 < \theta < 1 \), uniformly

\[ (18) \quad \int_{\frac{1}{\log T}}^T \frac{1}{t^\theta} \sum_1(t) \, dt = \Theta \left( \frac{T}{1-\theta} \right). \]

Thus in particular

\[ (19) \quad \int_{\frac{1}{\log T}}^T \frac{1}{t^\theta} \left( \frac{\xi(t)}{\xi} \right) |t^k dt = \Theta \left( \frac{T}{1-\theta} \right). \]

Since, as we shall presently see,

\[ (20) \quad \frac{1}{\log t} \left| \frac{\xi(t)}{\xi} \left( \frac{\xi}{\xi} + it \right) \right| > c > 0, \]
with some positive constant $c$, (19) lets us derive (5) from (4) with little effort and without degrading the remainder term.

If we write

$$ (20) \quad \xi \left( \frac{1}{2} + it \right) = e^{-i \delta(t)} X(t) $$

where

$$ (21) \quad \delta(t) = \frac{t}{2} \log \pi - \arg \Gamma \left( \frac{1}{2} + \frac{it}{2} \right), $$

so that

$$ (22) \quad \delta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O \left( \frac{1}{t^2} \right), $$

we see that

$$ (23) \quad \frac{\xi'}{\xi} \left( \frac{1}{2} + it \right) = -\delta'(t) - i \frac{X'}{X}(t), $$

or

$$ \Re \left. \frac{\xi'}{\xi} \left( \frac{1}{2} + it \right) \right| = -1. $$

From this (20) follows at once.

Hejhal also considers the distribution

$$ (24) \quad \log \left| \frac{\delta'(t)}{\log \log t} X'(t) \right| $$

and proves a result similar to (2) on RH.
While we obviously have
\[(25) \int_T^T \frac{1}{(\phi'(t))} \cdot \frac{X'(t)}{X(t)} \equiv 0 \left( \frac{T}{1-\epsilon} \right),\]
from (23) and (19), the analog of (20) does not hold, since \(X'(t)\) has many real zeros for \(9 < t < T\) (actually \(> AT \log T\)).

Using a theorem of Littlewood we can prove
\[\int_T^T \log \left| \frac{1}{(n'(t))} \cdot X'(t) \right| \, dt \geq -C \log T,\]
since also
\[\int_T^T \log |X(t)| \, dt = \Theta(T),\]
it follows that
\[(26) \int_T^T \log \left| \frac{1}{(n'(t))} \cdot \frac{X'(t)}{X(t)} \right| \, dt = \Theta(T),\]
and using (25) that
\[(27) \int_T^T \log \left| \frac{1}{(n'(t))} \cdot \frac{X'(t)}{X(t)} \right| \, dt = \Theta(T).\]
From this we can deduce that
\[
K(t) = \frac{\log \left( \log \log t \right)}{\sqrt{\log \log t}}
\]
and
\[
\mathcal{M}_{a, \delta}^* (T) = T \int_a^b e^{-\frac{\pi u^2}{T}} du + O \left( \frac{T}{\log \log T} \right).
\]

As we see the remainder term is degraded compared to the earlier cases.

More generally one can show if
\[
\beta_0 \neq (i) = \sum_{i=0}^{N} c_i i \left( \log^r t \right)^{-i} \xi \left( \frac{1}{2} + it \right),
\]
\[
N > 0,
\]
(3.1) \[ \sum_{q}^{T} \left| \frac{f(q)}{f(\frac{1}{2}+it)} \right|^\frac{1}{N} = O\left( \frac{T}{1-\theta} \right), \]

for \( 0 < \theta < 1 \), and that

(3.2) \[ \int_{T}^{1} \left| \log \left| \frac{f(q)}{f(\frac{1}{2}+it)} \right| \right| \, dt = O(T), \]

so that again if we define

(3.3) \[ u_f(t) = \frac{\log |f(q)|}{\sqrt{\pi \log \log t}} \]

and \( \mu_{a,b}^f(T) \) as before, then

(3.4) \[ \mu_{a,b}^f(T) = T \int_{-T}^{T} e^{-\pi u^2} \, du + O\left( \frac{T}{(\log \log T)^{\frac{1}{4}}} \right). \]

Similar results to those given above hold for the general class defined in (3.3) if one assumes that for the function in question we have

\[ N(C, T) = O\left( T^{1-\alpha (\tau-\frac{1}{2})} \log T \right), \]

with some positive constant \( \alpha \).

Essentially this can be proved to hold in those cases for which (in the notation of that paper) \( \Lambda \leq 1 \).
In [37] I have indicated how one can determine the distribution also of
\[
\frac{\log |F(\frac{1}{2} + it)|}{\sqrt{\log t}}
\]
where \( F \) is a finite linear combination of the type of functions considered there,
\[ F(x) = \sum_{i=1}^{n} c_i F_i(x) , \]
with distinct \( F_i(x) \).

One can show in general that
\[
\frac{\log |\log t \ F'(\frac{1}{2} + it)|}{\sqrt{\log t}}
\]
has the same distribution.

Similar results hold also for expressions involving the higher derivatives in analogy with (30).