B bounded homogeneous complex domain
points \( z = (z_1, \ldots, z_n) \), \( \mathbb{C} \) transitive group of complex analytic mappings
\( z \to g \cdot z \).

Automorphism factor \( \varphi_g(z) \), defined on \((\mathbb{B}, \mathbb{G})\) and such that

1. \( \varphi_{g_1} \cdot \varphi_{g_2}(z) = \varphi_{g_1}(\varphi_{g_2}(z)) \cdot \varphi_{g_2}(z) \),

assume \( \varphi_g(z) \) holomorphic and \( \neq 0 \) in \( \mathbb{B} \)
\( \varphi_g(z) \) bounded on compact subset of \((\mathbb{B}, \mathbb{G})\)
assume differentiable in \( \mathbb{G} \).

\( \frac{\varphi_g(z)}{f(z)} \) where \( f \) holomorphic \( \neq 0 \) in \( \mathbb{B} \)
is trivial automorphism factor.
Can show that if \( \mathbb{B} \) is irreducible
that is not a direct product of domains
fulfilling same conditions in lower dimension then

2. \( \varphi_g(z) = (\varphi_g(z))^{i} \cdot \frac{\varphi_g(z^2)}{f(z)} \)

where
\( \varphi_g(z) \) is Jacobian \( \left| \frac{\partial (\varphi_g(z))}{\partial z_j} \right| \).

same conclusion if in (1) we only require
it to hold up to a factor \( \varphi_{g_1} \cdot \varphi_{g_2} \) of abs.
value 1.
So we can restrict ourselves to factors
that are powers of Jacobian of mapping.
If \( \mathbb{B} \) is reducible \( \varphi_g(z) \) is product of powers
of the Jacobians of the mappings of the irreducible
factors.
Let \( K(z, \overline{z}) \) be Bergmann kernel function of \( \mathbb{B} \) and let \( dw_2 = k(z, \overline{z}) \, \prod_{i=1}^{n} dx_i \, dy_i \).

We look at Hilbert space \( H^n \)

\[
(f, f)_n = \int_{\mathbb{B}} \frac{|f(z)|^2}{(k(z, \overline{z}))^n} \, dw_2 < \infty,
\]

where \( n > 0 \) and assume it is not empty which in case of integral exists for \( f = 1 \), that is for \( n > n_0 \) when \( 0 < n_0 < 1 \). If we write

\[
T_q^n f(z) = (T_q(z))^n f(qz)
\]

then

\[
(f, h)_n = (T_q^n f, T_q^n h).
\]

Produce complete orthonormal system \( f_i(z) \), form

\[
k_n(z, \overline{z}) = \sum f_i(z) \overline{f_i(z)}
\]

(convergence proof along Bergmanns lines)

get that

\[
\frac{k_n(z, \overline{z})}{(k(z, \overline{z}))^n}
\]

is invariant for \( z \to qz, \overline{z} \to q\overline{z} \), since \( \mathbb{B} \) homogeneous get

\[
k_n(z, \overline{z}) = c(n) \left( k(z, \overline{z}) \right)^n.
\]
Get

\[ f(z) = \sum \frac{c(n)}{\text{vol}(B)} \int_B \frac{1}{k(z, \xi)} f(\xi) d\xi, \]

for \( f \) in \( \mathcal{H}_n \), easy to see that formula holds also if \( f \) not in \( \mathcal{H}_n \) as long as integral exists if we take abs. values.

In all cases \( c(n) \) is a polynomial of degree \( n \) (and \( n \) is largest power of \( c(n) \)).

Let \( \Gamma \) be discrete subgroup of \( G \), with compact fundamental domain \( \mathcal{D} \) in \( B \).

Form of weight \( n \), function in \( \mathcal{D} \), and such that

\[ \xi_{\gamma}(f\gamma(z)) f(\xi_\gamma(z)) = f(z) \]

for elements \( \gamma \) of \( \Gamma \) where \( |\xi_\gamma| = 1 \), say \( \xi_\gamma \) consistent automorphic factor for weight \( n \). If \( f \) holomorphic in \( B \), we say \( f \) is a regular form of weight \( n \). Question of determining for a given \( \Gamma \) for which weights \( n \) there exist consistent automorphic factors unsolved in general. Clear that if \( n \) is an integer we can always take \( \xi_\gamma = 1 \) and we have a consistent a. f. , and the name of \( n \) in
such that $(f_\gamma(z)) \gamma$ is single-valued on the group $G$. Two consistent a. f. for same $\gamma$ differ only by a factor $\chi(\gamma)$ where $\chi$ is a (one-dimensional) group character of $\Gamma$. If we for simplicity assume $\gamma$ integral and $\gamma_0^\gamma = 1$ in (4) and that $f$ is regular, we get from (3), writing $\beta = \sum_{\gamma \in \Gamma} \gamma^{-1} \gamma_0^\gamma$,

$$f(\gamma) = c(n) \sum_{\gamma \in \Gamma} \int \left( \frac{f(\gamma z, \gamma_0^\gamma)}{f(z, \gamma_0^\gamma)} \right)^2 f(z) \, dw$$

$$= c(n) \sum_{\gamma \in \Gamma} \int \left( \frac{f(\gamma z, \gamma_0^\gamma)}{f(z, \gamma_0^\gamma)} \right)^2 f(z) \, dw$$

$$= c(n) \int \frac{K_n(z, \gamma_0^\gamma)}{(k(z, \gamma_0^\gamma))^n} f(z) \, dw,$$

where we have put

$$K_n(z, \gamma_0^\gamma) = \sum_{\gamma \in \Gamma} (f_\gamma(z))^n (k(z, \gamma_0^\gamma))^2.$$

From this we get that the number of linearly independent regular forms of weight $\gamma$ (belonging to any a. f.) is

$$N(\gamma) = c(n) \int \frac{K_n(z, \gamma_0^\gamma)}{(k(z, \gamma_0^\gamma))^n} \, dw.$$

This can be computed exactly by combining the terms in $K_n$ where the $\gamma$ are conjugate within $\Gamma$. First approximation is

$$N(\gamma) = c(n) \sqrt{v(\gamma)} + o(\gamma^{m-1}).$$
We now specialize to the one dimensional case where $D$ is the unit circle, which as is well known, with the invariant metric is a model of the hyperbolic plane. Prefer to map unit circle into upper half-plane $H$, $z = x + iy$, $y > 0$, then the group $G$ consists of $z \rightarrow \frac{az + b}{cz + d}$, with $a, b, c, d$ real and $ad - bc = 1$. $(\frac{a}{c}, \frac{b}{d})$ real unimodular matrix 

Invariant metric
\[ ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad dw_2 = \frac{dx \ dy}{y^2} \]

and invariant Laplacian \[ \eta^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \eta^2 \Delta \]. Geodesics circles orthogonal to Real line.

Fundamental domain $D$ is now a polygon with even no. of sides corresponding in pairs under $\Gamma$. Area of a polygon with $n$ sides is \( (n-2)\pi - \sum \alpha \) when $\alpha$ are the interior angles. Apart from identity elements $e$ of $\Gamma$ fall in 3 classes, hyperbolic if $|\alpha + d| > 2$, elliptic if $|\alpha + d| < 2$ and parabolic if $|\alpha + d| = 2$. For the time being we still consider only case $D$ compact, then our parabolic elements are present in $\Gamma$. If there are $k$ inequivalent elliptic elements in $\Gamma$ (there can only be a finite number $< \text{number of vertices in polygon}$) of order $m_1, \ldots, m_k$ respectively and $g$ is the genus of the closed surface we get by identifying corresponding sides then

\[ A(D) = 2\pi \left( 2g - 2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{m_i} \right) \right) \]
If \( q^z = \frac{a z + b}{c z + d} \) then \( f_q(z) = (cz + d)^{-2} \).

Let \( N^{(n)} \) be the number of regular forms of weight \( n \) (a positive integer) then (5) takes the form

\[
(7) \quad N^{(n)} = \frac{2n-1}{4\pi} A(g) + O(1),
\]

(cif no elliptic elements the term \( O(1) \) is actually zero, otherwise it is periodic in \( z \)).

For a form of weight \( n \) we can determine the number of inequivalent zeros (zeros in a fundamental domain), say \( N_n \).

We see that

\[
y^n |f(z)|
\]

is invariant under \( \Gamma \). Let

\[
u = \log y^n |f(z)|, \quad n = 1,
\]

we have

\[
r A(g) = \iint (u y^2 \Delta u - u \Delta y^2 u) \frac{dx dy}{y^2}.
\]

If we remove a small geodesic circle with radius \( p \) around each zero of \( f(z) \) (for those on the boundary we only remove the part inside \( D \)), and call the resulting domain \( D^* \), we get using Green's formula that
\[
r A(\Theta^*) = \oint_{\Theta^*} (u y^2 \Delta - \nu y^2 \Delta u) \frac{dx dy}{y^2} = \oint_{\Theta^*} (u \frac{\partial v}{\partial n} - \nu \frac{\partial u}{\partial n}) ds,
\]

In last integral the contribution of \(\partial \delta\) cancels out (corresponding sides give opposite and equal contributions), so we are left with the contribution from the integrals over our small circles, this contribution is seen to be \(2\pi l + O(\epsilon)\) for a zero of order \(l\) not at a fixed point and for a zero at a fixed point of order \(l\) set is \(\frac{2\pi l}{m} + O(\epsilon)\) when \(m\) is in the order of the antipode of \(P\) leaving the point fixed.

Letting \(\epsilon \to 0\) we get

\[
r A(\Theta) = 2\pi N_2
\]

where \(N_2\) is the number of zeros counted so that at a fixed point of order \(m\) we count the multiplicity of the zero with the weight \(\frac{1}{m}\).

Thus

\[
N_2 = \frac{r A(\Theta)}{2\pi}
\]

and combining this with (7)

\[
N^{(n)} - N_2 = O(1) \text{ as } n \to \infty.
\]
Clearly if we have \( N^{(n)} \) linearly independent forms of weight \( r \), we can at a given point \( z_0 \) produce a linear combination, not vanishing identically, which has a zero of at least multiplicity \( N^{(n)} - 1 \) at \( z_0 \). If \( z_0 \) is a fixed point of order \( m \), we see that we can produce a form with a zero of at least multiplicity \( (N^{(n)} - 1) m \).

Let us denote such a form by \( f_\nu(z, z_0) \) and norm it so that

\[
\max_{\nu} |f_\nu(z, z_0)| = 1.
\]

We wish to show that as \( r \to \infty \)

\[
\frac{1}{N^{(n)}} \log |f_\nu(z, z_0)| \quad \text{tends to}
\]

a limit function \( q(z, z_0) \), such that \( q(z, z_0) \) is invariant under \( \Gamma \),

\[
y^2 \Delta q(z, z_0) = -\frac{2\pi}{A(\Sigma)} q(z, z_0) - \frac{2\pi}{A(\Sigma)} \log y
\]

is harmonic and regular for all \( z \neq z_0 \) with \( y \in \Sigma \), \( q(z, z_0) \) is also regular at \( z = z_0 \) if \( z_0 \) has fixed point of order \( m \).

If we renormalize \( q(z, z_0) \) by adding a suitable constant \( \delta \) so that for
the new function \( q^*(z, z_0) \) we have
\[
\oint_{\partial D} q^*(z, z_0) \frac{dx dy}{y^2} = 0,
\]
then a simple application of Green's Theorem shows that we have the
reciprocity relation for \( z_1 \neq z_2 \)
\[
q^*(z_1, z_2) = q^*(z_2, z_1).
\]

Constructing the analytic function which
\[
\frac{1}{m} \left( q(z, z_0) - \frac{2\pi}{A(z_0)} \log z \right)
\]
as real part
(if \( z_0 \) is a point of order \( m \))
and exponentiating we get a regular
form of weight \( \frac{2\pi}{mA(z_0)} \) which has
a simple zero at \( z_0 \) and the equivalent
points \( z \neq z_0 \), but no other zeros. We
call this the prime form \( p(z, z_0) \), it
is uniquely determined up to a factor
of absolute value 1. It is the funda-
mental any automorphic function or
form can be expressed as
\[
C \frac{\Gamma \Phi(z)}{\Gamma \Phi(z, \beta)}
\]
where \( \alpha \) runs over the zeros and
\( \beta \) over the poles of the form and \( C \)
is a constant.
Since \( P(z, z_0) \) is a form of weight
\[
\frac{2\pi}{\text{on } A(\mathbb{D})}
\]
if \( z_0 \) is a point of order \( m \), we see that we can produce forms of all weights that are integral multiples of
\[
\frac{2\pi}{[\text{on } m_1, \ldots, m_k] A(\mathbb{D})} = \frac{1}{[\text{on } m_1, \ldots, m_k] (2g-2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{m_i} \right))}
\]
where \([\text{on } m_1, \ldots, m_k]\) denotes the least common multiple of \( m_1, \ldots, m_k \).

Also, for a sufficiently large positive integral multiple of this expression there always exist regular forms of that weight.

If \( D \) is not compact, but has finite area, the fundamental domain will have a finite number of vertices on the boundary of \( H \) (on the real line or at \( \infty \)) these are called cusps and are connected with the presence of parabolic elements in \( \Gamma \) which leave these cusps fixed. We can still use our arguments with a little modification to prove the existence of a prime form even if we put \( z_0 \) at a cusp, only this prime form has no zeros in the interior of \( H \), so we can form arbitrary real powers of \( z^2 \), and
so produce regular forms for all positive real $r$. In this case there are forms and consistent a.f. for all real weights.

For compact $D$, the set of all consistent a.f. for a given weight that is an integral multiple of $(10)$ can be seen to depend on $2g$ continuous real parameters and a discrete parameter that can take

$$\left[\frac{m_1, \ldots, m_k}{m_1, \ldots, m_k}\right]$$

values. For noncompact $D$ with $\lambda(D) < \infty$ if we have $2e > 0$ inequivalent cusps it depends on $2g + 2e - 1$ continuous real parameters and a discrete parameter that can take $m_1, \ldots, m_k$ values.

For compact $D$ one sees that there are no forms of half odd weight if

$$\sum_{i=1}^{k} (2g - 2 + \sum_{i=1}^{k} (1 - \frac{1}{m_i}))$$

is an odd number. This is the case if some $m_i$ are even and the $m_i$ of highest parity occur in an odd number.

In this case, and only in this case, is the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ contained in the commutator subgroup of the matrix group $\Gamma$. 