Dear Professorbilt,

while trying to formulate clearly the question I was asking you before Chemistalk I was led to two more general questions, your opinion of these questions would be appreciated. I have not had a chance to think over these questions seriously and I would not ask them except as the continuation of a casual conversation. I hope you will treat them with the tolerance they require at this stage. After I have asked them I will comment briefly on their genesis.

It will take a little discussion but I want to define some Euler products which I will call Artin-Hecke L-series because the Artin L-series, the L-series with Grössencharakter, and the series introduced by Hecke into the theory of automorphic forms are all special cases of these series. The first question will be of course whether or not these series define automorphic functions with functional equations. I will say a few words about the functional equation later. The second question I will formulate later. It is a generalization of the question of whether or not abelian L-series are L-series with Grössencharakter. Since I want to formulate the question for automorphic forms on any reductive group I have to assume that certain results in the reduction theory can be pushed a little further than they have been so far.

Unfortunately I must be rather pedantic but let k be the rational field or a completion of it. Let T be a product of simple
are simply connected. The case that the product is empty and $E = 123^r$ not without interest. For a split Cartan subgroup $\tilde{\Gamma}$ and let $\tilde{L}$ be the lattice of weights of $\tilde{\Gamma}$. $\tilde{L}$ contains the roots. I want to define the conjugate group to $G$ and the conjugate lattice to $\tilde{L}$. It is enough to do this for a simple group $G$ we can then take direct products and direct sums. If $G$ is abelian and simple let $\tilde{L}'$ be any sublattice of $\tilde{L}$ and $\tilde{L}''$, the conjugate lattice, be the dual of $\tilde{L}'$ (that is, $\text{Hom}(\tilde{L}'', \tilde{Z})$).

If $\tilde{L}'$, the dual of $\tilde{L}$, let $G'$ be a one-dimensional subgroup whose lattice of weights is identified with $\tilde{L}'$. If $G'$ is simple and non-abelian let $\tilde{L}$ be the lattice generated by the roots and let $\tilde{L}''$ be the dual of $\tilde{L}'$. $\tilde{L}''$ contains $\tilde{L}'$, the dual of $\tilde{L}$. Choose for each root $\alpha$ an element $H_\alpha$ in the Cartan subalgebra corresponding to $\tilde{L}$ in the usual way so that $\alpha(H_\alpha)$. The linear functions $\alpha(x) = x(H_\alpha)$ generate $\tilde{L}'$. There is a unique simply connected group $G''$ whose lattice of weights is isomorphic to $\tilde{L}$ in such a way that the roots of $G''$ correspond to the elements $\alpha_\alpha$. Fix simple roots $\alpha_1, \ldots, \alpha_r$ of $G''$, then $\text{Ext}_3 \ldots, \text{Ext}_s$ can be taken as the simple roots of $G''$. Now return to the general case.

If $L$ is a lattice lying between $\tilde{L}'$ and $\tilde{L}$ we can associate to $L$ in a natural way a group $G$ containing $G$. The dual lattice $\tilde{L}''$ lies between $\tilde{L}'$ and $\tilde{L}$. It determines a group $G''$, containing $G$ which contains the conjugate of $G$ by $\tilde{L}''$. Let $\tilde{L}$ be the Lie algebra of $\tilde{L}$ and $\tilde{L}''$ be the Lie algebra of $\tilde{L}''$. Choose for each root $\alpha$ a root vector $X_\alpha$ so that the conditions of simplicity are satisfied. Also let $\tilde{L}$ be a split Cartan subalgebra of $\tilde{L}$ for each root $\alpha$. Choose a root vector $X_\alpha$ so that the conditions of simplicity are satisfied. Let $\tilde{A}$ be the group of automorphisms of $\tilde{L}$ which take $\tilde{L}$ to itself, and take $\tilde{L}, \tilde{L}, \tilde{L}'$ to themselves. Define $\tilde{G}$ in a similar fashion. $\tilde{A}$ is the centralizer in $\tilde{G}$ of $\tilde{L}$ so that $\tilde{L}$ and $\tilde{L}$ are
In order to define the local factors of the L-series I have to recall some facts about the Hecke algebra of \( G^0 \), when \( K \) is an unramified extension of the p-adic field. If we choose a maximal compact subgroup of \( G^0 \) in a suitable manner, then, according to Borel and Satake, the Hecke algebra is isomorphic to the subalgebra of elements in the group algebra of \( \mathcal{L}^0 \), the set of elements in \( \mathcal{L} \) finishing \( \mathcal{L}^0 \), which are invariant under the restricted Hecke group \( \mathcal{W}^0 G \) of \( G^0 \).

(Actually we have to stretch their results a little.) Thus any homomorphism \( \chi \) of the Hecke algebra into the complex numbers can be extended to a homomorphism \( \chi' \) of the group algebra of \( \mathcal{L} \) into the complex numbers. There is at least one element \( g \in \mathcal{L} \) so that

\[
f = \sum_{\lambda \in \mathcal{L}} a_\lambda \xi \lambda \quad (\xi \lambda \text{ written multiplicatively})
\]

then \( \chi'(f) = \sum a_\lambda \xi \chi' (\lambda) \).

The semi-direct product \( \mathcal{L} \rtimes G^0 \) is a complex group, let \( \pi \) be a complex representation of \( \mathcal{L} \). If \( \sigma \) is the Frobenius then

\[
\frac{1}{[G^0(1) - \pi(G^0)]} (x \text{ an indeterminate})
\]
is the local zeta function corresponding to $\chi$ and $\pi$. Hence to verify that it depends only on $\chi$ and not on $g$, if $\lambda$ is any weight let $n_\lambda$ be the lowest power of $\sigma$ which fixes $\lambda$ and if $n_\lambda \neq 0$ and $\pi h \in \pi$ let $\chi(n)$ be the trace of $\sigma$ on

$$\{ v \in \pi h : v = \xi(n) \}$$

for all $h \in T^\times$.

Then

$$\log \frac{1}{1 - x \pi(n, x)} = \sum_{\lambda \in \Lambda} \chi(\lambda) \frac{1}{1 - x \pi(\lambda, x)} = \sum_{\lambda \in \Lambda} \chi(\lambda) \sum_{n \geq 0} x^n \pi(\lambda, x^n)$$

Moreover if $\omega$ is a character of $G^\times$ we can always choose a representation of it which commutes with $\sigma$. Then the local zeta function does not change if $g^\lambda$ by $w^{-1} w$ so it equals

$$\frac{1}{[\chi, \omega]} \sum_{\lambda \in \Lambda} \pi(\lambda) \chi(\lambda) \xi(n) \sum_{\gamma \in G^\times} \xi(\gamma) (\sum_{k=0}^{\infty} \xi(\gamma^{-1} x^k)) \omega (\gamma)$$

Since

$$\sum_{\lambda \in \Lambda} \xi(\lambda) (\sum_{k=0}^{\infty} \xi(\gamma^{-1} x^k))$$

belong to the image of the Hecke algebra. The assertion is verified.

I don't know if it legitimate but let us assume that the character of the complex representation $\chi$ takes the semi-simple conjugacy classes in $G^\times$. Thus by the above theorem associate to each homomorphism $\chi : \mathfrak{g} \to \mathbb{C}$ the complex numbers the conjugacy class of the semi-simple
element \( g \). Conversely, given a semi-simple conjugacy class in \( G \times \mathbb{C}^G \), it contains, by Bäzler-Noether, an element in the normalizer of \( T \). Thus it even contains an element which takes positive roots into positive roots. Thus, if the projection of the conjugacy class on \( G \) (and behavior of \( g \)) is \( \sigma \), the conjugacy class contains an element of the form \( \sigma \cdot g \cdot \sigma^{-1} \).

As above, \( g \) determines a homomorphism of the Hecke algebra into the complex numbers. If this homomorphism \( \chi \) is completely determined by the local \( \mathfrak{g} \)-functions attached to \( \mathfrak{g} \), then it is completely determined by the conjugacy class, \( \sigma \cdot g \), and we have a one-to-one correspondence between homomorphisms of the Hecke algebra into the complex numbers and semi-simple conjugacy classes in \( G \times \mathbb{C}^G \) whose projection on \( G \) is \( \sigma \). It is enough to check that the value of \( \chi \) on an element of the form \( \sum_{\alpha \leq \mathfrak{h}} \chi(\alpha) \mathfrak{h}^\alpha \) when \( \sum_{\alpha \leq \mathfrak{h}} \mathfrak{h}^\alpha \) belongs to the positive Weyl chamber is determined by the local \( \mathfrak{g} \)-functions. This can be done by the usual sort of induction for \( \sum_{\alpha \leq \mathfrak{h}} \mathfrak{h}^\alpha \), in which case \( \mathfrak{h} \) is invariant under \( G \) and then the highest weight of a representation of \( \mathfrak{g} \times \mathbb{C}^G \) whose restriction to \( \mathbb{C}^G \) is irreducible.

Now I am going to try to define the Artin-Hecke \( L \)-series.

To do this let us fix for each \( p \) an embedding of \( \overline{\mathbb{Q}}_p \) into \( G \). Then the algebraic closure of \( \mathbb{Q}_p \) in \( \overline{\mathbb{Q}}_p \) we will have to come back later and check that these \( \overline{\mathbb{Q}}_p \) are independent of these choices. The choice will be implicit in the next paragraph.

Suppose we have a twisting from \( \overline{G} \) of \( G \) over the rationals. The twisting can be accomplished in two steps. First for a suitable Galois extension of \( K \) take a homomorphism \( \delta \) of \( G \) into \( \mathbb{C}^G \). Then take an inner twisting of \( \mathbb{C}^G \) by means of the
conclude $\lambda_2 | \lambda_0$. Let me assume the truth of the following:

(i) Suppose $G$ is a linear group acting on $V$. Let $V$ be a heredity lattice in $V_\mathbb{Q}$. Then the intersection of $G \mathbb{Q}_p$ with the stabilizers of $L \otimes \mathbb{Z}_p$ ($\mathbb{Z}_p$ is the ring of integers in $\mathbb{Q}_p$), so far almost all one of the maximal compact subgroups referred to above.

(ii) For almost all $p$, the restriction of $\lambda_2$ to $G_f(K \mathbb{Q}_p/\mathbb{Q}_p)$ is $\mathbb{Q}_p$ splits. Moreover, there is a $b$ in the intersection of $G_{K_p}$ with the stabilizers of $L \otimes \mathbb{Z}_p$ so that $a_0 = b\otimes b^{-1} \sigma \in G_{K_p}$.

Now take $p$ satisfies (ii) and (iii) of which does not ramify in $K$. Since $G_0 \mathbb{Q}_p = \{g \in G_{K_p} \mid g_{\sigma_0} = g \} \sigma_0 \in G_{\mathbb{Q}_p}$ the map $g \mapsto g_0$ is an isomorphism $G_{\mathbb{Q}_p}$ with $G_{K_p}$. Moreover, we can take $G_{Z_p}$ to be the intersection of $G_0 \mathbb{Q}_p$ with the stabilizers of $L \otimes \mathbb{Z}_p$ so the map takes $G_{Z_p}$ to $G_{K_p}$. The inductive map isomorphism of the Hecke algebras is independent of the choice of $b$. Now $G_{A_p} = \prod G_{K_p}$.

Suppose we have an automorphic form $\phi$ on $G \mathbb{Q}_p \backslash G_{A_p}$ which is an eigenfunction of the Hecke algebras for almost all $p$. Then, for almost all $p$, we have a homomorphism of the Hecke algebra into the complex numbers and thus a semisimple conjugacy class $a_p$ in $G_0 \mathbb{Q}_p \times G \leq G_0 \times G$. If $\pi$ is a complex representation of $G_0 \times G$, we define the Artin-Helton L-series

$$L(s, \pi, \phi) = \prod_{p} \frac{1}{1 - \left(1 - \left(\pi(p) - \pi(p)^{-1}\right)\phi(p)\right)_{p}^{s}}$$

(Product is taken over almost all $p$)
I have to check that these series are independent of the embedding of $\overline{Q}$ into $\overline{Q}_p$. For the moment fix $p$. We have used the original embedding to identify $\overline{Q}$ with a subfield of $\overline{Q}_p$, but we preserve this identification. Any other embedding is obtained by sending $x \mapsto x^2 \text{ with } \sigma \in \text{Gal}(\overline{A}/Q)$. If we use the original embedding to identify $\overline{Q}_p$ with a subgroup of $\overline{Q}$ then the map $\overline{Q}_p$ into $\overline{Q}$ given by the new embedding $\sigma \mapsto 2\sigma^2$ (I denote $\sigma$ with its image in $\overline{Q}$). The restriction of $\sigma$ to $\overline{Q}_p$ is replaced by $\sigma^2$ with $\sigma^2 = \sigma(12)$. Then $G_{Q_p}$ is replaced by $G_{Q_p}$. The map $g \mapsto g^4$ is an isomorphism of $G_{Q_p}$ with $G_{Q_p}$ if $g \in G_{Q_p}$. Then $g$ is the image of $g^4$ so this map commutes with the embedding $G_{Q_p}$ in the two groups. The new embedding $\sigma \alpha_{\sigma^2}$ is the image of $a_{\sigma^2} = a_{\sigma} a_{\sigma}^{-1} = a_{\sigma^2}^{-1} a_{\sigma}^{-1} = a_{\sigma^2} a_{\sigma}^{-1} = a_{\sigma^2} a_{\sigma}$ since $a_{\sigma^2} = a_{\sigma^2}$ for all $\sigma$ and $\alpha$. The map is $\sigma(\alpha^2) a_{\sigma} a_{\sigma}^{-1} \delta(\alpha^2)$. Thus

$$\overline{G}_{Q_p} = \{ g \in G_{Q_p} \mid g^4 \sigma(\alpha^2) a_{\sigma} a_{\sigma}^{-1} \delta(\alpha^2) = g \sigma(\alpha^2) a_{\sigma} a_{\sigma}^{-1} \delta(\alpha^2) \}$$

for $\sigma \in \text{Gal}(\overline{A}/Q)$. And the map $g \mapsto g^4(\alpha^2)$ is an isomorphism of $G_{Q_p}$ with $G_{Q_p}$ It commutes with the embeddings $\overline{G}_{Q_p}$ in the two groups, since $\overline{G}_{Q_p} = \{ g \in G_{Q_p} \mid g^4(\alpha^2) = g \text{ for all } g \}$. Moreover for almost all $p$ it takes $\overline{G}_{Q_p}$ to $\overline{G}_{Q_p}$. So then we choose for each $p$ a new embedding except a new faithful group $\overline{G}_A$. The above
maps define an isomorphism $\mathbb{Q}/\mathbb{Z}$ with $\mathbb{Q}$ which takes $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{Q}$ itself. Then we have a map $\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\mathbb{Q}/\math{
It is the representation contragredient to $T$.

Before I go into the second question let me just say that I have been making some experiments with Eiseinstein series and although the work is far from completed, it looks as though we will get some series of the above type which because of their relation to the Eiseinstein series will be isomorphic in the whole plane. It might even be possible to get a functional equation in a smaller number of cases from the functional equations of the Eiseinstein series. The definitions above are the result of trying to find some class of Euler products which will contain the ones coming from the Eiseinstein series but which are not restricted in any artificial fashion.

Now if $G = G(\epsilon)$ and the action of $\theta$ is trivial, and if is the representation $g \rightarrow g$ one can perhaps use the ideas of Tannagawa handle to get the above series. This leads to the second question.

Suppose we have $K$, $G$, and $\delta$ as above, and also $K'$, $G'$, and $\delta'$.

If $K \subseteq K'$ we have a homomorphism $\theta' \rightarrow \theta$. Suppose moreover that $\psi$ is a homomorphism of $G' \times G$ into $G \times G$ so that the following diagram commutes:

$$
\begin{array}{ccc}
G' \times G & \rightarrow & G' \\
\downarrow & & \downarrow \\
G \times G & \rightarrow & G
\end{array}
$$

If $\phi'$ is an automorphism from some inner form of $G' \times G'$, then the condition we had above then for almost and $\phi'$ defines a conjugacy class $\phi$ in $G' \times G'$. 
let $\phi$ be the map of $G'$ in $G_1 \times G_2$. The second question in the following. Is there a automorphic form $\phi$ associated to some inner form $G^*$ of $G^*$ such that for almost all the conjugacy classes associated to it is $\phi$.

Let me give some idea of what an affirmative answer to the question entails.

(i) Take $G_1' = G_1$ and let $G'$ be a split torus of rank equal to the rational rank of $G_1^*$ on which $G_1'$ acts trivially. Let $A$ be a maximal split torus of $G_1^*$. Since $G_1$ acts trivially on $A$

(ii) Now let me say a few words about the relation of the question to the Artin reciprocity law. For the rational field take $G_1' = G_1^*$ and let $\chi$ be an automorphic of $G_1'$. Let $G_1 = G_1'$ and let $G$ be a one-dimensional split torus in which $G_1$ acts trivially

(iii) to take $2 \times 1$ to $2 \times \chi(2)$. Then an affirmative answer...
$K/K_2$ is Galois and $K/K_2$ is abelian. Let $G = G(K/K_2)$, let $G_2$ be the elements of $G$ which fix $K_2$ and let $G_2'$ be the elements of $G$ which fix $K'_2$. Finally, suppose that $K_2$ is a characteristic field of $K_2$. Let $e = \log_2 G_2$ and $e' = \log_2 G_2'$, hence $e' \leq e$. Take $e = e_1 + e_2$ with $e_1 \in G_2$. Suppose that $e_1$ is such that $e_1, e_2 \in G'_2$. Then $G$ be the direct product $T_1 \times T_2$ of $e$ monodimensional split tori. Define $\xi$ by $(e_1, e_2) \in T_2 = e_{\sigma} 	imes e'_{\sigma}$. It is easy to check that $\psi(e_1, e_2) = \psi(e_2)$. Moreover $G'$ is just the multiplicative group of $K_2$. Also $G' = G$. Define $\psi_1(e_1)$ by $\psi_1(e_2) = \psi_1(e_2) \in T_2$. Then $\psi_1(e_2) = \psi_1(e_2) \in T_2$. Define $\omega$ by

$$\omega(\sigma 	imes \xi) = \sigma \times (X(\psi_1(e_1)) \times \cdots \times X(\psi_1(e_n)))$$

Then

$$\omega(\sigma 	imes \xi) \omega(\xi 	imes \sigma) = \sigma \times \prod_{i=1}^e \chi(\psi_1(e_i)) \psi_1(e_i)$$

$$= \omega(\sigma \xi \sigma).$$
By the way if the $\xi_i, 1 \leq i \leq r$ are replaced by $\xi_i' = \mu_i^{-1} \xi_i$ with $\mu_i \in G_1$ then $p_i' \circ \sigma = \mu_i^{-1} p_i \circ \sigma \mu_i$ and

$$\omega'(\sigma \times 2) = (\chi(\mu_1) \times \cdots \times \chi(\mu_r))^{-1} \omega(\sigma \times 2) (\chi(\mu_1) \times \cdots \times \chi(\mu_r))$$

so the map does not depend in an essential way on the choice of representation.

I will take $\phi'$ to be a constant. By the Artin reciprocity law there is associated to $\chi$ a character of $K_1^* \backslash K_1$ that is an automorphic form on $G_Q \backslash G_A$. I claim that $\phi$ is the automorphic form which provides an affirmative answer to the question.
To show this we make use of the freedom we have in the choice of coset representatives. Let \( p \) be a prime which does not ramify \( k \).

Fix an embedding of \( \overline{k} \) in \( \overline{\mathbb{Q}_p} \). We identify \( \mathbb{Q}_p \) with its image. Let \( \overline{\mathbb{Q}_p}^* \) be the punctured disk of \( \overline{k} \). Choose \( \mu_1, \ldots, \mu_r \) in \( \overline{\mathbb{Q}_p}^* \) so that the map \( x \mapsto x \mu_j \) extends to a continuous map of the completion \( \overline{k} \) with respect to \( \mathfrak{p}_j \) into \( \overline{\mathbb{Q}_p} \). Let \( L_j = \mu_1 \overline{\mathbb{Q}_p}^* \mu_1^{-1} \mathfrak{p}_j \) and let \( \mathfrak{p}_j = L_j \mathfrak{p}_j \). If \( \sigma \) is the Frobenius, the automorphisms \( \mu_j \sigma \) are \( \mathfrak{p}_j \)-adic, as \( k \) is a finite extension. From a set of representatives for the cosets of \( \mathfrak{p}_j \), \( \mathfrak{p}_j = \mu_j \sigma \mathfrak{p}_j \) and \( \mathfrak{p}_j \sigma \mathfrak{p}_j \) are in the Frobenuis conjugacy class of \( k \) corresponding to \( \mathfrak{p}_j \).

On the other hand, \( G_{\mathfrak{p}_j} \) is the set of elements of the form \( \prod_{j=1}^r \prod_{k=0}^{d-2} x_k \), with \( x_k \) a non-zero element in \( \mathfrak{p}_j \). The restriction of \( \phi \) to such an element is, by (1),

\[
\overline{\mathbb{Q}_p}^* \chi(x_1 \sigma \sigma^{-1} \sigma_j) = \frac{1}{\prod_{j=1}^r} \chi(x_1 \sigma \sigma^{-1} \sigma_j) = \prod_{j=1}^r \chi(x_1 \sigma \sigma^{-1} \sigma_j)
\]

where \( x_1 = \sigma \sigma_j(x_1) \). Since \( \chi \sigma \sigma_j = \chi \sigma_j \), the associated conjugacy is the one determined by any element

\[
\sigma \prod_{j=1}^r \prod_{k=0}^{d-1} \sigma_j \sigma_k
\]

such that

\[
\prod_{j=1}^r \prod_{k=0}^{d-1} \sigma_j \sigma_k \sigma_j^{-1} \sigma_k^{-1} = \prod_{j=1}^r \chi(x_1 \sigma \sigma^{-1} \sigma_j)
\]
Looking above we see that $\omega(p)$ is such an element.

Finally I want to comment on the implications an affirmative answer to the second question might have for the problem of finding a splitting law for non-abelian extensions. I had planned to discuss arbitrary ground fields but imagine now that I have to take the ground field to be $\mathbb{Q}$. However one could probably go back and reformulate the two questions in the context of groups over a number field. The first question is not sensitive to the choice of ground field but the second is. I did not appreciate this until now, since it is would be gained by rewriting the letter I content myself with taking the ground field to be $\mathbb{Q}$.

Let $K$ be a Galois extension of $\mathbb{Q}$ and let $G = \text{Gal}(K/\mathbb{Q})$.

We want a method of finding for almost all $p$ the Frobenius, conjugacy class $\sigma(p)$ in $G$. Thus we have to find trace $\text{Tr}(\sigma(p))$ of the conjugacy class of $\sigma(p)$ in $\text{GL}_n(\mathbb{C})$. We seek a representation $\tilde{\rho}$ of $G$ such that $\text{Tr}(\tilde{\rho}(g))$ is the trace $\text{Tr}(\rho(g))$. As before I will take $G' = G$, $G' = \text{SL}_n$ and $p$ to be a constant function $\text{Tr}$ and take $G = \text{GL}_n$. Let us check that $G$ is also $\text{GL}_n$.

Take $\tilde{G} = \tilde{G} = A \times \text{SL}_n$ where $A$ is a one-dimensional splitting $\mathbb{Q}$-

split torus. Then

$$\text{cL} = \tilde{\text{cL}} = \{(z, z_1, \ldots, z_m) \mid z, z_1, \ldots, z_m \in \mathbb{Z}, \sum_{i=1}^{m} z_i = 0\}$$

$$\text{cL} = \text{L} = \{(z, z_1, \ldots, z_m) \mid z_1, z_2, \ldots, z_m \in \mathbb{Z}, \sum_{i=1}^{m} z_i = 0\}$$

$$\tilde{\text{cL}}' = \tilde{\text{L}} = \{(m, z_2, z_2, \ldots, z_m, z_1, -z_1, -z_2, \ldots, -z_m) \mid z, z_1 \in \mathbb{Z}\}$$

The pairing is given by

$$\langle (z, z_1, \ldots, z_m), (y_1, y_1, \ldots, y_m) \rangle = oz \sum_{i=1}^{m} z_i y_i,$$
In any case \( G = G_{G} = G_{G'} \). Define \( \omega \) by

\[
\omega(\sigma \times \lambda) = \sigma \times \Pi(\sigma)
\]

The action of \( G' \) on \( G_{G} \) is to be trivial. Since \( \omega(\sigma_{G}) = \sigma_{G} \) is the conjugating class of \( \sigma_{G} \times \Pi(\sigma_{G}) \), which of course determines the conjugating class of \( \Pi(\sigma_{G}) \), all we need is a method of finding \( \sigma_{G} \).

Suppose there is an automorphic form \( \phi \) on some inner term of \( G_{G} \), which provides an affirmative answer to the above question. To find \( \sigma_{G} \), we need to calculate the eigenvectors of a finite number of elements of the Hecke algebra \( H_{p} \) corresponding to the eigenfunction \( \phi \). Choose a finite subset of primes containing the infinite prime so that \( \tilde{G}_{s} = \prod G_{s} \tilde{G}_{Q} \), and \( \tilde{G}_{s} = \prod G_{s} \tilde{G}_{Q} \).

Then \( \tilde{G}_{A} = \tilde{G}_{Q} \tilde{G}_{s} \tilde{G}_{s} \) and \( \phi \) is a function on \( \tilde{G}_{Q} / \tilde{G}_{A} / \tilde{G}_{s} \).

Suppose \( p \neq s \) and \( f \) is the characteristic function of \( \tilde{G}_{Q} / \tilde{G}_{p} \), which is the disjoint union \( \bigcup \tilde{G}_{Q} / \tilde{G}_{p} \). If \( g \in \tilde{G}_{Q} / \tilde{G}_{p} \),

\[
\chi(f) \phi(g) = \int_{\tilde{G}_{Q} / \tilde{G}_{p}} \phi(gh) f(gh) \, dh
\]

\[
= \prod_{i=1}^{n} \phi(g_{i}) = \prod_{i=1}^{n} \phi(\lambda_{i} g_{i})
\]

Choose \( \lambda_{i} \) in \( \tilde{G}_{Q} / \tilde{G}_{p} \) so that \( \lambda_{i} \tilde{a}_{i} \in \tilde{G}_{s} \) and let \( b_{i} \) be the projection of \( \lambda_{i} \) on \( \tilde{G}_{s} \). If \( \chi(f) \) is the automorphic eigenvalue of \( f \),

\[
\chi(f) \phi(g) = \prod \phi(\tilde{a}_{i}^{*} g_{i})
\]

\[
= \prod \phi(b_{i}^{*} g)
\]
Now roughly speaking the elements $\bar{a}_p, \bar{a}_n$ are obtained by solving some diophantine equations involving $\pi$ as a parameter. 

Then $\Phi(h) = \phi(g)$ depends upon the congruence properties of $\bar{a}_p$ modulo powers of the finite primes $\pi, n$ and the projection of $\bar{a}_n$ on $\bar{T}_{G_0} = G_0$, for each $g \in G$, $\phi(h)$ as a function of $h$ in the connected component of $G_0$.

If $\bar{a}_n$ were rational we would get a good splitting law. It would be rather complicated but in principle not worse than the splitting law of Dedekind-Hasse for the splitting field of a cubic equation. However because of the strong approximation in $\Phi(h)$ will probably not be rational unless $m = 1, a$. Thus we could only get a transcendental splitting law.

Nonetheless if we took $G$ to be the symplectic group in $2n$ variables and $G_0$ to be the orthogonal group in $2n+1$ variables then strong approximation would be easier because $G$ has inner forms for which $G_0$ is compact and we might hope to obtain laws about embeddings of $G_0$ in $G$.

Yours truly,

R. Langlands

Postscript: Let $n$ be a quadratic extension of $Q$. Let $af^2 = af = af G_2(\bar{x} / \bar{a})$. Let $G = G = G_2(\mathbb{R})$, where $\mathbb{R}$ and $\mathbb{R}$ are some dimension split tori. If $\zeta$ is the non-trivial element of $G$, let $\zeta = \zeta(2)$ and let $\zeta$ act trivially on $G$. Define $\psi$ by

$$\psi(1 \times (t, 1, t)) = (t, 1, t, 1)$$

$$\psi(\sigma \times (1, 1, 1)) = (\sigma(t, 1, t, 1))$$

$G_0$ is just the identity group in $K$. Take $\sigma$ to be a Gaussian character. It is not inconceivable that they work, Hecke and Maass on the relation between...
I seen with Grössen character from a quadratic field and automorphic forms will provide an affirmative answer to the second question in this talk.