Dirichlet character \( \chi \) mod \( q \), \( \chi(m) = 0 \) for \( (m, q) \neq 1 \)
\[
\chi(m) = \chi(cn) \quad \text{if} \quad m \equiv cn \pmod{q}.
\]
totally multiplicative
\[
\chi(cn) \chi(cn) = \chi(cn^2).
\]
\[\chi(1) = 1.\]
If \( q = \prod p_i^{a_i} \), then

a character \( \chi \) mod \( q \) has a unique factorization
\[
(1) \quad \chi(n) = \prod p_i \chi_i(n),
\]
where \( \chi_i \) is a character mod \( p_i^{a_i} \).

If \( \chi(m) \) for \( (m, q) = 1 \) does not coincide with a character mod \( p_i^{a_i-1} \), we say that \( \chi_i \) is a primitive character mod \( p_i^{a_i} \). Finally, if in \((1)\) every \( \chi_i \) is primitive mod \( p_i^{a_i} \), we say \( \chi \) is a primitive character mod \( q \).

The number of distinct characters mod \( q \) is \( \phi(q) = q \prod \left( 1 - \frac{1}{p} \right) \) and the

number of primitive characters of
\[
\phi^*(q) = q \prod \left( 1 - \frac{2}{p} \right) \prod \left( 1 - \frac{1}{p} \right)^2
\]

The character \( \chi(1) = 1 \) if \( (m, q) = 1 \) is called the principal character and denoted by \( \chi_0(n) \). We have \( \sum_{m \equiv 0 \pmod{q}} \chi(m) = 0 \) for \( \chi \neq \chi_0 \)
\[\chi_0(n).\]
We also have
\[ \sum_{m \equiv \ell (\text{mod } q)} X(m) = 0 \quad \text{for } \ell \equiv 1 (\text{mod } q) \]
\[ \phi(q) \quad \text{for } \ell \equiv 1 (\text{mod } q). \]

Gauss or Jacobi - sums

Define
\[ T_{\chi}(m) = \sum_{\ell \pmod{q}} X(\ell) e^{2\pi i \frac{m\ell}{q}}, \]
(2)

If \( \chi \) is a primitive character mod \( q \) and \( (m, q) > 1 \) then \( T_{\chi}(m) = 0 \).

(Proof: if \((m, q) > 1\) there is a \( p \mid (m, q) \), combine terms where \( \ell \) belongs to same residue class mod \( \frac{q}{p} \), and show sum is zero.)

If \((m, q) = 1\), then
\[ T_{\chi}(m) = \overline{X}(m) \sum_{\ell \pmod{q}} X(\ell \ell') e^{2\pi i \frac{m\ell}{q}} = \overline{X}(m) T_{\chi}(1) \]

on writing \( T_{\chi} \) for \( T_{\chi}(1) \), \( T_{\chi}(m) = \overline{X}(m) T_{\chi} \)

Thus
\[ \sum_{\ell \pmod{q}} |T_{\chi}(\ell)|^2 = \phi(q) |\overline{T_{\chi}}|^2 \]

But also
\[ \sum_{\ell \pmod{q}} |T_{\chi}(\ell)|^2 = \sum_{\ell \pmod{q}} \sum_{k \pmod{q}} X(\ell) \overline{X}(k) e^{2\pi i \frac{m(\ell-k)}{q}} \]
\[ = q \sum_{\ell} |X(\ell)|^2 = q \phi(q) \]
\[ \sum_{\ell} |X(\ell)|^2 = q \phi(q) \]

So
\[ |\overline{T_{\chi}}|^2 = q \].

As \( \overline{T_{\chi}} = \chi(1) \overline{T_{\chi}} \), we get
\[ T_{\chi} \overline{T_{\chi}} = \chi(1) \phi(q) \].
Poisson Summation formula:
If \( f(u) = \int_{-\infty}^{\infty} f(u) \sin(\pi u) \, du \),
then (under conditions always satisfied in the cases when we use this formula later) we have
\[
\sum_{m=-\infty}^{\infty} \hat{f}(m) = \sum_{m=-\infty}^{\infty} f(m),
\]

There is a generalization, if \( \chi \) is a primitive character mod \( q \), then
\[
(3) \quad \sum_{m} \chi(m) \hat{f}\left( \frac{m}{q} \right) = \sum_{m} \chi(m) \hat{f}\left( \frac{m}{q} \right),
\]
where \( |\chi| = 1 \).

We have
\[
\sum_{m} \chi(m) \hat{f}\left( \frac{m}{q} \right) = \sum_{\chi(q)} \chi(e) \sum_{m} \hat{f}\left( \frac{m+e}{q} \right).
\]

From classical Poisson formula
\[
\sum_{m} \hat{f}\left( \frac{m}{q} \right) = \frac{1}{\sqrt{q}} \sum_{\chi} \hat{f}\left( \frac{\chi}{q} \right) e^{-\pi i \frac{\chi}{3}}
\]
\[
= \sum_{m} \chi(m) \hat{f}\left( \frac{m}{q} \right) = \frac{1}{\sqrt{q}} \sum_{\chi} \hat{f}\left( \frac{\chi}{q} \right) \chi(-1)
\]
\[
= \frac{\chi(-1) \hat{f}_X}{\sqrt{q}} \sum_{m} \chi(m) \hat{f}\left( \frac{m}{q} \right),
\]
which proves (3) with \( \sum_{m} e^{-\frac{2\pi i m q}{q^2}} = \frac{\chi(-1) \hat{f}_X}{\sqrt{q}} \).
Poisson Summation formula:
\[ \hat{f}(v) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i v n} \]

Then (under suitable conditions)
\[ \sum_{n} f(n) = \sum_{n} \hat{f}(n) \]

There is a generalization if \( \chi \) is a primitive character mod \( q \), then
\[ \chi \sum_{n} \chi(n) \hat{f}(\frac{mn}{q}) = \chi \sum_{n} \chi(n) \hat{f}(\frac{mn}{q}) \]
where \( \chi = \chi \hat{\chi} \) and \( |\chi| = 1 \).

We have
\[ \sum_{n} \chi(n) \hat{f}(\frac{mn}{q}) = \sum_{n} \chi(n) \sum_{n} \hat{f}(\frac{mn}{q}) \]

From classical Poisson formula
\[ \sum_{m} \hat{f}(\frac{mn+\epsilon}{q}) = \frac{1}{\sqrt{q}} \sum_{m} \hat{f}(\frac{mn}{q}) e^{-2\pi i \frac{\epsilon}{q}} \]

So
\[ \sum_{n} \chi(n) \hat{f}(\frac{mn}{q}) = \frac{1}{\sqrt{q}} \sum_{n} \hat{f}(\frac{mn}{q}) \sum_{n} \chi(n) \]
\[ = \frac{\chi(-1)^{2} x}{\sqrt{q}} \sum_{n} \chi(n) \hat{f}(\frac{mn}{q}) \]
which proves (3) with \( \chi \frac{1}{\sqrt{q}} = \frac{x(-1)^{2} x}{\sqrt{q}} \)}
For \( \sigma > \sigma_0 \), define
\[
\zeta(s, \chi) = \sum_{n=1}^{\infty} \frac{x(n)}{n^s} = \prod_{p} \left( 1 - x(p)p^{-s} \right)^{-1},
\]
we assume \( \chi \) is a primitive character mod \( q \); and include the case \( q = 1 \), with \( x(n) = 1 \) for all integers \( n \). In that case we may write \( \zeta(s) \) instead of \( \zeta(s, \chi) \).

If we want to extend the domain of definition beyond \( \sigma > 1 \), we can write \( \zeta(s, \chi) \) as a Stieltjes integral
\[
\zeta(s, \chi) = \int_{\frac{1}{2}}^{\infty} \frac{d \chi(x)}{x^s};
\]
where we have written
\[
\sum_{n \leq x} x(n).
\]

Then, by partial integration,
\[
\zeta(s, \chi) = \left[ \frac{\chi(x)}{x^s} \right]^{\infty}_{\frac{1}{2}} + s \int_{\frac{1}{2}}^{\infty} \frac{\chi(x)}{x^{s+1}} dx
\]
\[
= s \int_{\frac{1}{2}}^{\infty} \frac{\chi(x)}{x^{s+1}} dx.
\]

For \( q > 1 \), \( |d \chi(x)| \) is bounded < \( q \); so \( \zeta(s, \chi) \) is regular for \( \sigma > 0 \) and
\[
|\zeta(s, \chi)| < Aq q (1+\xi) \text{ for } \sigma > \delta > 0.
\]
If \( q = 1 \) we have \( S(x) = \lfloor x \rfloor \), so
\[
S(x) = \sum_{m=1}^{\infty} a_n x^n = 5 \int_{-\infty}^{\infty} \frac{e^{-\pi x^2}}{x^{2r+1}} \, dx =
\]
\[
= 5 \int_{-\infty}^{0} \frac{e^{-\pi x^2}}{x^{2r+1}} \, dx - 5 \int_{0}^{\infty} \frac{e^{-\pi x^2}}{x^{2r+1}} \, dx =
\]
\[
= \frac{e^{-\pi x^2}}{x^{2r+1}} \bigg|_{-\infty}^{0} - 5 \int_{0}^{\infty} \frac{e^{-\pi x^2}}{x^{2r+1}} \, dx
\]
Thus \( S(x) = \frac{1}{2r-1} \) is regular for \( 0 < x < 1 \) and bounded by \( A_n (1 + |x|) \).

To effect analytic continuation of these functions in the whole complex plane, we go back to the generalized Poisson formula (3). We have
\[
f^n(x) = x^\frac{1}{2} e^{-\pi u^2 x} \text{ then } f^n(x) = x^\frac{1}{2} e^{-\pi u^2 x},
\]
and of
\[
g^n(u) = u^\frac{3}{2} e^{-\pi u^2 x} \text{ then } g^n(u) = i u^\frac{3}{2} e^{-\pi u^2 x},
\]
If \( \chi(-1) = 1 \), we put the above \( f^n(u), f^n(u) \)
in (3) and get for \( n = 1 \) since \( f \) and \( f^n \) are even
\[
\sum_{n=0}^{\infty} s^n(x) x^{-\frac{1}{2} - \frac{\alpha_2^2}{x}} = \sum_{n=0}^{\infty} \chi(u) x^{-\frac{1}{2} - \frac{\alpha_2^2}{x}}
\]
which we may rewrite as

\[ (4') \sum_{\chi} x^\frac{1}{4} \theta_\chi(x) = \overline{\sum_{\chi} x^{-\frac{1}{4}} \theta_\chi(\frac{1}{x})}, \]

where \( \theta_\chi(x) = \sum_{n=1}^{\infty} \chi(n) e^{-\pi \frac{n^2}{x}}. \)

For an odd character where \( \chi(-1) = -1 \), we put the functions \( g \) and \( \tilde{g} \) in (3) and get when we multiply by \( g \sqrt{x} \),

\[ \sum_{\chi} \frac{8}{3} \chi(m) x \frac{1}{2} n e^{-\pi \frac{n^2}{x}} = \]

\[ = i \overline{\sum_{\chi} \frac{8}{3} \chi(n) x^{-\frac{3}{4}} e^{-\pi \frac{n^2}{x}}} \]

\[ = i \sum_{\chi} \frac{8}{3} \chi(n) x^{-\frac{3}{4}} e^{-\pi \frac{n^2}{x}}, \]

\[ \sum_{\chi} x^{\frac{1}{4}} \theta_\chi^*(x) = i \overline{\sum_{\chi} x^{-\frac{3}{4}} \theta_\chi^*(\frac{1}{x})}, \]

where we have put

\[ \theta_\chi^*(x) = \sum_{n=1}^{\infty} \chi(n) e^{-\pi \frac{n^2}{x}}. \]

For \( q=1 \), we get putting \( t \) and \( \tilde{t} \) in the classical Poisson formula, that

\[ x^\frac{1}{4} (1 + 2 \sum_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}}) = x^{-\frac{3}{4}} (1 + 2 \sum_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}}), \]

since the presence of the constant term in the brackets will cause some difficulty.
We have for $a > 0$, $\sigma > 0$:

\[
\int_0^\infty x^\frac{\sigma}{2} e^{-ax} \frac{dx}{x} = \frac{\Gamma(\sigma)}{a^{\sigma}}.
\]

Consider first case $\sigma > 1$; $\chi$ even, we have for $\sigma > 1$:

\[
\int_0^\infty x^{\frac{\sigma}{2}} \Theta_{\chi}(x) \frac{dx}{x} = \pi^{-\frac{\sigma}{2}} \frac{\varphi_{\frac{\sigma}{2}}}{\varphi_{\frac{1}{2}}} \Gamma\left(\frac{\sigma}{2}\right) \sum_{n=1}^\infty \chi(m) \alpha_{m}^{-\sigma/2}
\]

\[
= \pi^{-\frac{\sigma}{2}} \frac{\varphi_{\frac{\sigma}{2}}}{\varphi_{\frac{1}{2}}} \Gamma\left(\frac{\sigma}{2}\right) \mathcal{L}(s, \chi).
\]

Here the integral on the left-hand side exists for all complex $s$, and shows that $\pi^{-\frac{\sigma}{2}} \frac{\varphi_{\frac{\sigma}{2}}}{\varphi_{\frac{1}{2}}} \Gamma\left(\frac{\sigma}{2}\right) \mathcal{L}(s, \chi)$ is an integral function, and consequently $\mathcal{L}(s, \chi)$ is an integral function.

Furthermore, we have $\lambda_s(\chi')$:

\[
\xi \pi^{-\frac{s}{2}} \frac{\varphi_{\frac{s}{2}}}{\varphi_{\frac{1}{2}}} \Gamma\left(\frac{s}{2}\right) \mathcal{L}(s, \chi) = 3 \chi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} x^{-\frac{s}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x} = \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x}
\]

\[
= \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x} = \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x}
\]

\[
= \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x} = \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x}
\]

\[
= \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x} = \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x}
\]

\[
(7) \quad \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x} = \xi \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \Theta_{x}(\frac{1}{x}) \frac{dx}{x}
\]
We now consider the case when $\nu$ is odd, and consider
\[
\sum_{00}^{\infty} x^{\frac{1}{2 \nu}} \Theta_{x}^{(*)} (x) \frac{d x}{x} = \pi^{-\frac{\nu}{2}} q^{\frac{1}{2}} \ll (\frac{1}{2}, x) \sum_{n=1}^{\infty} x(n) m^{-x} \ni \frac{1}{2} \frac{\pi}{2} q \ll (\frac{1}{2}, x).
\]

Using now (5') in the same way as we before used (4'), we get further for $\nu (\nu - 1) = -1$, that

\[(8') \quad \overline{\frac{1}{\nu}} \pi^{-\frac{\nu}{2}} q^{\frac{1}{2}} \ll (\frac{1}{2}, x) =
\]

\[
= \frac{1}{\overline{\nu}} \pi^{-\frac{\nu}{2}} q^{\frac{1}{2}} \ll (\frac{1}{2}, x) \overline{\ll (1 - \nu, x)},
\]

writing $\nu' = \overline{\nu}$, this takes the form

\[(8') \quad \overline{\nu} \pi^{-\frac{\nu}{2}} q^{\frac{1}{2}} \ll (\frac{1}{2}, x) =
\]

\[
= \overline{\nu} \pi^{-\frac{\nu}{2}} q^{\frac{1}{2}} \ll (\frac{1}{2}, x) \overline{\ll (1 - \nu, x)}.\]

Consequences:

Writing $\xi (\nu, x)$ for the left-hand-side of (7) and (8'), the functional equation can be written in the form

\[(9) \quad \xi (\nu, x) = \xi (1 - \nu, \overline{x}), \]

\[
\overline{\xi (\nu, x)} = \xi (1 - \nu, \overline{x}).\]
9.

We can conclude that \( \xi(s, \kappa) \) is an integral function of \( s \), which is real on the line \( \sigma = \frac{1}{2} \).

All zeros of \( \xi(s, \kappa) \) lie in the strip \( 0 \leq \sigma \leq 1 \).

We still need to deal with the case \( q = 1 \), that is \( \xi(s, 1) \). Formula

\[
\frac{1}{x^\frac{1}{2}} \left( 1 + 2 \sum_{\nu=1}^{\infty} e^{-\pi \nu^2} \right) = x^{-\frac{1}{2}} \left( 1 + 2 \sum_{\nu=1}^{\infty} e^{-\frac{\pi \nu^2}{x}} \right)
\]

can not be dealt with in the same way because of the presence of the constant terms in the brackets.

The two operators \( x^{\frac{1}{2}} \frac{d}{dx} x^{-\frac{1}{2}} \) and \( x^{\frac{3}{2}} \frac{d}{dx} x^{-\frac{3}{2}} \) commute (actually they are both of the form \( x^{1-\alpha} \frac{d}{dx} x^\alpha = x \frac{d}{dx} + \alpha \)), their product

\[
D = \left( x \frac{d}{dx} \right)^2 - \frac{1}{16}
\]

will annihilate the constant terms on both sides of (6).

\( D \) is also easily seen to be self-adjoint.
with respect to the measure \( \frac{dx}{x} \).

Writing \( \Theta(x) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x} \),

we now consider, first for \( \sigma \geq 1 \),

\[
\int_0^\infty x^{\frac{\sigma}{2} - \frac{1}{4}} D\left(x^{\frac{1}{4}} \Theta(x)\right) \frac{dx}{x} =
\]

\[
= 2 \sum_{n=1}^{\infty} \int_0^\infty x^{\frac{\sigma}{2} - \frac{1}{4}} D\left(x^{\frac{1}{4}} e^{-\pi n^2 x}\right) \frac{dx}{x}.
\]

Here, for \( \alpha > 0 \)

\[
\int_0^\infty x^{\frac{\sigma}{2} - \frac{1}{4}} D\left(x^{\frac{1}{4}} e^{-\alpha x}\right) \frac{dx}{x} =
\]

\[
= \int_0^\infty x^{\frac{\alpha}{4}} - \alpha x D\left(x^{\frac{1}{4}}\right) \frac{dx}{x} =
\]

\[
= \frac{\Gamma\left(\frac{\alpha}{4}\right)}{\alpha} \int_0^\infty x^{\frac{\alpha}{4}} e^{-\alpha x} \frac{dx}{x} =
\]

\[
= \frac{\Gamma\left(\frac{\alpha}{4}\right)}{\alpha} \Gamma\left(\frac{\alpha}{4}\right) \alpha^{-\frac{\alpha}{4}}.
\]

Thus

\[
\int_0^\infty x^{\frac{\sigma}{2} - \frac{1}{4}} D\left(x^{\frac{1}{4}} \Theta(x)\right) \frac{dx}{x} =
\]

\[
= \frac{\Gamma\left(\frac{\alpha}{4}\right)}{\alpha} \pi^{-\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right) \sum_{n=1}^{\infty} n^{-\alpha} =
\]

\[
= \frac{\Gamma\left(\frac{\alpha}{4}\right)}{\alpha} \pi^{-\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right) \xi(\alpha).
\]