\( \xi(\alpha, x) = \xi(1-\alpha, x) \)

for \( x \) primitive and even, where \( \alpha > 0 \) and

\[ \xi(\alpha, x) = \sum_{\chi} \frac{-\alpha}{\phi} \frac{\xi}{\xi} \chi \left( \frac{x}{\varphi} \right) \zeta(1-\alpha, x), \]

and for \( x \) odd

\[ \xi(\alpha, x) = \sum_{\chi} \frac{-\alpha}{\phi} \frac{\xi}{\xi} \chi \left( \frac{x}{\varphi} \right) \zeta(1-\alpha, x), \]

and finally for the case \( \alpha = 1 \)

\[ \xi(1) = \xi(1-1, x), \]

where

\[ \xi(1) = \Delta(\alpha-1) \frac{-\alpha}{\phi} \frac{\xi}{\xi} \chi \left( \frac{x}{\varphi} \right) \xi(x). \]

The \( \xi \) are all integral functions with all of their zeros on the line \( \Re(s) = \frac{1}{2} \), and symmetrically to the line \( \Re(s) = \frac{1}{2} \).

**Lemma.** If \( f(z) \) is analytic in \( \Re(z) \)

\( 1 \leq R \); \( f(z) \neq 0 \) and \( |f(z)| < M \)

for \( |z| \leq R \), then the number of zeros of \( f(z) \) in the circle \( |z| < R \)

where \( R < R \).
is bounded by
\[
\frac{M}{\log \left| f(0) \right|} \log \frac{R}{R^*}.
\]

Denote the zeros of \( f(z) \) in the circle \( |z| = R \) by \( \alpha_i \), and write \( R \in (0, 1) \),
write
\[
q(z) = \prod \frac{z - \alpha_i}{R - \alpha_i^* R}
\]
We have \( |q(0)| = \prod \frac{\alpha_i^*}{R} \leq \left( \frac{2}{\pi} \right)^N \)
if \( N \) is the number of zeros.
Since \( \frac{f(z)}{q(z)} \) is regular in \( |z| \leq R \)
and \( |q(z)| = 1 \) on the boundary, we
have
\[
\left( \frac{2}{\pi} \right)^N |f(0)| \leq \frac{f(0)}{q(0)} \leq N,
\]
which gives the bound above.

Using Stirling's formula
\[
\log P(z) = \left( z - \frac{1}{2} \right) \log z - z + \log \sqrt{2\pi} + O(z^{-1}),
\]
valid outside any angle which contains
the negative real axis in its interior, we
get easily from our earlier estimations...
of \( \zeta(s, x) \) and \( \xi(s) \) for \( 0 \leq s \leq 1 \) that for \( |s - 2| < 2R \), \( \xi(s) \) is bounded by \( O(R^R) \) and \( \xi(s, x) \) by \( O(R \log R R) \). So the number of zeros in \( |s - 2| < R \) is found to be \( O(R \log R R) \).

More precise results are obtained by following the variation of the argument of \( \xi(s) \) or \( \xi(s, x) \) around the rectangle with the vertices

\[ 2, 2+iT, -1+iT, -1 \]

We shall denote the number of zeros in this rectangle by

\[ N(T) \] for \( \xi(s) \) (or \( \xi(s, x) \)) and

\[ N(T, x) \] for \( \xi(s, x) \) (or \( \xi(s, x, y) \)) and counting zeros that may lie on the short sides of the rectangle with one-half their multiplicity.
We may consider the variation of the argument of \( \xi(s) \) or \( \xi(s, x) \) from \( \frac{1}{2}, 2, 2 + \delta T, \frac{1}{2} + i T \), and the upper half of the rectangle will add the same amount by virtue of the functional equation.

A factor
\[
\pi^{-\frac{1}{2}} \frac{q^s}{\phi(s)} \left( \frac{q^s}{\phi(s)} \right) \text{ or } \pi^{-\frac{1}{2}} \frac{q^s}{\phi(s)} \Gamma\left( \frac{s+1}{2} \right)
\]
gives in both cases that the argument increases from \( \frac{1}{2} \) to \( \frac{1}{2} + i T \) by
\[
\frac{T}{2} \left( \log \frac{q^s}{2\pi} - 1 \right) + O(1).
\]
Also the variation of the argument of \( \xi(s) \) or \( \xi(s, x) \) on \( \sigma = 2 \) is seen to be bounded since
\[
\Re \xi(s, x) \geq 1 - \sum_{p \leq x} \frac{1}{p^2} = 2 - \frac{\pi^2}{6} > \frac{1}{3}.
\]

It remains to estimate the variation of the argument of \( \xi(s) \) or \( \xi(s, x) \) on the rectangle \( 2 + iT, \frac{1}{2} + iT \) and \( \frac{1}{2}, 2 \). We assume at first that one zero \( \rho = \beta + i \gamma \) has \( \gamma = T \) and we look at
\[ f(z) = L(2 + iT - z, x) + L(2 + iT - z, x), \]

we have \( |f(0)| > \frac{M}{3} \) and in the circle \( |z| \leq \frac{T}{4} \) we have
\[ |f(z)| < \frac{c}{f(1 + T)}. \]

Thus by our earlier lemma, the number of zeros of \( f(z) \) in \( |z| \leq \frac{T}{4} \)
is bounded by
\[ n < c \log f(2 + T). \]

But the zeros on the positive real axis are simply the points \( \text{where} \]
\[ L(2 + iT - z, x) \] is purely imaginary.

Between these the argument of \( L \)
can vary at most \( \pi \) so the total variation of \( L(0 + iT, x) \) when \( 0 \) goes
from \( 2 + \frac{T}{2} \) to \( 0 \leq (n + 2) \pi \)
\[ = O(\log f(2 + T)). \]

A similar argument on the sketch \( (\frac{T}{2}, 2) \) gives \( O(\log f) \).

The total variation around the rectangle is thus
\[ T \left( \log \frac{q^T}{2\pi} - 1 \right) + \Theta \left( \log q^{(2+T)} \right) \]

dividing by \( 2\pi \), we find

\[ NCT, \mathcal{X} = \frac{1}{\sqrt{\pi}} \left( \log \frac{q^T}{2\pi} - 1 \right) + \Theta \left( \log q^{(2+T)} \right) \]

which holds whether \( \mathcal{X} \) is even or odd, and also in the case \( \mathcal{O} = 1 \), for \( NCT \).

Now clear that

\[ \xi(s) = c'e^{c\mathcal{X}} \sqrt{\frac{1}{\pi}} (1 - \frac{\mathcal{X}}{\mathcal{P}}) e^{\frac{\mathcal{X}}{\mathcal{P}}} \]

and

\[ \xi(s, \mathcal{X}) = c'e^{c\mathcal{X}} \sqrt{\frac{1}{\pi}} (1 - \frac{\mathcal{X}}{\mathcal{P}}) e^{\frac{\mathcal{X}}{\mathcal{P}}} \]

where in each case \( \mathcal{P} \) runs through the zeros of the function on the left-hand side and \( c' \) and \( c \) constants and the \( c' \mathcal{X} \) and \( c \mathcal{X} \) constants depending on \( \mathcal{X} \) only. Product-formulas can also be given for \( \xi(s) \) and \( L(s, \mathcal{X}) \). We shall instead look at the logarithmic
derivative
\[
\frac{\xi'}{\xi} \lambda \gamma = c'' - \frac{1}{s-\xi} + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right) + \sum_{\alpha=1}^{8} \left( \frac{1}{s-2\alpha} + \frac{1}{2\alpha} \right),
\]
and for \( \lambda \) primitive and even
\[
\frac{L'}{L} (\lambda, \chi) = c'' + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right) + \frac{1}{8},
\]

and \( \chi \) primitive odd,
\[
\frac{L'}{L} (\lambda, \chi) = c'' + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right) + \sum_{\alpha=1}^{8} \left( \frac{1}{s-2\alpha} + \frac{1}{2\alpha} \right),
\]

from our results about \( \pi (2, \chi) \)

It is easy to show
\[
\sum_{\rho} \left| \frac{1}{s-\rho} + \frac{1}{\rho} \right| = O \left( \log^2 (2 + \Re(s)) \right)
\]

for \( \Re(s) > \sigma > 1 + \delta \),
and
\[
\sum_{\rho} \left| \frac{1}{s-\rho} + \frac{1}{\rho} \right| = O \left( \log (2 + \Re(s)) \right)
\]
Let \( \alpha > 1 \), we have,

\[
\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s+1}}{s(s+1)} \, ds = 0 \text{ for } 0 \leq x \leq 1, \quad x-1 \text{ for } x > 1.
\]

Thus if

\[
\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s+1}}{s(s+1)} \, ds = \sum_{\nu \in \mathbb{N}} \frac{(x-\nu)c_{\nu}}{\nu x}
\]

is absolutely convergent for \( \sigma > 1 \), then

\[
\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s+1}}{s(s+1)} \, ds = \sum_{\nu \in \mathbb{N}} \frac{(x-\nu)c_{\nu}}{\nu x} = \int_0^x \left( \sum_{\nu \in \mathbb{N}} \frac{c_{\nu}}{\nu t} \right) \, dt.
\]

We define \( \Lambda(x) = \operatorname{log} p \) for \( m = p^n, n > 0 \).

(Non-Harmonic function).

Then

\[
\sum_{\nu} \frac{\Lambda(\nu)}{\nu^s} = -\frac{\zeta'}{\zeta}(s),
\]

and

\[
\sum_{\nu} \chi(\nu) \frac{\Lambda(\nu)}{\nu^s} = -\frac{L'}{L}(s, \chi).
\]

Write \( \psi(x) = \sum_{m \leq x} \Lambda(m) \),

and

\[
\psi_{\chi}(x) = \sum_{m \leq x} \chi(m) \Lambda(m),
\]
\[
\int_0^x \psi(t) \, dt = \sum_{\alpha} \left( x - \alpha \right) \Lambda(\alpha)
\]
\[
= \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{x^{s+1}}{s(2\pi i)} \left( -\frac{1}{s} \right) \, ds.
\]

Here we may use the previous expansion for \(-\frac{1}{s}\) and can integrate term by term since integral exists if we take absolute values everywhere. We get easily
\[
\int_0^x \psi(t) \, dt = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho (\rho+1)} + O(x),
\]
and similarly
\[
\int_0^x \psi(x) \, dt = - \sum_{\rho} \frac{x^{\rho+1}}{\rho (\rho+1)} + O(x),
\]
where the \(\rho\) run through the zeros of \(\xi(s)\) or \(\xi(s, x)\) respectively.

To proceed we need to show that neither \(\xi(s)\) or any of the \(\xi(s, x)\) have a zero with real part 1. For \(\sigma > 1\) we have
\[ R(-3 \frac{3}{2} \sigma + n, \lambda) = 4 \frac{1}{\lambda} (\sigma + i t, \chi) - \frac{1}{\lambda} (\sigma + 2i \epsilon, \chi^2) \geq 0. \]

If either \( t \neq 0 \) or \( \chi^2 \neq \chi_0 \), we get easily a contradiction if \( 1 + i \epsilon \) is a zero of \( L(\sigma, \chi) \), by letting \( \sigma \to 1 \)
and seeing that the left-hand side of the above equation then would tend to \(-\infty\).

In the case \( t = 0 \), \( \chi^2 = \chi_0 \), we look at
\[ \xi(\sigma) \zeta(\sigma, \chi) \]
and see that
\[ \xi(\sigma) \zeta(\sigma, \chi) = \sum \frac{c_m}{m^s} \text{ with } c_m \geq 0, c_m > 0? \]

If \( \zeta(\sigma, \chi) \) has a zero at \( s = 1 \), then
\[ \xi(\sigma) \zeta(\sigma, \chi) \]
is an integral function.

As its power-series around \( s = 2 \) converges everywhere, the \( \ell^\text{th} \)
coefficient is in absolute value
\[ \frac{1}{\ell!^{(s)}} \xi(s) \zeta(s, \chi) \]
\[ \geq \frac{1}{\ell!} \sum_{m = 1}^{\infty} \frac{c_m \log m^\ell}{m^2} \frac{1}{k!} \sum_{m = 1}^{\infty} \frac{c_m \log^2 m}{m^4} \]
comparing this to the k'th coefficient in the power series of \( g(z) \) around \( z = 2 \) we find that to be in absolute value
\[
\frac{1}{k!} \sum_{n} \frac{2^k \log^k n}{a^k y^k}
\]
but this power series can not converge beyond \( s = \frac{1}{2} \) since \( g(z) \) has a pole there. This gives us a contradiction so \( L(1, s) \neq 0 \).

From this we now easily conclude
\[
\int_{0}^{x} y(t) \, dt = \frac{x^2}{2} + o(x^2)
\]
and
\[
\int_{0}^{x} y_x(t) \, dt = o(x^2).
\]
All that is needed is to show that in each case
\[
\sum_{\rho} \frac{x^\beta + 1}{(\rho(\rho + 1))} = o(x^2)
\]
where \( \rho = \beta + i \gamma \).

We can always choose \( T \) so large that
\[
\sum_{|\gamma| > T} \frac{x^\beta + 1}{(\rho(\rho + 1))} < \frac{\varepsilon}{x}
\]
if \( \varepsilon \) is a given positive quantity.
Then
\[ \sum_{p} \frac{x^{\beta+1}}{\phi(p)(p+1)} < \sum_{\gamma | T} \frac{x^{\beta+1}}{\phi(p)(p+1)} + \varepsilon x^2, \]

Since in the finite sum \(\gamma | T\), all exponents \(\beta+1 < 2\), this will be \(\varepsilon x^2\) for \(x\) sufficiently large. So
\[ \sum_{p} \frac{x^{\beta+1}}{\phi(p)(p+1)} < \varepsilon x^2 \]
for \(x > x_0\), which proves the asymptotic relations.

If we write
\[ \psi_{q,2}(x) = \sum_{\substack{1 \leq n \leq x \\ \Omega(n) = 2(q)\delta(n)}} \Lambda(n) \]

We see that
\[ \psi_{q,2}(x) \] differs only slightly from
\[ \frac{1}{\phi(q)} \sum_{q' | q} \sum_{q'' | q'} x(q') \psi_{n}(x) \]
(The difference being \(\leq \psi(q) \log x\)
\(n(q)\) being \(m\) of primefactors of \(q\).)
From this we get easily
\[ \int_0^x \Psi_{q,e}(t) \, dt = \frac{1}{\phi(q)} \frac{x^2}{2} + o(x^2) \]

in addition to our previous
\[ \int_0^x \Psi(t) \, dt = \frac{x^2}{2} + o(x^2) \]

From these relations
\[ \Psi(x) = x + o(x) \]

and
\[ \Psi_{q,e}(x) = \frac{1}{\phi(q)} x + o(x) \]

follow easily, we have
\[ \frac{1}{h} \int_{x-h}^{x+h} \Psi(t) \, dt \leq \Psi(x) \leq \frac{1}{h} \int_{x-h}^{x+h} \Psi(t) \, dt \]

\[ x - h - \frac{1}{h} o(x^2) \leq \Psi(x) \leq x + h + \frac{1}{h} o(x^2) \]

Take \( h = \frac{3}{2} x \) and \( x \) so large

that \[ \frac{3}{2} x o(x^2) < \frac{3}{2} x \], and we get

\[ x - 3x \leq \Psi(x) \leq x + 3x \]

for \( x > x_0 \). Similarly for \( \Psi_{q,e}(x) \).