We shall consider further the possibility of the presence of an exceptional zero, a real zero $\beta > 1 - \frac{1}{\log q}$ for the primitive quadratic character modulo $q$. We saw last time that $\beta > 1 - \frac{c}{\sqrt{q} \log^3 q}$, with a constant that could be effectively determined. It was also mentioned that the connection with the class number gives $\beta > 1 - \frac{c'}{\sqrt{q} \log q}$, and that work of Goldfeld, Gross and Zagier gives $\beta > 1 - \frac{c'}{\sqrt{q}}$, and I believe this has now been improved to $\beta > 1 - \frac{c}{\sqrt{q}}$. In each case the result is obtained by finding a lower bound for $L(1, \chi)$ and then deriving a lower bound for $1 - \beta$ by using that $L'(\sigma, \chi) = O(\log^2 q)$ for $\sigma > 1 - \frac{1}{\log q}$. 
We shall now turn to Siegel's theorem, a result that is in some ways vastly superior in the order of the lower bound for $1 - \beta$ in terms of $q$, but also very inefficient for most purposes in that we have no way of estimating the constants that enter as coefficients.

We begin with the inequalities:

$$\xi(s) = \frac{s}{s-1} + O((1+e)^{\frac{1}{2}+\epsilon} \log (2+e)),\]$$

and

$$L(\rho, \chi) = 1 + O((1+\epsilon)^{-\frac{1}{2}+\epsilon} \log (2+e)).$$

We assume we have a primitive quadratic character mod $q$, such that $L(s, \chi)$ has a real zero $\beta_1 = 1 - \delta$ with $\delta$ small, and shall see what we can conclude about $L(1, \chi)$ for a primitive quadratic character $\chi$ mod $q$, $q \neq q_1$.

We consider

$$f(\alpha) = \xi(s) L(\alpha, \chi) L(\alpha, \chi^2) L(\alpha, \chi^3).$$
We have \[
\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s+1)}{\Gamma(s+1)} f(s+\beta_1) \, ds = \sum_{n=1}^{\infty} \frac{x^n}{a_n \beta_1} \geq x-1 > \frac{x}{2}.
\]

Moving the path of integration to \(\sigma = \frac{4}{5} - \beta_1 = -\frac{1}{5} + \delta\), we get that the integral above also equals (taking into account the residue at \(s=1-\beta_1 = \delta\))

\[
\frac{x^{4+\delta}}{\delta (1+\delta)} L(1, x) \frac{1}{\delta} L(1, x) L(1, x x_1) + O(x^{\frac{4}{5}+\delta} (\frac{1}{x} + \delta)).
\]

Comparison of the two results gives
\[
\frac{x}{4} < x^{1+\delta} \frac{L(1,\chi) L(1,\chi') L(1,\chi\chi_1)}{\delta} + A x^{\frac{\theta}{3} + \delta} \frac{1}{\beta}. 
\]

We now choose
\[
x > \frac{4A x^{\frac{\theta}{3} + \delta}}{L(1,\chi)} (9q)^{\frac{1}{2}},
\]
or
\[
x^{\frac{1}{2} - \delta} > \frac{4A (9q)^{\frac{1}{2}}}{L(1,\chi)}.
\]

We may assume \( \delta < \frac{1}{60} \), and so take
\[
x = \left( \frac{4A}{L(1,\chi)} \right)^{\frac{1}{3}} (9q)^{\frac{1}{2}}.
\]

Then we have
\[
\frac{x}{4} < x^{1+\delta} \frac{L(1,\chi) L(1,\chi') L(1,\chi\chi_1)}{\delta}
\]
or
\[
L(1,\chi) L(1,\chi') L(1,\chi\chi_1) > \frac{5}{4} x^{-\frac{\delta}{12}} A (9q)^{1-2\delta}
\]

Since
\[
L(1,\chi) = O(\log q), L(1,\chi') = O(\log q),
\]
we get
\[
L(1,\chi) > c(\delta) (9q)^{-\frac{45}{4}}.
\]
From $L'(u, x) = O(\log^2 q)$, for $\sigma > 1 - \frac{1}{\log q}$, we then get

$$1 - \beta > c'(6) q^{-5\delta}.$$ 

Now, either the $\beta$ have an absolute upper bound $\theta < 1$, or we can find $q_1$ with $\beta_1$ arbitrarily close to one. In the second case we may choose $x_1$ such that $1 - \beta_1 < \frac{\delta}{5}$ for any given $\varepsilon > 0$. We then get

$$1 - \beta > c(\varepsilon) q^{-\varepsilon},$$

while in the first case we of course have $1 - \beta > 1 - \theta > 0$. We could of course also phrase the result as

$$1 - \beta > q^{-\varepsilon}, \quad \text{for } q > q_0(\delta),$$

where $\varepsilon$ can be chosen positive and arbitrarily small.
In either formulation, the constants $C(\varepsilon)$ and $\eta(\varepsilon)$ exist, but we can not give any bound for their size.

If on the other hand we could actually find a $C(\varepsilon, x, \beta)$ with a $\beta$ very close to 1, we could obtain an effective estimate (and quite a bit better than indicated by our proof of Siegel's result, since we did not try to obtain the best factor in front of $5 \log 2$ in the exponent of $\varepsilon$).

This essentially concludes the material in analytic prime number theory that I had planned to talk about. I have not touched on developments like:

Better (that is: wider) zero-free regions along $\sigma = 1$. These can be obtained using the theory of
exponential sums to estimate sums of the type
\[ \sum_{n=1}^{N} e^{i \alpha n}, \sum_{n=1}^{N} \chi(n) e^{i \alpha n}, \]
Such estimates lead, for instance, for \( \zeta(s) \) to a zero free region of the type
\[ \sigma > 1 - \frac{\alpha}{(\log (1+\varepsilon))^6}, \alpha > 0, \]
with a \( \varepsilon \ll 1 \). This gives us the remainder terms
\[ f(x) = x + O \left( x e^{-\alpha'(\log x)^{\frac{1}{1+c}}} \right), \]
and
\[ \frac{x}{\log x} + O \left( x e^{-\alpha'(\log x)^{\frac{1}{1+c}}} \right), \]
Another development of great importance for many applications are the so-called "density theorems".