3/7/79

Dear Fulton,

Your proof of "Zariski's theorem" is beautiful! It seems to me that a paraphrase of your proof gives the result with the topological π_1, over $\mathbb{C}$. This letter is to make it clear to me; I expect you already observed it.

Let $C \subset \mathbb{P}^2(\mathbb{C})$ be a nodal curve. Let D be an irreducible component of C, and $\tilde{D}$ = normalisation of D. Let $V(\tilde{D})$ be a "tubular neighbourhood" of $\tilde{D}$ in $\mathbb{P}^2(\mathbb{C})$: $\tilde{D} \hookrightarrow V(\tilde{D}) \xrightarrow[\text{étale}]{\varphi} \mathbb{P}^2(\mathbb{C})$, ~~and~~

<u>Aim</u>: $\pi_1(V(\tilde{D}) - \varphi^{-1}(C)) \twoheadrightarrow \pi_1(\mathbb{P}^2 - C)$ (1)

If (1) is granted, Abhyankar argument works.

To get (1), my method has been to take your proof, and to systematically look at what it was implying on the algebraic π_1, by applying your statements to ramified coverings - then remove "algebraic". This translates your key statement into:

Th Let Z be ~~a [illegible]~~ smooth connected <u>locally closed</u> subvariety of $(\mathbb{P}^d)^n$, with $\dim Z > d(n-1)$. The diagonal Δ has then a fundamental system of neighbourhood $V(\Delta)$, such that $Z \cap V(\Delta)$ is connected, and

$$\pi_1(Z \cap V(\Delta)) \twoheadrightarrow \pi_1(Z)$$

<u>Th $\Rightarrow$ Aim</u>: take $d = 2$, $n = 2$, $Z = (\mathbb{P}^2 - C) \times (D - \text{double points of } C \text{ on } D)$.

Put also $Z_1 = (\mathbb{P}^2 - C) \times \tilde{D}$ (above $(\mathbb{P}^2)^2$);

$$\begin{tikzcd} V(\Delta) \cap Z \arrow[r] \arrow[d] & V(\Delta) \cap Z_1 \arrow[d] \\ Z \arrow[r] & Z_1 \end{tikzcd} \quad ; \text{ on } \pi_1 : \quad \begin{tikzcd} {} \arrow[d, "(\mathrm{Th})"] & {} \arrow[d, two heads] \\ {} \arrow[r, two heads, "(\text{clear})"'] & {} \end{tikzcd}$$

and $V(\Delta) \cap Z_1 \approx V(\tilde{D}) - \varphi^{-1}(C)$; ~~as I have done~~ we hence get more than aimed for:

$$\pi_1(V(\tilde{D}) - \varphi^{-1}(C)) \twoheadrightarrow \pi_1(\mathbb{P}^2 - C) \times \pi_1(\tilde{D})$$

In this case, the π_0 statement is trivial, and only π_1 matters. Because of this, I will only try to prove part of (Th):

$$\pi_1(\text{some component of } V(\Delta)\cap Z) \twoheadrightarrow \pi_1(Z) \quad ;$$

the π_0 should however not be difficult to get (anyway, you have it)

As in your proof, one compare

$$\begin{tikzcd}
\mathbb{P}^{nd} \supset L^d & \text{and} & (\mathbb{P}^d)^n \supset \Delta \\
& W \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

W (= join of two compactifications of $\mathbb{A}^{nd}$)

and, once an infinity hyperplane is choosen in $\mathbb{P}^d$, and $\mathbb{P}^{nd}$, with

$$\mathbb{P}^{nd} - \infty = \mathbb{A}^{nd} = (\mathbb{A}^d)^n = (\mathbb{P}^d - \infty)^n ,$$

$$\begin{cases} \varphi_1 = \text{to blow up } (\infty\text{-hyperplane}) \cap \text{closure of } \infty \text{ factors } \mathbb{A}^d \text{ of } (\mathbb{A}^d)^n = \mathbb{A}^{nd} & /\text{they are disjoint from } L/ \\ \varphi_2 = \text{" " "} \ \infty\text{-hyperplane of } \Delta \end{cases}$$

Hence: for $V_2(\Delta)$ neighborhood of Δ in $(\mathbb{P}^d)^n$, there is $V_1(\Delta)$, neighborhood of L^d in $\mathbb{P}^{nd}$, with

$$\underline{\varphi_1^{-1}(V_1(L^d)) \subset \varphi_2^{-1}(V_2(\Delta))}$$

We may replace Z by $Z\cap \mathbb{A}^{nd}$ (because $\pi_1(Z\cap\mathbb{A}^{nd}) \twoheadrightarrow \pi_1(Z)$), and hence to prove (Th), it suffices to consider $\mathbb{P}^{nd}\supset L^d$, instead of $(\mathbb{P}^d)^n\supset\Delta$

(Th*) Let $Z\subset\mathbb{P}^N$ be a smooth connected locally closed subvariety, L a linear subspace, and assume $\dim Z > \operatorname{codim} L$. Then, L has a fundamental system of neighborhood with $Z\cap V(L)$ connected and $\pi_1(Z\cap V(L)) \twoheadrightarrow \pi_1(Z)$

As before, I will only care for π_1.

A neighboorhood of L in $\mathbb{P}^N$ contains all L' close to L. Hence the reformulation

Th** Same assumption, with now L general (= in ~~a~~ suitable (non empty) Zariski open subset of the grassmanian). Then, $\pi_1(Z \cap L) \twoheadrightarrow \pi_1(Z)$

For Th**, one can proceed by induction, and be reduced to the case where L is an hyperplane. I hope this case to be in the litterature; at least one can reduce this case to the one treated by Lê and Hamm: un théorème de Zariski de type Lefschetz, Ann. sc. ENS $\underline{6}$ 3 (1973) p 317-366, where they take $Z = (\mathbb{P}^N -$ some hypersurface$)$: to get the result for our Z, project it generically onto a linear subspace $\mathbb{P}^{\dim Z}$, and use the complement of the ramification locus as the Z of Lê: his result gives what I need for hyperplanes through the center of projection.

In fact, much more than Th** should be true

Conjecture (perhaps well known to Lê): One consider

Z smooth, connected (not assumed proper), $\quad f : Z \longrightarrow \mathbb{P}^N$, $\quad L$ linear subspace.

Z smooth outside of $f^{-1}(L)$ should be enough

and $\qquad \pi_i(f^{-1}V(L)) \longrightarrow \pi_i(Z)$

for ~~small~~ tubular neighboorhood $V(L)$ of L, of usual shape. This map should be bijective for

$$i < \dim f(Z) - \operatorname{codim} L - \sup_k \left(2k - \operatorname{codim} \text{ in } f(Z) \text{ of the locus where dim fibre} \geq k + \text{dim generic fibre of } Z \to f(Z)\right)$$

and surjective for $i = \dim f(Z) - \operatorname{codim} L$

Here are my reasons to hope for it.

a) a proof by Morse theory, if ok for a generic L, should work as well for any L, when L is replaced by $V(L)$. Let us take for L a general hyperplane.

b) For the similar statement in cohomology, and generic L, one should prove a vanishing of the low

$$H^i_c(\mathbb{P}^N - L,\ Rf_* \mathbb{Z}) \quad ,$$

related by duality to the higher

$$H^i(\mathbb{P}^N - L,\ Rf_! \mathbb{Z}) ,$$

handled by Leray spectral sequence : ~~dim Rf~~ dim support $R^i f_! \mathbb{Z}$ is checked by ~~dim~~ $(R^i f_! \mathbb{Z})_s = 0$ if $i > 2 \dim f^{-1}(s)$, and $H^i(\mathbb{P}^N - L, \mathcal{F}) = 0$ for $i >$ dim support of $\mathcal{F}$.

Yours sincerely

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