AMS/IP Studies in Advanced Mathematics
Volume 1, 1997

Local behavior of Hodge structures at infinity.

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This article is an expanded version of a letter to D. Morrison, inspired by his article [3]. I give a description of "very degenerating" variations of Hodge structures, which is well adapted to the variations arising in the mirror fairy tale. I also try to ascertain how "motivic" this description is.

0. Introduction

In [3], D. Morrison describes some of the results of [1] in a Hodge theoretic language. One ingredient is the variation of Hodge structures defined by a family of Calabi–Yau manifolds near a point (in a completed parameter space) of "maximal degeneracy". We interpret "maximal degeneracy" as leading to the appearance of variations of mixed Hodge structures of a special type (Hodge-Tate), and give a description of such mixed variations in terms of invertible holomorphic functions. The logarithmic derivatives of those functions are what appears in Morrison's paper.

I thank E. Cattani for a very helpful letter directing me to parts of [2] I had missed, and the referee for suggesting many improvements.

1. Notations

For V a polarized variation of Hodge structures of weight w on a complex variety S, we use the following notations.

$V_{\mathbb{Z}}$: underlying local system of free abelian groups. For $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathcal{O}$ (sheaf of holomorphic functions), $V_R := V_{\mathbb{Z}} \otimes R$.

$\mathbb{Z}(k)$: $(2\pi i)^k\mathbb{Z}$, viewed as a Hodge structure of type $(-k, -k)$

Also: the corresponding constant variation.

ψ: $V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}(-w)$: the polarization form, symmetric for w even and alternating for w odd.

At each point $s \in S$, the Hodge structure of the fiber V_s of V at s is given by a decomposition

$$V_{s\mathbb{C}} = \bigoplus_{p+q=w} V_s^{p,q}$$

with $V_s^{q,p}$ the complex conjugate of $V_s^{p,q}$. The Weil operator C: multiplication by i^{p-q} on $V_s^{p,q}$, is real and the form

$$(2\pi i)^w \psi(Cx, \bar{y})$$

is hermitian symmetric and positive definite.

The holomorphic vector bundle with integrable connection having $V_{\mathbb{C}}$ as sheaf of horizontal sections is identified with its sheaf $V_{\mathcal{O}}$ of holomorphic sections. We denote

∇: its connection.

F: the Hodge filtration. At each point, $F_s^p = \bigoplus_{a \geq p} V_s^{a,b}$. It is a holomorphic filtration and $\nabla F^p \subset \Omega^1 \otimes F^{p-1}$ (Griffiths transversality).

Typical example

Let $(X_s)_{s \in S}$ be a family of nonsingular projective varieties parametrized by S, i.e. a projective and smooth map $f: X \to S$ (with $X_s = f^{-1}(s)$). Take

$$V_{\mathbb{Z}} := R^w f_* \mathbb{Z}$$

modulo torsion. It is the local system of the

$$H^w(X_s)/\text{torsion},$$

and $V_s^{p,q} = H^{p,q}(X_s)$.

For X_s purely of dimension n, and $w = n$, on the primitive part of $H^n(X_s)$ (the kernel of the cup-product with the first Chern class of the given ample line bundle), a polarization form is given by

$$(-1)^{n(n-1)/2} \frac{1}{(2\pi i)^n} \int x \cup y.$$

The construction of a polarization form on the whole of the cohomology is by reduction to that case, applied to linear sections of X_s.

2. Asymptotics

Let D be the open unit disc, and $D^* := D - \{0\}$. We take $S = D^{*n}$ ($n \geq 1$) and recall known results on the behavior of V_s for $s \to 0$. To the extent possible, our references will be to the survey paper [2].

2.1. Monodromy

Let T_j be the monodromy of $V_{\mathbb{Z}}$ around $s_j = 0$. The T_j commute and are known to be quasi-unipotent. For simplicity, we assume they are even unipotent. We define

$$N_j := -\log(T_j)$$

(note the minus sign). Sign convention, explained for $n = 1$: T is the effect of pushing an element of $(V_{\mathbb{Z}})_s$ horizontally along the path $\exp(2\pi i u)s$ ($0 \leq u \leq 1$).

2.2. Canonical extension

For any complex local system $H_{\mathbb{C}}$ on D^{*n}, with unipotent monodromy, we continue to write $H_{\mathcal{O}}$ for the *canonical extension* of the vector bundle $H_{\mathcal{O}}$ to D^n. It is characterized by the property that, in a local basis near 0, the matrix of 1-forms defining the connection has logarithmic poles along the $s_j = 0$, with nilpotent residues.

Consider on $H_{\mathcal{O}}$ the new connection

$$\nabla^c = \nabla \frac{-1}{2\pi} i \sum N_j \frac{ds_j}{s_j}. \tag{2.2.1}$$

The horizontal sections for ∇^c are the sections of the form $\exp(\log s \cdot N/2\pi i)h$ for h ∇-horizontal. The connection ∇^c has no monodromy and turns $H_{\mathcal{O}}$ into a constant bundle. We will say *constant* to mean horizontal for ∇^c. The canonical extension of $H_{\mathcal{O}}$ is obtained by extending it as a constant bundle.

The construction of ∇^c depends on the choice of coordinates near 0, while the canonical extension does not. In fact, the constant vector bundle attached to $H_{\mathcal{O}}$ should be viewed as living on the tangent space T_0 of D^n at 0. Local coordinates s'_j (with s'_j/s_j invertible) define (a) an isomorphism φ between a neighborhood of 0 in D^n and a neighborhood of 0 in T_0, (b) a constantification of $H_{\mathcal{O}}$. If we use φ to transplant the local system $H_{\mathbb{C}}$ and the constantification of $H_{\mathcal{O}}$ to T_0, the result is independent of the local coordinates used.

If a filtration F of the canonical extension $H_{\mathcal{O}}$ is given, the corresponding *nilpotent orbit* is the constant filtration of $H_{\mathcal{O}}$ which coincides with F at 0. Again, to be coordinate free, it should live on T_0.

2.3. Nilpotent orbit

For a polarized variation V on D^{*n}, the nilpotent orbit theorem states that

(a) The Hodge filtration F extends to the canonical extension $V_{\mathcal{O}}$ as a filtration (still denoted F) by locally direct factors ([2] 2.1 (i)),

(b) The corresponding nilpotent orbit is again, in a neighborhood of 0, a variation of Hodge structures of weight w polarized by ψ ([2] 2.1 (ii)).

Conversely, let $(V_{\mathbb{Z}}, F_{\mathrm{nilp}})$ be a polarized variation of Hodge structures on D^{*n} which is a nilpotent orbit: the filtration F_{nilp} is assumed constant (for ∇^c, see (2.2)). Let F be a new filtration of $V_{\mathcal{O}}$ on D^n, agreeing with F_{nilp} at $s = 0$, and obeying transversality:

$$\nabla F^p \subset \Omega^1 \otimes F^{p-1}.$$

Assume further that $\psi(F^p, F^{w-p-1}) = 0$ for all p. Then $(V_{\mathbb{Z}}, F)$ is, near 0, a variation of Hodge structures on D^{*n}, polarized by ψ. See [2], 2.8.

2.4 Weight filtration

There exists on $V_{\mathbb{Q}}$ a unique increasing filtration W, the *weight filtration*, such that $N_j W_k \subset W_{k-2}$ and that for any positive linear combination

$$N = \sum \lambda_j N_j$$

$(\lambda_j > 0)$, N^k induces an isomorphism from $Gr^W_{w+k}(V_{\mathbb{R}})$ to $Gr^W_{w-k}(V_{\mathbb{R}})$. Further, $(V_{\mathbb{Z}}, W, F_{\mathrm{nilp}})$ is a variation of mixed Hodge structures ([2] 2.3).

The construction of W is compatible with passage to the dual, as well as to tensor products. It follows that if ψ is a polarization form, then W_{w-i} and W_{w+i-1} are mutually orthogonal, and ψ induces a perfect pairing $Gr^W(\psi)$ between $Gr^W_{w-i}(V)$ and $Gr^W_{w+i}(V)$. Further, the induced pairings, and

$$Gr^W(N)\colon Gr^W_k(V) \to Gr^W_{k-2}(V)$$

have positivity properties reminiscent of those of the cohomology of a nonsingular projective variety of dimension k, with the cup-product pairing to H^{2k}, the role of N being played by the cup-product with the first Chern class of an ample line bundle. In the terminology of [2] 1.16, $(V, W, F_{\mathrm{nilp}})$ is polarized by ψ and N. The reason for this analogy is not understood.

3. Warning

In general, W and the original Hodge filtration F do not define a mixed Hodge structure in a neighborhood of 0. Consider, for instance, a general pencil $f\colon X \to \mathbb{P}^1$ of hypersurfaces of degree d in $\mathbb{P}^M$. The fiber $X_{(\lambda,\mu)}$ at the point with projective coordinates (λ, μ) is the hypersurface with equation

$$\lambda F + \mu G = 0$$

(F, G general forms of degree d). Let S be the set of critical values of f. On $\mathbb{P}^1 - S$, we consider the variation

$$V_{\mathbb{Z}} = R^{M-1} f_* \mathbb{Z},$$

and restrict it to a punctured disc D^* centered at a critical value (λ_0, μ_0).

We assume that

(a) M is even, hence X_z is of odd dimension $w = M - 1$.

The cohomology $H^w(X_z, \mathbb{Z})$ is torsion-free, primitive (i.e. annihilated by cup product with the hyperplane section class) and the polarization form is given by the cup product. One knows that the global monodromy, image of $\pi_1(\mathbb{P}^1 - S, z)$, is Zariski dense in the corresponding symplectic group. Provided that $H^w(X_z, \mathbb{Z}) \neq 0$, i.e. $d \neq 1, 2$, N is nilpotent of rank one. The weight filtration is

$$0 \subset W^{w-1} = \mathrm{Im}(N) \subset W^w = \ker(N) = V_{\mathbb{Z}}.$$

(b) $M \geq 4$ and d is large enough, so that $V_{\mathbb{C}} \neq F^{(w-1)/2}$.

If (W, F) were a mixed Hodge structure on D^*, Gr^W_{w-1}, being one-dimensional, would have to be of type $((w-1)/2, (w-1)/2)$:

$$\mathrm{Im}(N) \subset F^{(w-1)/2}.$$

By analytic continuation, any image of $\mathrm{Im}(N)$ by the global monodromy group would still be in $F^{(w-1)/2}$. Those images span V, by Zariski density of the global monodromy group in the symplectic group, contradicting (b).

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We now consider a case where (W, F) is a mixed Hodge structure. Fix a base point $s_0 \in D^{*n}$, let G be the group of automorphisms of $V_{s_0\mathbb{Q}}$ respecting the form ψ, and let G^0 be the connected component of the identity of G, viewed as a linear algebraic group: it is a special orthogonal group for w even, a symplectic group for w odd. The filtrations W, F and F_{nilp} correspond to isotropic flags:

$$W_{w-i}^{\perp} = W_{w+i-1}$$
$$(F^i)^{\perp} = F^{w-i+1}$$

and similarly for F_{nilp}. The subgroups of $G^0_{\mathbb{C}}$ stabilizing them are parabolic subgroups.

Proposition 4.1. *If the parabolic subgroups $P(F_{\mathrm{nilp}})$ and $P(W)$ defined by F_{nilp} and W satisfy*

$$\mathrm{Lie}\ P(F_{\mathrm{nilp}}) + \mathrm{Lie}\ P(W) = \mathrm{Lie}\ G_{\mathbb{C}}, \tag{4.1.1}$$

then, near 0, (W, F) is a mixed Hodge structure, with the same Hodge numbers as (W, F_{nilp}).

Remark. As the

$$\exp\left(\sum u_j N_j\right) \in G_{\mathbb{C}}$$

respect W and permute transitively the F_{nilp} taken at various points, the validity of (4.1.1) does not depend on the point s_0 at which it is considered.

Proof. The map

$$u \to s_0 \exp(2\pi i u)$$

maps a product of upper half-planes to D^{*m}. The pullback of the local system $V_{\mathbb{Z}}$ is a trivial local system, and we can consider the Hodge filtration F at $s_0 \exp(2\pi i u)$ as a filtration $\phi(u)$ of the fixed vector space $V_{s_0\mathbb{C}}$.

If one wants to view $V_{\mathcal{O}}$ as a constant vector bundle as in 2.2, $\phi(u)$ is to be replaced by $\exp(-uN)\phi(u)$. In particular, if we write F_{nilp} for F_{nilp} at s_0,

$$F_{\text{nilp}} = \lim \exp(-uN)\phi(u).$$

Here the limit is taken as the infimum of the $\operatorname{Im}(u_j) \to \infty$, i.e. $s_0 \exp(2\pi i u) \to 0$. From (4.1.1), it then follows that for $\inf \operatorname{Im}(u_i)$ large, the pair of filtrations

$$(W, \exp(-uN)\phi(u))$$

is conjugate to (W, F_{nilp}) by $g \in G(\mathbb{C})$ close to 1. In particular,

$$\dim\left(W_a \cap F^b_{\text{nilp}}\right) = \dim\left(W_a \cap \exp(-uN)\phi^b(u)\right).$$

As $\exp(-uN)\phi(u)$ is close to F_{nilp}, this equality of dimensions implies that the filtration induced by $\exp(-uN)\phi(u)$ on

$$Gr^W_a(V_{s_0\mathbb{C}})$$

tends to that induced by F_{nilp}. As N_j respects W and acts trivially on

$$Gr^W_a(V_{s_0\mathbb{C}}),$$

the filtrations $\phi(u)$ and $\exp(-uN)\phi(u)$ induce the same filtration on $Gr^W_a(V_{s,\mathbb{C}})$. For $\inf \operatorname{Im}(u_j)$ large enough, this induced filtration defines a Hodge structure of weight a; a, being close to the Hodge filtration $Gr^W_a(F_{\text{nilp}})$. It follows that (W, F) is mixed Hodge, with the same Hodge numbers as (W, F_{nilp}).

5. Variant

Fix some Hodge structures H_α and some horizontal tensors t_α, integral of type $(0,0)$, in some tensor powers

$$\bigotimes^{a(\alpha)} V_{\mathbb{Z}} \otimes \bigotimes^{b(\alpha)} V_{\mathbb{Z}}^{\vee} \otimes H_\alpha.$$

One can then repeat 4.1 and its proof with G replaced by the subgroup fixing the t_α.

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We now assume that the mixed Hodge structure $(V_{\mathbb{Z}}, W, F_{\text{nilp}})$ is *Hodge-Tate*, i.e. such that

$$Gr_k^W \quad \text{is} \quad \begin{cases} \text{of type } (\ell,\ell) & \text{for } k = 2\ell \\ 0 & \text{for } k \text{ odd.} \end{cases} \tag{6.1}$$

This should be viewed as an assumption of maximal degeneracy. It means that $W_{2k} = W_{2k+1}$ and that the filtrations F_{nilp} and W are opposite in the sense that the map $F^p_{\text{nilp}} \oplus W_{2p-2} \to V_{\mathcal{O}}$ is an isomorphism for all p. If we set

$$V^{(p)}_{\text{nilp}} = F^p_{\text{nilp}} \cap W_{2p},$$

$V_{\mathcal{O}}$ is then the direct sum of the $V^{(p)}_{\text{nilp}}$, with

$$F^p_{\text{nilp}} = \bigoplus_{a \geq p} V^{(p)}_{\text{nilp}}, \quad W^{2p} = W^{2p+1} = \bigoplus_{a \leq p} V^{(p)}_{\text{nilp}}.$$

The assumption (4.1.1) holds and 4.1 is in fact easier to prove in this case. In terms of the nilpotent orbit trivialization of $V_{\mathcal{O}}$, the filtrations W and F_{nilp} are constant. As W and F_{nilp} are opposed, and F and F_{nilp} coincide at 0, W and F remain opposed in a neighborhood of 0:

$$\begin{gathered} V_{\mathcal{O}} = \oplus V^{(p)} \qquad \text{with} \\ F^p = \bigoplus_{a \geq p} V^{(a)}, \quad W^{2k} = W^{2k+1} = \bigoplus_{a \leq k} V^{(a)}, \end{gathered} \tag{6.2}$$

implying that $(V_{\mathbb{Z}}, W, F)$ is mixed Hodge and Hodge-Tate. We will assume D^{*m} has been shrunk so that $(V_{\mathbb{Z}}, W, F)$ is mixed Hodge on the whole of D^{*m}.

Our aim is to classify, i.e. to give another description, of variations such as the above.

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In the category of mixed Hodge structures $\operatorname{Hom}(\mathbb{Z}(0), \mathbb{Z}(1)) = 0$ (notations: see 1): an extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$ has no nontrivial automorphism as extension. The extensions are classified by $\mathbb{C}^*$:

$$\operatorname{Ext}^1(\mathbb{Z}(0), \mathbb{Z}(1)) = \mathbb{C}^*, \tag{7.1}$$

with $q \in \mathbb{C}^*$ corresponding to the extension

$$0 \to \mathbb{Z}(1) \xrightarrow{\alpha} H \xrightarrow{\beta} \mathbb{Z}(0) \to 0.$$

$$(7.2)\qquad \begin{aligned} H_{\mathbb{C}} &= \mathbb{C}^2, \quad \text{basis} \quad e_0, e_1 \\ W_{-2} &= \mathbb{C}e_1,\ F^0 = \mathbb{C}e_0 \\ H_{\mathbb{Z}} &= 2\pi i\mathbb{Z}\cdot e_1 + \mathbb{Z}\cdot(e_0 + \log q\, e_1) \subset H_{\mathbb{C}} \\ \alpha(2\pi i) &= 2\pi i\, e_1,\ \beta(e_0) = 1 \end{aligned}$$

Note that the choice of the determination of log q does not matter.

Equivalent description: $H_{\mathbb{Z}}$ has a basis f_0, f_1 with $\alpha(2\pi i) = f_0$, $\beta(f_1) = 1$, $F^0 \subset H_{\mathbb{C}}$ the kernel of $x + \frac{\log q}{2\pi i}\, y$: $H_{\mathbb{C}} \to \mathbb{C}$.

Variations of mixed Hodge structures H over S that are extensions of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$ are similarly classified by invertible holomorphic functions. The extension H defined by q has $H_{\mathcal{O}} = \mathcal{O}e_0 \oplus \mathcal{O}e_1$, with basis e_0 (spanning F^0) and e_1 (spanning W_{-2}) and integral lattice and structure maps α, β as in (7.2). The connection is given by

$$(7.3)\qquad \nabla = d + \begin{pmatrix} 0 & 0 \\ -\frac{dq}{q} & 0 \end{pmatrix}$$

We now take $S = D^{*n}$. Let n_j be the order of q along $s_j = 0$:

$$T_j(\log)q = \log q + n_j \cdot 2\pi i.$$

The constant sections of $H_{\mathcal{O}}$, in the sense of 2.2, are spanned by e_1 and

$$e_0 + \log(q/\prod s_j^{n_j})e_1.$$

It follows that F extends to the canonical extension of $H_{\mathcal{O}}$, remaining a supplement to W_{-2}, if and only if q belongs to the group $\mathcal{O}^*_{\text{mer}}(D^{*m})$ of invertible holomorphic functions on D^{*m}, meromorphic along the $s_j = 0$. The filtration F is constant with respect to the trivialization of $H_{\mathcal{O}}$ given in 2.2 if and only if q is a monomial $a \cdot \prod s_i^{n_i}$. The nilpotent orbit attached to H (cf. 1.3) is again an extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$. Its invariant is the monomial q_0 such that q/q_0 is holomorphic invertible on D^m with value 1 at 0.

The definitions given do not require the choice of which is i and which is $-i$ in $\mathbb{C}$. If one is willing to make that choice, one could rather say that the extensions of $\mathbb{Z}$ (type $(0,0)$) by $\mathbb{Z}$ (type $(-1,-1)$) are parametrized by $\mathbb{C}^*$. Formulas (7.2) (7.3) should be changed as follows

$$(7.2)^* \qquad H_{\mathbb{Z}} = \mathbb{Z}e_1 + \mathbb{Z}\left(e_0 + \frac{\log q}{2\pi i}e_1\right) \subset H_{\mathbb{C}}$$

$$\alpha(1) = e_1 \quad \beta(e_0) = 1$$

$$(7.3)^* \qquad \text{replace } -\frac{dq}{q} \text{ by } -\frac{1}{2\pi i}\frac{dq}{q}$$

More generally, for A and B free abelian groups, viewed as purely of type (p,p) and $(p-1,p-1)$, one has for the corresponding constant local system on S

$$\operatorname{Ext}^1_{\text{mixed Hodge}}(A,B) = \operatorname{Hom}(A,B)\otimes \mathcal{O}^*(S) \tag{7.4}$$

The underlying vector bundle with connection $(H_{\mathcal{O}},\nabla)$ of the extension with class E is

$$\begin{aligned} H_{\mathcal{O}} &= A_{\mathcal{O}}\oplus B_{\mathcal{O}} \\ \nabla &= d-\tfrac{1}{2\pi i}d\log E \end{aligned} \tag{7.5}$$

For $S = D^{*m}$, the filtration F extends to the canonical extension, remaining opposite to W at 0, if and only if

$$E\in \operatorname{Hom}(A,B)\otimes \mathcal{O}^*_{\text{mer}}(D^{*m}).$$

The group $\mathcal{O}^*_{\text{mer}}(D^{*n})$ contains the group of monomials $\{a\cdot\prod s_j^{n_j}\}$, and retracts to it by a unique homomorphism nilp such that $f/\operatorname{nilp}(f)$ is holomorphic at zero, where it takes the value 1.
Let F_{nilp} be the constant filtration (2.2) agreeing with F at 0. The nilpotent orbit $(H_{\mathbb{Z}},F_{\text{nilp}})$ is again an extension of A by B. Its invariant is $\operatorname{nilp}(E)$, in

$$\operatorname{Hom}(A,B)\otimes(\text{monomials}).$$

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With SS6 as the motivating example, we now consider on D^{*m} variations of mixed Hodge structures $(V_{\mathbb{Z}},W,F)$ of Hodge-Tate type (6.1) such that (8.1), (8.2) below hold.

8.1.

$Gr^W_{2k}(V_{\mathbb{Z}})$ is a constant local system.

In (8.1), W denotes the filtration of $V_{\mathbb{Z}}$ induced by the filtration W of $V_{\mathbb{Q}}$. We will identify the constant local system

$$Gr^W_{2k}(V_{\mathbb{Z}})$$

with its constant value: a free abelian group. The conditions (8.1) imply that the monodromy is unipotent.

8.2.

F extends to the canonical extension of $V_{\mathcal{O}}$, and, at 0, the filtrations F and W are opposed.

By (8.1) (8.2), the quotient

$$W_{2k}(V_{\mathbb{Z}})/W_{2k-4}(V_{\mathbb{Z}})$$

is an extension of $Gr^W_{2k}(V_{\mathbb{Z}})$ (type (k,k)) by $Gr^W_{2k-2}(V_{\mathbb{Z}})$ (type $(k-1,k-1)$) of the type considered in 7. It has an extension class

$$E_k \in \mathrm{Hom}(Gr^W_{2k}(V_{\mathbb{Z}}), Gr^W_{2k-2}(V_{\mathbb{Z}})) \otimes \mathcal{O}^*_{\mathrm{mer}}(D^{*m}).$$

Let E be the sum of those classes

$$E \in \mathrm{End}(Gr^W(V_{\mathbb{Z}}))_{-2} \otimes \mathcal{O}^*_{\mathrm{mer}}(D^{*m}). \tag{8.3}$$

As in (6.1), the weight and Hodge filtration of $V_{\mathcal{O}}$ define a direct sum decomposition $V_{\mathcal{O}} = \oplus V^{(p)}$. One has

$$V^{(p)} \xrightarrow{\sim} Gr^W_{2p}(V_{\mathcal{O}}) = Gr^W_{2p}(V_{\mathbb{Z}}) \otimes \mathcal{O},$$

giving

$$V_{\mathcal{O}} \simeq Gr^W(V_{\mathbb{Z}}) \otimes \mathcal{O}. \tag{8.4}$$

Lemma 9. *The isomorphism (8.4) transforms the connection ∇ of $V_{\mathcal{O}}$ into the connection*

$$\nabla = d - \frac{1}{2\pi i} d \log E$$

The connection ∇ of $V_{\mathcal{O}}$ respects W, induces the trivial connection on

$$Gr^W(V_{\mathcal{O}}) = Gr^W(V_{\mathbb{Z}}) \otimes \mathcal{O}$$

and maps F^p to F^{p-1}. It is hence given by a 1-form with values in endomorphisms of degree -2 of $Gr^W(V_{\mathbb{Z}}) \otimes \mathcal{O}$. To compute it, it suffices to compute the induced connections on the quotient W_{2k}/W_{2k-4}, and we apply (7.5).

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If we replace F by F_{nilp}, the constant filtration (2.2) agreeing with F at 0, one obtains the nilpotent orbit $(V_{\mathbb{Z}}, W, F_{\mathrm{nilp}})$ attached to $(V_{\mathbb{Z}}, W, F)$. By SS7, the corresponding invariant $E((V_{\mathbb{Z}}, W, F_{\mathrm{nilp}}))$ is deduced from $E = E((V_{\mathbb{Z}}, W, F))$ by applying

$$\mathrm{nilp}\colon \mathcal{O}^*_{\mathrm{mer}}(D^{*m}) \to \{a\, s_1^{n_1} \ldots s_m^{n_m}\} \tag{10.1}$$

The group of monomials is an extension of $\mathbb{Z}^m$ (the exponents) by $\mathbb{C}^*$. If one projects further to $\mathbb{Z}^m$, the information retained is the $Gr^W(1-T_i) = Gr^W(N_i)$: by reduction to the case of extensions of $\mathbb{Z}$ by $\mathbb{Z}(1)$, one checks that, applying v_i, the order along $s_i = 0$, one has

$$(10.2) \qquad Gr^W(1-T_i) = Gr^W(N_i) = -v_i(E) \in \mathrm{End}(Gr^W(V_{\mathbb{Z}}))_{-2}$$

The integrability of ∇ can be written

$$(10.3) \qquad d\log E \wedge d\log E = 0 \quad \text{in} \quad \mathrm{End}(G^W(V_{\mathbb{Z}}))_{-4} \otimes \Omega^2.$$

For $(V_{\mathbb{Z}}, W, F_{\mathrm{nilp}})$, i.e. for the dominant part E_0, it corresponds to the commutativity of the $Gr^W(N_i)$.

Let us consider pairs

$$(V_{\mathbb{Z}}, W, F_{\mathrm{nilp}}), E)$$

of a mixed Tate nilpotent orbit $(V_{\mathbb{Z}}, W, F_{\mathrm{nilp}})$, whose invariant will be denoted E_{nilp}, and of

$$E \in Gr^W(V_{\mathbb{Z}})_{-2} \otimes \mathcal{O}^*_{\mathrm{mer}}(D^{*n}),$$

such that

$$d\log E \wedge d\log E = 0 \quad \text{and}$$
$$E_{\mathrm{nilp}} = \mathrm{nilp}(E).$$

We turn those pairs into a category by defining a morphism f from

$$((V_{\mathbb{Z}}, W, F_{\mathrm{nilp}}), E)$$

to

$$((V_{\mathbb{Z}}, W', F'_{\mathrm{nilp}}), E')$$

to be

$$f\colon V_{\mathbb{Z}} \to V'_{\mathbb{Z}},$$

compatible with W and F_{nilp}, and such that

$$E' \circ Gr^W(f) = Gr^W(f) \circ E$$

Theorem 11. *To $(V_{\mathbb{Z}}, W, F)$ as in SS8, let us attach*

(a) *the corresponding nilpotent orbit $(V_{\mathbb{Z}}, W, F_{\mathrm{nilp}})$.*

(b) $E \in \mathrm{End}(Gr^W(V_{\mathbb{Z}}))_{-2} \otimes \mathcal{O}^*_{\mathrm{mer}}(D^{*m})$.

This construction is an equivalence from the category of variations $(V_{\mathbb{Z}}, W, F)$ as in SS8 to the category of pairs as above.

Remark. The functor of the theorem is compatible with tensor products, if the E of $V_{\mathbb{Z}} \otimes V'_{\mathbb{Z}}$ is defined to be $E \otimes \mathrm{Id}_{V'} + \mathrm{Id}_V \otimes E'$. It is also compatible with duals (with E going to the opposite of its transpose) and, combining the two, with inner Hom.

Proof. A morphism of V to V' is a section of $\mathrm{Hom}(V, V')_{\mathbb{Z}}$ in W_0 and F^0. Thus, to show that the function is fully faithful, it suffices to show that a global section v of $V_{\mathbb{Z}}$ is in W_0 and F^0 if it is in W_0, in F^0_{nilp} and if its image $\bar{v}$ in $Gr^W_0(V_{\mathbb{Z}})$ is killed by E: $E\bar{v}$ trivial in $Gr^W_{-2}(V_{\mathbb{Z}}) \otimes \mathcal{O}^*$.

Being global and horizontal, v is constant, in the sense of (2.2). Let $v^{(0)}$ be its component in $V^{(0)}$ (6.2). Via the isomorphism of $V^{(0)}$ with $Gr^W_0(V_{\mathbb{Z}}) \otimes \mathcal{O}$, it corresponds to $\bar{v}$ in $Gr^W_0(V_{\mathbb{Z}})$. By Lemma 9, as $E\bar{v}$ is trivial, $v^{(0)}$ is horizontal. The two constant sections v and $v^{(0)}$ coincide at 0, as v is in F^0_{nilp}. It follows that they are equal, and $v = v^{(0)} \in F^0$.

It remains to show that any $((V^{\mathrm{nilp}}_{\mathbb{Z}}, W_{\mathrm{nilp}}, F_{\mathrm{nilp}}), E)$ is the image by the functor of some $(V_{\mathbb{Z}}, W, F)$. The sought after $V_{\mathcal{O}}$, with its weight and Hodge filtration, and connection, is

$$V_{\mathcal{O}} = Gr^W(V^{\mathrm{nilp}}_{\mathbb{Z}}) \otimes \mathcal{O}$$
$$\nabla = d - \frac{1}{2\pi i}\, d \log E.$$

Its canonical extension is $Gr^W(V^{\mathrm{nilp}}_{\mathbb{Z}}) \otimes \mathcal{O}$ on D^m, and the constantification (2.2) gives

$$\phi\colon Gr^W(V^{\mathrm{nilp}}_{\mathbb{Z}}) \otimes \mathcal{O} \to V_{\mathcal{O}}$$

with $Gr^W(V^{\mathrm{nilp}}_{\mathbb{Z}})$ mapping to constant sections. The map ϕ is horizontal if, at the source, one uses

$$\nabla_0 = d - \frac{1}{2\pi i}\, d \log E_{\mathrm{nilp}}.$$

The vector bundle with connection

$$Gr^W(V^{\mathrm{nilp}}_{\mathbb{Z}}) \otimes \mathcal{O}, \ \ \nabla_0$$

is the one underlying

$$(V^{\mathrm{nilp}}_{\mathbb{Z}}, W^{\mathrm{nilp}}, F_{\mathrm{nilp}}).$$

The sought after $(V_{\mathbb{Z}}, W, F)$ is given by

$$V_{\mathbb{Z}} = \phi(V^{\mathrm{nilp}}_{\mathbb{Z}}) \subset V_{\mathcal{O}}.$$

Applying the functor, it gives back $(V^{\mathrm{nilp}}_{\mathbb{Z}}, W^{\mathrm{nilp}}, F_{\mathrm{nilp}})$. To check that it gives back E, it suffices to check it on the quotients W_{2k}/W_{2k-4}. This case in turn reduces to that of extensions of $\mathbb{Z}$ by $\mathbb{Z}(1)$, which is left to the reader.

12

We now apply Theorem **11** in the situation of SS6. To a polarized variation V on D^{*m}, with $(V, W, F_{\mathrm{nilp}})$ of Hodge-Tate type, we attach the corresponding polarized nilpotent orbit (V, F_{nilp}) and $E \in \mathrm{End}(Gr^W(V_{\mathbb{Z}}))_{-2} \otimes \mathcal{O}^*_{\mathrm{mer}}(D^{*m})$.

The fact that ψ is a morphism $V \otimes V \to \mathbb{Z}(-w)$, and hence induces a morphism $(V_{\mathbb{Z}}, W, F) \otimes (V_{\mathbb{Z}}, W, F) \to \mathbb{Z}(-w)$, translates into

$$\psi(Ex, y)\psi(x, Ey) = 1 \quad \text{in} \quad \mathcal{O}^*_{\mathrm{mer}}(D^{*m}) \tag{12.1}$$

for $x \in Gr^W_k(V_{\mathbb{Z}})$, $y \in Gr^W_\ell(V_{\mathbb{Z}})$ and $k + \ell = 2$.

Conversely, given

(a) a polarized nilpotent orbit (V, F_{nilp}) with $(V, W, F_{\mathrm{nilp}})$ of Hodge-Tate type,

(b) $E \in \mathrm{End}(Gr^W(V_{\mathbb{Z}}))_{-2} \otimes \mathcal{O}^*_{\mathrm{mer}}(D^{*n})$ with $d \log E \wedge d \log E = 0$, E compatible with E_0 attached to $(V, W, F_{\mathrm{nilp}})$: $E - E_0$ holomorphically trivial at 0, and E obeying (12.1), we obtain by **11** a mixed variation (V, W, F). By (12.1), $\psi\colon (V_{\mathbb{Z}}, W, F) \otimes (V_{\mathbb{Z}}, W, F) \to \mathbb{Z}(-w)$ is a morphism. If we forget W, $(V_{\mathbb{Z}}, F)$ admits (V, F_{nilp}) as nilpotent orbit. By 2.3, $(V_{\mathbb{Z}}, F)$ is a polarized variation near 0.

13. Example

Let E be a family of elliptic curves parametrized by D^*, and consider the variation $H_1(E_s)$ $(s \in D^*)$. Assume the monodromy is unipotent and nontrivial. In that case, one can uniquely write

$$E_s = \mathbb{C}^*/q(s)^{\mathbb{Z}},$$

with $q(s)$ holomorphic invertible on D^* and tending to 0 with s. We are in the situation of (4.1) and the mixed variation $(H_1(E_s), W, F)$ is the extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$ with invariant $q(s)$.

In this case, the mixed Hodge structures $(H_1(E_s), W, F)$ are subquotients of the homology of algebraic varieties: they are "motivic". Indeed, the extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$ with invariant q $(q \neq 1)$ is the homology (H_1) of $\mathbb{P}^1(\mathbb{C}) - \{1, q\}$ with the points 0 and ∞ identified. To check this, apply (P. Deligne, Théorie de Hodge III, Publ. Math. IHES **44** (1974), pp. 5–78, construction 10.3.8).

More generally, for a family of polarized abelian varieties A on D^{*m}, and their H_1, 4.1 applies and the resulting mixed Hodge structures are motivic.

I do not expect this to hold in general. Let V be a direct factor of $H^w(X_s)$, for X a family of projective nonsingular varieties over D^{*n}. The nilpotent orbit

$$(V, W, F_{\mathrm{nilp}})$$

should be motivic but, even if $(V, W, F_{\mathrm{nilp}})$ is Hodge-Tate, the (V_s, W, F) should not always be. If they were, it would conflict with another conjecture: for an iterated extension E of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)^n$ by $\mathbb{Z}(2)$, defining extension classes

$$q_{2,i} \in \mathrm{Ext}^1(\mathbb{Z}(0), \mathbb{Z}(1)^n) = \mathbb{C}^{*n}$$
$$q_{2,i} \in \mathrm{Ext}^1(\mathbb{Z}(1)^n, \mathbb{Z}(2)) = \mathbb{C}^{*n},$$

if E is motivic, then $\prod\{q_{1,i}, q_{2,i}\} \in K_2(\mathbb{C})$ is trivial. As a consequence,

$$\sum \frac{dq_{1i}}{q_{1i}} \wedge \frac{dq_{2i}}{q_{2i}} = 0$$

in $\Omega^2_{\mathbb{C}/\mathbb{Q}}$ and, in the case $n = 1$, q_1 and q_2 are algebraically dependent over $\mathbb{Q}$.

Let us consider the family of quintic threefolds

$$\sum_1^5 X_i^5 - 5\lambda \prod_1^5 X_i = 0.$$

Let G be the multiplicative group of 5-uples of fifth roots of 1 with $\prod_1^5 \zeta_i = 1$. It acts on each quintic in the family: $(X_i) \mapsto (\zeta_i X_i)$, with the quotient of G by the diagonal (all ζ_i equals) acting faithfully. Let V be the part of H^3 fixed by G. By the computations of [1], V is of rank 4 and the monodromy group is Zariski dense in Sp_4: it contains a unipotent transformation u with $u - 1$ of rank one (monodromy around $\lambda = \zeta$, for $\zeta^5 = 1$), and one with $u - 1$ of rank 3, i.e. one Jordan block (monodromy around $\psi = \infty$). As the Zariski closure of the monodromy is known to be semi-simple, and contained in Sp_4, this leaves no choice.

Near $\lambda = \infty$, the nilpotent orbit is of Hodge-Tate type, so that E gives rise to an iterated extension of $\mathbb{Z}(-3)$ by $\mathbb{Z}(-2)$ by $\mathbb{Z}(-1)$ by $\mathbb{Z}(0)$. Let q_1 (resp. q_2) be the class of the resulting extension of $\mathbb{Z}(-3)$ by $\mathbb{Z}(-2)$ (resp. of $\mathbb{Z}(-2)$ by $\mathbb{Z}(-1)$). If, for every value of λ near ∞, q_1 and q_2 are algebraically dependent over $\mathbb{Q}$, a Baire category argument and analytic continuation show that $F(q_1, q_2) = 0$ identically for some polynomial F.

For all but finitely many points (q_1^0, q_2^0) on the algebraic curve $F(q_1, q_2) = 0$, in a neighborhood of (q_1^0, q_2^0), the equation $F(q_1, q_2) = 0$ amounts to $q_2/q_2^0 = \varphi(q_1/q_1^0)$ for some function φ depending on (q_1^0, q_2^0). We write $\varphi[q_1^0, q_2^0]$ for that function. It maps 1 to 1, and its ∞-jet at 1, $\varphi_\infty[q_1^0, q_2^0]$, depends algebraically on (q_1^0, q_2^0), i.e. for each k, the k-jet $\varphi_k[q_1^0, q_2^0]$ does so.

The ∞-jet $\varphi_\infty[q_1^0, q_2^0]$ carries the following information. Fix λ_0 near ∞, giving rise to (q_1^0, q_2^0). Fix on V a new integral basis e_i $(0 \le i \le 3)$, with e_i in $F^i \cap W_{-2i}$ and projecting in $Gr^W_{-2i}(V) \simeq \mathbb{Z}(-i)$ to the integral generator. In this new basis V_{λ_0} is the split mixed Hodge structure $\bigoplus_0^3 \mathbb{Z}(-i)$. The ∞-jet $\varphi_\infty[q_1^0, q_2^0]$ tells how, in this basis, the Hodge filtration changes with λ, up to a reparametrization $\lambda \mapsto \varphi(\lambda)$.

Let $\mathbb{P}(V)$ be the projective space of rays in V. The symplectic form ψ gives on $\mathbb{P}(V)$ a "null-system" of lines, corresponding to the Lagrangian subspaces of V.

The variable Hodge filtration F gives a curve Γ in $\mathbb{P}(V)$: the trajectory of F^3. At a general point of Γ, the tangent line is a null line, corresponding to F^2. The weight filtration gives an isotropic flag w. The previous discussion gives: for each $\gamma \in \Gamma$, let us consider the relative position of the ∞ order jet Γ_γ of Γ at γ, and of w. Here, "relative position" is an element of the quotient by $\mathrm{Sp}(4)$ of the space of pairs (∞-order jet of curve, isotropic flag). For variable γ, this relative position is controlled by a $\varphi_\infty[q_0, q_1]$, hence stays on a 1-dimensional algebraic subspace. If we analytically continue and come back to γ, we see that (Γ_γ, w') is again in this subspace, for a Zariski dense set of w', hence for all w'. As the space of all isotropic flags is 4-dimensional, there is a 3-dimensional space of w' such that all (Γ_γ, w') are in the same relative position. The ∞-jet Γ_γ is hence stable by a subgroup H of $\mathrm{Sp}(4)$, fixing γ, and of dimension ≥ 3. The curve Γ spans $\mathbb{P}(V)$, hence cannot be pointwise fixed. It must be an orbit of H, hence is an algebraic curve. Contradiction: no algebraic curve has a group of automorphisms of dimension ≥ 3, all fixing a point γ.

14. Example

The following occurs in the study of mirror symmetry.

Let V be a polarized variation of Hodge structure on D^{*n}, of the following type:

(14.1) weight -3 (like an H_3), Hodge numbers h^{pq} $(p = 0, -1, -2, -3): 1, n, n, 1$

(14.2) polarization form ψ of discriminant 1

(14.3) unipotent monodromy, and associated nilpotent orbit of Hodge-Tate type. Rank of the $Gr_i^W(V_\mathbb{Z})$ $(i = 0, -2, -4, -6): 1, n, n, 1$. The form ψ is a perfect duality between Gr_{-i}^W and $Gr_{-(6-i)}^W$.

(14.4) For 1 a basis of $Gr_0^W(V_\mathbb{Z})$, the $N_i(1)$ form a basis of $Gr_{-2}^W(V_\mathbb{Z})$.

Fix a basis $1 \in Gr_0^W(V_\mathbb{Z})$. By definition of the weight filtration and (14.4), $Gr^W(V_\mathbb{Q})$, as a module over the polynomial algebra $\mathbb{Q}[N_1, \ldots, N_n]$, is generated by 1. As a quotient of $\mathbb{Q}[N_1, \ldots, N_n]$, it inherits an algebra structure, with unit 1 and with N_i = multiplication by $N_i(1)$. The integral graded module $Gr^W(V_\mathbb{Z})$ is a subalgebra: as $Gr^W(V_\mathbb{Z}) = 0$ for $i \neq 0, -2, -4, -6$, it suffices to check that it is stable by multiplication by $v \in Gr_i^W(V_\mathbb{Z})$, $i = 0$ or -2 which is clear: $G_0^W(V_\mathbb{Z}) = \mathbb{Z} \cdot 1$ and, if L is the free module spanned by $N_1, \ldots, N_n$,

$$L \xrightarrow{\sim} Gr_0^W(V_\mathbb{Z}) : \ell \to \ell(1).$$

The form induced by ψ can be recovered from the linear form t on $Gr_{-6}^W(V_\mathbb{Z})$: $t(v) := \psi(1, v)$. Indeed, for $\ell \in Gr_{-2}^W(V_\mathbb{Z})$, $\lambda \in Gr_{-4}^W(V_\mathbb{Z})$, one has

$$\psi(\ell, \lambda) = -\psi(1, \ell\lambda) = -t(\ell\lambda).$$

The algebra structure, and t, can be recovered from the symmetric trilinear form $\phi(x,y,z) := t(xyz)$ on $L = Gr^W_{-2}(V_{\mathbb{Z}})$: the algebra is

$$\mathbb{Z} \oplus L \oplus L^\vee \oplus \mathbb{Z},$$

the first $\mathbb{Z}$ spanned by 1, the second by $1'$ with $t(1') = 1$. The product of $\ell \in L$ with $\lambda \in L^\vee$ is $\lambda(\ell) \cdot 1'$ while that of $x, y \in L$ is the linear form $z \mapsto \phi(x,y,z)$.

We now consider the extension classes

$$E_0 \in \operatorname{Ext}^1\left(Gr^W_0(V_{\mathbb{Z}}), G^W_1(V_{\mathbb{Z}})\right) = L \otimes \mathcal{O}^*_{\text{mer}} D^{(*n)}) = \operatorname{Hom}\left(L^\vee, \mathcal{O}^*_{\text{mer}}(D^{*n})\right)$$

$$E_1 \in \operatorname{Ext}^1\left(Gr^W_{-2}(V_{\mathbb{Z}}), Gr^W_{-4}(V_{\mathbb{Z}})0\right) = L^\vee \otimes L^\vee \otimes \mathcal{O}^*_{\text{mer}}(D^*)$$
$$= \operatorname{Hom}\left(L \otimes L, \mathcal{O}^*_{\text{mer}}(D^{*n})\right)$$

$$E_2 \in \operatorname{Ext}^1\left(Gr^W_{-4}(V_{\mathbb{Z}}), Gr^W_{-6}(V_{\mathbb{Z}})\right) = L \otimes \mathcal{O}^*_{\text{mer}}(D^*) = \operatorname{Hom}\left(L^\vee, \mathcal{O}^*_{\text{mer}}(D^{*n})\right)$$

The valuation of the E_k along $z_i = 0$ are given by N_i (10.2). For E_0, recalling that $L = \mathbb{Z}^n$, we find $E_0 = (q_1^{-1}, \ldots, q_n^{-1})$ and the q_i form a system of local coordinates near 0. For $\lambda = (a_1, \ldots, a_n) \in L^\vee = \mathbb{Z}^n$, we write $q^\lambda := \prod q_i^{a_i}$. We have then

$$E_0 : \lambda \longmapsto q^{-\lambda}$$
$$E_1 : \ell, m \longmapsto q^{-\ell \cdot m}\ U_{\ell,m}, \quad \text{with } U_{\ell,m} \text{ invertible on } D^n.$$
$$E_2 : \lambda \longmapsto q^{-\lambda};$$

compatibility with ψ imposes $E_0(\lambda) = E_2(\lambda)$, $E_1(\ell, m) = E_1(m, \ell)$.

Let us expand the $U_{\ell,m}$ as an infinite product, extended over the $\lambda \in L^\vee$, $\lambda = (a_1, \ldots, a_n)$, with $a_i \geq 0$, $\lambda \neq 0$.

$$U_{\ell,m} = c_{\ell,m} \prod_\lambda (1 - q^\lambda)^{e(\ell,m,\lambda)},$$

with $c_{\ell,m}$ a constant, bimultiplicative in ℓ, m and $e(\ell, m, \lambda)$ complex, biadditive in ℓ, m. The integrability condition 10.3 reduces to

$$e(e_i, e_j, \lambda)\lambda_a = e(e_a, e_j, \lambda)\lambda_i,$$

Thus we obtain an expansion

$$E_1(\ell, m) = c_{\ell,m} q^{-\ell.m} \prod_\lambda (1 - q)^{\lambda(\ell)\lambda(m)e(\lambda)}.$$

In the mirror story, $\mathbb{Z}$, L, $L^\vee$, $\mathbb{Z}$ become the H^0, H^2, H^4, H^6 of a Calabi–Yau threefold, the basis of L some basis of H^2 contained in the ample cone, and λ runs over the cohomology class of rational curves (in H^4), with $e(\lambda)$, the number of rational curves in a given class, suitably defined.

For the components of E_1 in the given basis of $L = \mathbb{Z}^n$, and the dual basis of $L^\vee$, this gives

$$c_{i,j} \prod (q_a)^{-(e_i e_j)_a} \prod_\lambda \left(1 - \prod q_a^{\lambda_a}\right)^{e(\lambda)\lambda_i\lambda_j},$$

having as logarithmic derivative

$$-\sum_a \frac{dq_a}{q_a} \left\{ (e_i e_j)_a + \sum_\lambda \frac{e(\lambda)\lambda_i\lambda_j\lambda_a \prod q_b^{\lambda_b}}{1 - \prod q_b^{\lambda_b}} \right\}.$$

15. Integrality

For the degenerating elliptic curve 1.13 over D^* (coordinate z), the formal power series $\sum a_n z^n$ Taylor expansion of q as a function of z has a purely algebraic sense: if the coefficients c_α of an equation for E are expanded as formal power series in z: $c_\alpha = \sum c_{\alpha,n} z^n$, then the a_n are polynomials in the $c_{\alpha,m}$. Further, the story makes sense in any characteristic, and is compatible with reduction modulo p. I expect similar integrality to hold in general.

Bibliography

[1] Ph. Candelas, X. C. de la Ossa, P. S. Green and L. Parkes, A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. Nuclear Phys. B359 (1991) 21–74. Reprinted in: *Essays on Mirror Manifolds*, ed. S. T. Yau, pp. 31–95. International Press, Hong Kong 1992.

[2] E. Cattani and A. Kaplan, Degenerating variations of Hodge structures, in *Théorie de Hodge*, Astérisque 179-180 (1989) pp. 67–96.

[3] D. Morrison, Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians, J. Amer. Math. Soc. 6 (1993) 223–247.

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March, 1993