Notes on Booker's Paper

by

Peter Sarnak* (2002)†

The main calculation is carried out in a different notation (to check it!)

Consider the case of ρ being a 2-dimensional odd irreducible Galois representation.

$$L(s,\rho) = \sum_{n=1}^{\infty} \lambda_\rho(n) n^{-s}\,,\; L(s,\tilde{\rho}) = \sum_{n=1}^{\infty} \lambda_{\tilde{\rho}}(n) n^{-s}$$

For $\alpha \in \mathbb{Q}^*$ set

$$L(s,\rho,\alpha) = \sum_{n=1}^{\infty} \lambda_\rho(n)\, e(n\alpha)\, n^{-s}\,, \quad \text{etc.}$$

$$\Lambda(s,\rho) = (2\pi)^{-s}\,\Gamma(s)\,L(s,\rho)$$

then

$$\Lambda(s,\rho) = \epsilon\, N^{\frac{1}{2}-s}\,\Lambda(1-s,\tilde{\rho})\,.$$

Set

$$F(z) = \sum_{n=1}^{\infty} \lambda_\rho(n)\, e(nz) \;\text{ for }\; z \in \mathbb{H} \;\text{ and}$$

$$G(z) = \sum_{n=1}^{\infty} \lambda_{\tilde{\rho}}(n)\, e(nz) \;\text{ for }\; z \in \mathbb{H}\,.$$

A standard calculation of Hecke shows that $\Lambda(s,\rho)$ is entire iff

$$F(z) = \frac{\epsilon}{\sqrt{N}z}\, G\left(\frac{-1}{Nz}\right) \tag{1}$$

To examine the possible poles of $L(s,\rho,\alpha)$ consider the relation (1) (we assume that $\Lambda(s,\rho)$ is entire) when z approaches the "cusp" α and hence $-\frac{1}{Nz}$ approaches the "cusp" $-1/N\alpha$.

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†The paper "Poles of Artin L-functions and the strong Artin conjecture" has now appeared in the *Annals of Math.*, **158**, (2003), 1089-1098.

Set $z = \alpha + iy, \quad y \downarrow 0.$

$$\frac{-2\pi i}{N(\alpha + iy)} = \frac{-2\pi i}{N\alpha} \left(1 - \frac{iy}{\alpha} - \frac{y^2}{\alpha^2}\right) + O(y^3) \tag{2}$$

In particular,

$$\Re\left(\frac{-2\pi i}{N(\alpha + iy)}\right) = \frac{-2\pi y}{N\alpha^2} + O(y^3)$$

and hence for $\eta > 0$ arbitrarily small (and $\lambda_{\tilde{\rho}}(n) = O_\epsilon(n^\epsilon)$),

$$\sum_{n \geq y^{-1-\eta}} \lambda_{\tilde{\rho}}(n)\, e\left(\frac{-n}{N(\alpha + iy)}\right) = O(y) \tag{3}$$

We have from (1) and (2)

$$(\alpha + iy) \sum_{n=1}^{\infty} \lambda_\rho(n)\, e(n\alpha)\, e^{-2\pi ny}$$

$$= \frac{\epsilon}{\sqrt{N}} \sum_{n \leq y^{-1-\eta}} \lambda_{\tilde{\rho}}(n)\, e\left(\frac{-n}{N(\alpha + iy)}\right) + O(y)\,. \tag{4}$$

$$= \quad \frac{\epsilon}{\sqrt{N}} \sum_{n \leq y^{-1-\eta}} \lambda_{\tilde{\rho}}(n)\, e\left(\frac{-n}{N\alpha}\right) e^{-2\pi ny/N\alpha^2} \left(1 + \frac{2\pi i n y^2}{N\alpha^3}\right)$$

$$+ O\left(y^{1-2\eta}\right)$$

$$= \frac{\epsilon}{\sqrt{N}} \sum_{n=1}^{\infty} \lambda_{\tilde{\rho}}(n)\, e\left(\frac{-n}{N\alpha}\right) e^{\frac{-2\pi ny}{N\alpha^2}} \left(1 + \frac{2\pi i n y^2}{N\alpha^3}\right) + O\left(y^{1-2\eta}\right)\,. \tag{5}$$

Set

$$H(y) = (\alpha + iy) \sum_{n=1}^{\infty} \lambda_\rho(n)\, e(n\alpha)\, e^{-2\pi ny}$$

$$- \frac{\epsilon}{\sqrt{N}} \sum_{n=1}^{\infty} \lambda_\rho(n)\, e\left(\frac{-n}{N\alpha}\right) e^{\frac{-2\pi ny}{N\alpha^2}} \left(1 + \frac{2\pi i n y^2}{N\alpha^3}\right)\,. \tag{6}$$

Then according to (5) we have that

$$\left.\begin{array}{l} H(y) \;=\; O(y^{1-2\eta}) \text{ as } \quad y \downarrow 0 \\ \text{and clearly } H(y) \text{ is rapidly} \end{array}\right\} \tag{7}$$

decreasing as $y \to \infty$.

Hence,

$$\widetilde{H}(s) \;=\; \int_0^\infty H(y)\, y^s \, \frac{dy}{y} \quad \text{is holomorphic in } \Re(s) \;>\; 1 \;+\; 2\eta\,. \tag{8}$$

Note that if we set

$$H_1(y) \;=\; \sum_{n=1}^{\infty} \lambda_\rho(n)\, e(n\alpha)\, e^{-2\pi n y}\,.$$

and

$$H_2(y) \;=\; \sum_{n=1}^{\infty} \lambda_{\tilde{\rho}}(n)\, e\left(\frac{-n}{N\alpha}\right)\, e^{\frac{-2\pi n y}{N\alpha^2}}$$

then

$$H(y) \;=\; \alpha H_1(y) \;+\; iy\, H_1(y) \;-\; \frac{\epsilon}{\sqrt{N}}\, H_2(y) \;+\; \frac{\epsilon i y^2}{\sqrt{N}}\, H_2'(y)\,.$$

The idea now is that if $L(s,\rho,\alpha)$ has a pole at s_0 with $0 < \Re(s_0) < 1$ (say a simple pole and no other poles) then

$$H_1(y) \;\sim\; A y^{-s_0} \quad \text{as} \quad y \downarrow 0\,.$$

From (7) it follows that

$$\frac{\epsilon}{\sqrt{N}}\, H_2(y) \;\sim\; \alpha A y^{-s_0}$$

(since the other terms in $H(y)$ are $O(1)$.)

But then

$$iyH_1(y) \sim iAy^{-s_0+1}$$

while

$$\frac{\epsilon}{\sqrt{N}}\, i\, \frac{y^2}{\alpha}\, H_2'(y) \sim -s_0 i A y^{-s_0+1}$$

So these last can cancel only if $s_0 = 1$.

To formalize this (since $L(s,\rho,\alpha)$ and $L(s,\tilde{\rho},-1/N\alpha)$ may have many poles) we compute $\widetilde{H}(s)$ from (6). We find that

$$\begin{aligned}\widetilde{H}(s) &= \alpha\,\Lambda\,(s,\rho,\alpha) + i\,\Lambda\,(s\,+\,1,\rho,\alpha) \\ &- \frac{\epsilon}{\sqrt{N}}\,\Lambda\,(s,\tilde{\rho},-\frac{1}{N\alpha})\,(N\alpha^2)^s - \frac{i\epsilon}{\alpha\sqrt{N}}\,(s+1)\,\Lambda\,(s+1,\tilde{\rho},-\frac{1}{N\alpha})\cdot(N\alpha^2)^{s+1} \qquad (9)\end{aligned}$$

where

$$\Lambda(s,\rho,\beta) \;=\; (2\pi)^{-s}\,\Gamma(s)\sum_{n=1}^{\infty}\lambda_\rho(n)\,e(n\beta)m^{-s}\,.$$

From Brauer and the passage from additive to multiplicative characters we have that $\Lambda(s,\rho,\beta)$ and $\Lambda(s,\tilde{\rho},\beta)$ are merormorphic and have no poles in $\Re(s)\geq 1$ and $-1<\Re(s)<0$.

Now suppose that $\Lambda(s,\rho,\alpha)$ has a pole at $s=s_0$ with $0<\Re(s_0)<1$. Say

$$\Lambda(s,\rho,\alpha) \;=\; \frac{A_0}{(s-s_0)^k} \;+\;\cdots$$

with $k\geq 1$ and $A_0\neq 0$.

Since $H(s)$ is holomorphic in $\Re(s)>1$ and the 2^{nd} and 4^{th} terms in (9) are as well, we have that

$$(N\alpha^2)^s\,\Lambda\,(s,\tilde{\rho},-\frac{1}{N\alpha}) \;=\; \frac{B_0}{(s-s_0)^k} \;+\;\cdots$$

with B_0 satisfying

$$\alpha A_0 \;=\; \frac{\epsilon B_0}{\sqrt{N}}\,. \qquad (10)$$

Now consider the potential pole at $s=s_0-1$ of $\widetilde{H}(s)$. At such a point the 1^{st} and 3^{rd} terms in (9) don't have poles. The 2^{nd} and 4^{th} have expansions

$$i\,\Lambda\,(s+1,\rho,\alpha) \;=\; \frac{iA_0}{(s-(s_0-1))^k} \;+\;\cdots$$

and

$$-\frac{\epsilon i}{\alpha\sqrt{N}}\,s_0\,\frac{B_0}{(s-(s_0-1))^k} \;+\;\cdots$$

respectively.

Since these must cancel we have that

$$iA_0 \;=\; \frac{\epsilon i B_0 s_0}{\alpha\sqrt{N}} \qquad (11)$$

Thus, from (10) and (11) we see that $s_0 = 1$.

Thus, the only pole that $\Lambda(s, \rho, \alpha)$ can accommodate is at $s_0 = 1$. The passage to $\Lambda(s, \rho \otimes \chi)$ shows that the same is true for these twisted L-functions. Since ρ is irreducible, these don't have poles at $s = 1$ (or on $\Re(s) = 1$ or $\Re(s) = 0$). Thus, $\Lambda(s, \rho \otimes \chi)$ is entire.

The case of even Galois representations can be analyzed in a similar fashion. Say, ρ is self-dual for example and that

$$\Lambda(s, \rho) \;=\; \pi^{-s}\,\Gamma^2\left(\frac{s}{2}\right)\,L(s, \rho) \;=\; \epsilon N^{\frac{1}{2}-s}\,\Lambda(1-s, \rho)\,.$$

Then Hecke's argument leads to

$$\begin{aligned} F(z) &:= \sum_{n=1}^{\infty} \lambda_\rho(n) y^{1/2}\, K_0(2\pi n y)\, \cos(2\pi n x) \\ &= \epsilon F(-1/Nz)\,. \end{aligned}$$

iff $\Lambda(s, \rho)$ is entire.

Now proceed with an analysis of the behavior of $F(z)$ as $z \to \alpha$ on the $\ell.h.s.$ above and $-1/Nz \to -\frac{1}{N\alpha}$ on the right. I haven't carried out the details.

June 11, 2004:gpp.