1. 2. lecture

(1) Loose ends ; Phragmén Lindelöf

(A) $$\frac{F'}{F}(s) = c + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{n=0}^{\infty}\left(\frac{\lambda_i}{\lambda_i s+\mu_i+n} - \frac{\lambda_i}{\mu_i+n}\right) - \frac{m}{s-1}$$

$\frac{1}{2} \le \sigma \le 2$; $t > 2$

(B) $$\frac{F'}{F}(s) = \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + O(\log t)$$

(C) $$R\,\frac{F'}{F}(s) = \sum \frac{\sigma - \frac{1}{2}}{(\sigma-\frac{1}{2})^2 + (t-\gamma)^2} + O(\log t)$$

Lemma $\exists\ T_m$; $m < T_m < m+1$

σ_m ; $-m-1 < \sigma_m < -m$

$$\frac{F'}{F}(s) = O(\log^2 m) \text{ for } \sigma \ge \sigma_m ;\ t = \pm T_m$$

and $|t| < T_m$ $\sigma = \sigma_m$

c small constant.

$$|\gamma - T_m| > \frac{c}{\log m}$$

$$\left|\sigma_m + R\,\frac{n+\mu_i}{\lambda_i}\right| > c$$

$x > 2$

$$\frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha+i\infty} \frac{y^z}{z^2}\,dz = \begin{cases} \log y ; & y \ge 1 \\ 0 ; & 0 < y \le 1 \end{cases}$$

$$\theta_x(n) = \begin{cases} 1 & \text{for } 1 < n \le x \\ \dfrac{\log \frac{x^2}{n}}{\log x} & \text{''}\quad x \le n \le x^2 \\ 0 & n \ge x^2 \end{cases}$$

write. $b_n\,\theta_x(n) = b_x(n)$

Prove

$$\frac{F'}{F}(s) = -\sum_{n<x^2} \frac{\Lambda_x(n)\log n}{n^s} + \frac{1}{\log x}\sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}$$

$$\underbrace{- m\,\frac{x^{1-s} - x^{2(1-s)}}{(1-s)^2 \log x} + \frac{1}{\log x}\sum_{n,i} \frac{x^{-\left(\frac{n+\mu_i}{\lambda_i}+s\right)} - x^{-2\left(\quad\right)}}{\left(\frac{n+\mu_i}{\lambda_i}+s\right)^2}}_{O\left(\frac{x^{-\sigma}}{t^2}\right)}$$

$x < t^2$ $\quad + \; O($

Proof consider $\quad \alpha > 1$

$$J = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{F'}{F}(z)\,\frac{x^{2(z-s)} - x^{(z-s)}}{\log x\,(z-s)^2}\,dz$$

compute in two ways directly term by term

$$J = -\sum_{n \le x^2} \frac{\Lambda_x(n)}{n^s}\log n$$

secondly moving path of integration over singularities (using lemma with T_m, σ_m) we get equal to sum over residues at simple poles (only ones) $z = s$; $z = \rho$; $z =$ trivial zero a possible pole at $z = 1$ get.

$$J = \frac{F'}{F}(s) + \frac{1}{\lg x} \sum_{\rho} \frac{x^{2(\rho - s)} - x^{\rho - s}}{(s-\rho)^2}$$

$$+ \frac{1}{\lg x} \sum_{n, i} \frac{x^{-2\left(\frac{n+\mu_i}{\lambda_i} + s\right)} - x^{-\left(\frac{n+\mu_i}{\lambda_i} + s\right)}}{\left(\frac{n+\mu_i}{\lambda_i} + s\right)^2}$$

$$- \frac{m}{\lg x} \frac{x^{2(1-s)} - x^{1-s}}{(s-1)^2}$$

or:

$$\frac{F'}{F}(s) = - \sum_{n \le x^2} \frac{b_x(n) \lg n}{n^s} + \frac{1}{\lg x} \sum_{\rho} \frac{x^{\rho - s} - x^{2(\rho - s)}}{(s-\rho)^2}$$

$$+ \frac{1}{\lg x} \sum_{n, i} \frac{x^{-\left(\frac{n+\mu_i}{\lambda_i} + s\right)} - x^{-2(\quad)}}{\left(s + \frac{n+\mu_i}{\lambda_i}\right)^2}$$

$$- \frac{m}{\lg x} \frac{x^{1-s} - x^{2(1-s)}}{(s-1)^2}$$

R. H.

assume $2 < x < t^2$

$\sigma_1 = \frac{1}{2} + \frac{1}{\log x}$; $\sigma \geq \sigma^* \geq \sigma_1$

$$\frac{F'}{F}(\sigma + it) = - \sum_{n \le x^2} \frac{b_x(n) \lg n}{n^{\sigma + it}}$$

$$+ 2\omega \frac{x^{\frac{1}{2} - \sigma}}{\log x} \sum_{\gamma} \frac{1}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} + O\left(\frac{x^{\frac{1}{2} - \sigma}}{\lg x}\right)$$

$|\omega| < 1$

$|\omega'| < 1$

$$2\omega' \frac{x^{\frac{1}{2} - \sigma}}{(\sigma^* - \frac{1}{2}) \lg x} \sum_{\gamma} \frac{\sigma^* - \frac{1}{2}}{(\sigma^* - \frac{1}{2})^2 + (t-\gamma)^2}$$ / real ℓ.

$|2\omega x^{\frac{1}{2}-\sigma_1}| < \frac{2}{e} < 1$ φ

get estimate for $\sum \frac{\sigma^* - \frac{1}{2}}{(\sigma^* - \frac{1}{2})^2 + (t-\gamma)^2}$

and

$$\frac{F'}{F}(s) = -\sum_{n < x^2} \frac{b_x(n) \log n}{n^{\sigma+it}} +$$

$$+ O\left(\frac{x^{\frac{1}{2}-\sigma}}{(\sigma^* - \frac{1}{2}) \log x} \left| \sum_{n \le x^2} \frac{b_x(n) \log n}{n^{\sigma^* + it}} \right| \right)$$

$$+ O\left(\frac{x^{\frac{1}{2}-\sigma} \log t}{(\sigma^* - \frac{1}{2}) \log x} \right)$$

$(\sigma^* - \frac{1}{2}) \log x \ge 1$
can be omitted.

for $\sigma \ge \sigma_1$, integrate to get

$$\log F(\sigma+it) = \sum_{n \le x} \frac{b_x(n)}{n^{\sigma+it}} +$$

$$+ O\left(\frac{x^{\frac{1}{2}-\sigma}}{(\sigma^* - \frac{1}{2}) \log^2 x} \left| \sum_{n \le x^2} \frac{b_x(n) \log n}{n^{\sigma^* + it}} \right| \right)$$

$$+ O\left(\frac{x^{\frac{1}{2}-\sigma} \log t}{(\sigma^* - \frac{1}{2}) \log^2 x} \right)$$

Next assume $\frac{1}{2} \le \sigma \le \sigma_1$

write $s_1 = \sigma_1 + it$;

have

$$\frac{F'}{F}(s) - \frac{F'}{F}(s_1) - (s-s_1)\left(\frac{F'}{F}\right)'(s_1)$$

$$= \sum_{\rho} \frac{(s-s_1)^2}{(s-\rho)(s_1-\rho)^2} + O\left(\frac{\log t}{(\sigma_1-\frac{1}{2})^2 \log^2 x}\right)$$

$$= O(\log t)$$

$$= O\left(\frac{\sigma_1-\frac{1}{2}}{|\sigma-\frac{1}{2}-i\eta_t|} \sum_{\gamma} \frac{\sigma_1-\frac{1}{2}}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2}\right)$$

$$\eta_t = \min_{\gamma} |t-\gamma|$$

$$\left(\frac{F'}{F}\right)'(s_1) = -\sum \frac{1}{(s_1-\rho)^2} + O(1)$$

$$\log F(s) - \log F(s_1)$$

$$= (s-s_1)\frac{F'}{F}(s_1) - \frac{(s-s_1)^2}{2}\left(\frac{F'}{F}\right)'(s_1)$$

$$+ O\left(\left(1+\log^+\left(\frac{1}{\eta_t}\right)\right) \sum_{\gamma} \frac{\sigma_1-\frac{1}{2}}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2}.\right)$$

gives finally

$\frac{1}{2} \le \sigma \le \sigma_1$

$$\log F(s) = \sum_{n \le x^2} \frac{b_x(n)}{n^{\sigma_1 + it}}$$

$$+ O\left(\frac{1}{\eta x} \left| \sum_{n \le x^2} \frac{b_x(n) \eta n}{n^{\sigma_1 + it}} \right| \left(1 + \log^+ \frac{1}{\eta_t \eta x}\right)\right)$$

$$+ O\left(\frac{\log t}{\log x} \left(1 + \log^+ \frac{1}{\eta_t \eta x}\right)\right).$$

Clearly term with η_t must be present since $\log|F(s)|$ may be infinite if $\eta_t = 0$ but if we look at imaginary part (argument) it can be omitted.

We use

$$\Im\left(\frac{F'}{F}(s) - \frac{F'}{F}(s_1)\right) = \Im \sum_\rho \frac{\sigma - \sigma_1}{(s-\rho)(s_1-\rho)}$$

$$+ O\left(\frac{\sigma - \sigma'}{t}\right) = \sum \frac{(\sigma - \sigma_1)(\sigma + \sigma_1 - 1)(t - \gamma)}{|s-\rho|^2 |s_1 - \rho|^2} + O(\sigma_1 - \tfrac{1}{2})$$

$$\le 2 \sum \frac{(\sigma_1 - \frac{1}{2})^2}{|s_1 - \rho|^2} \cdot \frac{|t-\gamma|}{|s-\rho|^2}$$

$$\int_{\frac{1}{2}}^{\sigma_1} \frac{|t-\gamma|}{|s-\rho|^2} \, d\sigma = O(1)$$

$$\int \frac{(t-\gamma)}{|s-\rho|^2} \, d\sigma$$

$$\int_{-\infty}^{\infty} \frac{|t-\gamma| \, du}{u^2 + |t-\gamma|^2} \le \pi$$