History ; own, Huber, lattice point. ost. Patterson, Lecture in 1971

Questions. for $SL(2,R)$

in General questions are best asked in terms of ~~Structure~~ Groups G and discrete subgroup Γ, assume $\Gamma \backslash G$ finite volume.

max. compact subgroup. A max noncomp. abelian subgroup

$G = KAK$. ; | $G = KAN$; N groups in the cusps.

cusp: parabolic. for $SL(2,R)$ $\gamma = \theta_1 \begin{pmatrix} R & 0 \\ 0 & R^{-1} \end{pmatrix} \theta_2$; how are angles distributed.

$c \neq 0$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{c} \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix}$

for $|c| \leq x$ how many are $\frac{a}{c}$ and $\frac{d}{c}$ distributed modulo 1.

We The general analysis best carried out directly on group. using functions of two arguments g_1 and g_2 defined by series

$$\sum_{\gamma \in \Gamma} f(g_1^{-1} \gamma g_2)$$

where. We shall instead here operate with two points z, ζ in H. and certain poincaré series that represent automorphic forms (not necessarily analytic) in both variables.

quickly recall. $z = x+iy$; $\zeta = \xi + i\eta$ ($y>0$, $\eta>0$)

$ds^2 = \frac{|dz|^2}{y^2}$; inv operator $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

invariant of two points.

$$u(z,\zeta) = \frac{|z-\bar{\zeta}|^2}{4y\eta} = \frac{e^{\rho}+e^{-\rho}+2}{4}$$

where $\rho = d(z,\zeta)$ invariant distance.

Area of circle $A(z,\zeta) = 4\pi\,(u(z,\zeta)-1)$

$gz = \frac{az+b}{cz+d}$; define $\varepsilon_g(z) = e^{2i\arg(cz+d)} = \frac{cz+d}{c\bar{z}+d}$

$$y_{gz} = \frac{y}{|cz+d|^2} \quad \text{here.}$$

operator equation

$$\frac{d^k}{d_g z^k} = (cz+d)^{k+1} \frac{d^k}{dz^k} (cz+d)^{k-1}$$

We say $f(z)$ is a form for Γ of ~~weight~~ index k, or (Γ, k) form

if $f(\gamma z) = \varepsilon_\gamma(z)^k f(z)$ for $\gamma \in \Gamma$.

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for $k \le h$ put.

$$D_{h,k} = (2i)^{h-k}\, y^{1-k} \frac{d^{h-k}}{dz^{h-k}}\, y^{h-1}$$

for $k \ge h$

$$D_{h,k} = (-2i)^{h-k}\, y^{k+1} \frac{d^{k-h}}{d\bar{z}^{k-h}}\, y^{-h-1}$$

<u>(Γ, k) form</u>

so that $D_{h,k} = \overline{D_{-h,-k}}$

then

$$D_{h,k}(qz) = \varepsilon_q(z)^h D_{h,k}(z)\, \varepsilon_q(z)^{-k}$$

then if f is form of index k for Γ

$D_{h,k} f$ is form of index h.

The operator

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2iky\frac{\partial}{\partial x}$$

carries a form of index k for Γ into another. Can show that if f is eigenfunction of Δ_k then $D_{h,k} f$ is eigen~~value~~ function of Δ_h with same eigenvalue (but $D_{h,k} f$ may of course vanish identically for some f.

Look at spectrum of eigenfunctions of Δ_k which are ~~forms of~~ (Γ, k) forms and for which $\iint_D |f|^2 \frac{dx\,dy}{y^2} < \infty$

with eigenvalue in form $\frac{1}{4} + r^2$. (whether r

The analysis of the spectrum for general k is essentially the same as for k = 0, only the spectrum of eigenvalues only changes when an eigenfunction is annihilated when passing from one level to another (apart from power of y)
(see that only analytic or antianalytic functions can be annihilated. So that say

for k > 0 as k increases ⋮

for k > 0 we have a finite no of eigenvalues of form ~~(k-l)(1+l-k)~~ $l(1-l)$ where $0 \leq l \leq k$. multiplicity of each equals no of regular analytic forms of weight l for Γ . similarly

for k < 0 ....

For k = 0

Ran ~~the~~ Hejhals ~~Springer~~ lecture, orthonormal eigenf. u_r^0 .

$$\sum f(u(\gamma z, \zeta)) = \frac{1}{2} \sum h(r) \, u_r^0(z) \overline{u_r^0(\zeta)}$$

with
$$h(r) = \iint_H y^{\frac{1}{2}+ir} f(u(z,i)) \frac{dx\,dy}{y^2}$$

For the other levels similar theories occur. will refer to $u_r^k(z)$ as orthonormal system of level k.

true analytic. constant. $\frac{1}{\sqrt{A(D)}}$ present only for ~~cts~~ level zero.

$h > k$ and $D_{k,h}\, u_n^h(z) = \lambda\, ~~D_{h,k}~~\, u_n^k(z)$

with ~~$|\lambda| =$~~

$$|\lambda|^2 = \frac{\Gamma(k+\frac{1}{2}+i\nu)\,\Gamma(k+\frac{1}{2}-i\nu)}{\Gamma(h+\frac{1}{2}+i\nu)\,\Gamma(h+\frac{1}{2}-i\nu)}$$

In cases when the eigenfunction disappears for level h; $\lambda = 0$ by this formula.

Experience shows that it is most convenient to work with dirichlet series

The general analytic vehicle we ~~shall choose is given by the~~ should need would seem to be given by

$$\sum_{\gamma \in \Gamma} \left(u(\gamma z, \zeta)\right)^{-s} \cdot e^{\,ih \arg \frac{\gamma z - \zeta}{\gamma z - \bar\zeta} + ik \arg \frac{\bar z - \gamma^{-1}\bar\zeta}{z - \gamma^{-1}\zeta}}$$

which converges for $\Re s > 1$ and is of level k in z and $-h$ in ζ. However in general (for $h \neq k$) this function is singular at $z = \zeta$ and so not very usable. Therefore we choose instead for ~~k~~ $h \geq k$

$$L^{k,h}(z,\xi;s)$$

$$= \sum_{\gamma\in\Gamma} \left(u(\gamma z,\xi)\right)^{-s}\left(1-\frac{1}{u(\gamma z,\xi)}\right)^{\frac{h-k}{2}} \cdot e^{ih\arg\frac{\gamma z-\xi}{\gamma z-\bar\xi} + ik\arg\frac{\bar z-\gamma^{-1}\bar\xi}{z-\gamma^{-1}\bar\xi}}$$

$$= \sum_{\gamma\in\Gamma} \varepsilon_\gamma^{-k}(z)\left(u(\gamma z,\xi)\right)^{-s}\frac{(\gamma z-\xi)^{h-k}\,|\gamma z-\bar\xi|^{2k}}{(\gamma z-\bar\xi)^{h+k}}$$

which does no such singularity

and for ~~k~~ $h \leq k$, we define

$$K^{k,h}(z,\xi;s) = L^{h,k}(\xi,z;\bar s)$$

For compact $\Gamma|H$ can show

for $h \geq k$

$$K^{k,h}(z,\xi;s) =$$

$$4\pi \sum_{\lambda} \varepsilon \left|\frac{\Gamma(h+\frac{1}{2}+i\nu)}{\Gamma(k+\frac{1}{2}+i\nu)}\right| \frac{\Gamma(s-\frac{1}{2}-i\nu)\,\Gamma(s-\frac{1}{2}+i\nu)}{\Gamma(s-k)\,\Gamma(s+h)}\, u_\lambda^{k}(z)\,\overline{u_\lambda^{h}(\xi)}$$

$\nearrow \varepsilon_{h,k,\lambda}$

The constant eigenfunction $\frac{1}{\sqrt{A(D)}}$ occurs only for $k = h = 0$

which it gives a term $\frac{4\pi}{A(D)} \frac{1}{s-1}$

which is only pole in region $\sigma \geq 1$

results. (assume first no eigenvalues between $0 < \; < \frac{1}{4}$

$$\sum_{u(z,\zeta) \leq X} e^{ih \arg \frac{\gamma z - \zeta}{\gamma \bar z - \bar\zeta} + ik \arg \frac{\bar z - \gamma^{-1} \bar\zeta}{z - \gamma^{-1}\zeta}}$$

$$= \frac{4\pi}{A(D)} X + O(X^{\frac{2}{3}})$$

for $h = k = 0$ $\quad \left(X^{\frac{2}{3}} + h^2 X^{\frac{1}{2}} \right)$

$$= O(X^{\frac{2}{3}}) \quad \left(X^{\frac{2}{3}} + (1+|k|)^2 X^{\frac{1}{2}}\right)$$

for $h = k \neq 0$ $\quad X^{\frac{2}{3}} + (1+|h|)(1+|k|) X^{\frac{1}{2}}$

$$= O(X^{\frac{2}{3}}) = O\left(X^{\frac{2}{3}} + \left((1+|h|)(1+|k|) X^{\frac{1}{2}}\right)\right)$$

for $h \neq k$.

if eigenvalues between 0 and $\frac{1}{4}$ of for $\rho_i(1-\rho_i)$ with $\rho \to \rho_i > \frac{1}{2}$ we get same remainder terms but have some leading terms $c_{\rho_i} X^{\rho_i} u_i(z) \overline{u_i(\zeta)}$

(use)

$$\sum_{n \leq R} |u^k_n(z)|^2 \ll_\rho (1+|k|+R)^2 \; ; \; n \leq R \sim \frac{A(D)}{2\pi} R^2$$

$$4\pi X^{\rho_i} \frac{\Gamma(2\rho_i - 1)}{\Gamma(\rho_i - k)\Gamma(\rho_i + h)} u^k_{\rho_i}(z) \overline{u^h_{\rho_i}(\zeta)}$$

automorphic functions for $m \neq 0$ square integrable
can be expanded in eigenfunctions
for $m = 0$ is Eisenstein series
~~finite poles~~
now look at fourier ~~transform~~ expansion of $u_\delta^m(z)$
in terms of $e^{2\pi i n x}$. Coeff of
$e^{2\pi i n x}$ involves series.

$$\sum_{\substack{c \neq 0 \\ \gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0}} \frac{e^{2\pi i \frac{m a + n d}{c}}}{|c|^{2s}} = Z^{m,n}(s)$$

meromorphic. for $\sigma \geq \frac{1}{2}$ has poles at most
in points $s = \frac{1}{2} + i r_j$, but for $m, n \neq 0, 0$
no pole at $s = 1$. for $m\,n = 0$ no poles on $\sigma = \frac{1}{2}$.
Eisenstein series.

$$\sum_{|c| < X} (X - |c|)\, e^{2\pi i n \frac{a}{c}} = \sum_{1 > \rho_i > \frac{1}{2}} c_i^{(n)} X^{1+2\rho_i} + o(X^2)$$

thus if no $1 > \rho_i > \frac{1}{2}$.

$$\sum_{|c| < X} (X - |c|)\, e^{2\pi i n \frac{a}{c}} = o(X^2)$$

for modular group taking $n = 1$ this is

$$\sum_{q < X} (X - q)\, \mu(q) = o(X^2).$$

which implies $\sum_{n \leq X} \mu(n) = o(X)$

AN EARLIER p. 8 ???

non compact, continuous spectrum but essentially identical results .... end-

application to.

consider fn discr. $-D$.

$$4AC - B^2 = D$$

(1) with $|A+C| < X$

number is $\frac{h(D)\, X}{\sqrt{D}} + O(X^{\frac{2}{3}})$

and. $$\sum_{\substack{4AC-B^2=D \\ |A+C|<X}} e^{ik \arg(A-C+i\sqrt{B})} = O(~~(1+|k|)\sqrt{X}~~)$$

$$= O(\text{[illegible]}\, X^{\frac{2}{3}} + |k|\sqrt{X})$$

S. J. Patterson Mathematika 25 (A lattice point problem in hyperb. plane.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a}{c}, \quad \frac{d}{c} \mod 1$ fn $|C| \leq \underline{\underline{X}}$,

analytic vehicle fn $m > 0$

$$U^{(m)}(z;s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y_\gamma^{s}\, e^{2\pi i m \gamma z}$$

fn $m < 0$

$$U^{m} = \sum y_\gamma^{s}\, e^{2\pi i m \gamma \bar{z}}$$