1

$$D_{h,k}(qz) = \varepsilon_q(z)^h \, D_{h,k}(z) \, \varepsilon_q(z)^{-k}$$

thus if f is form of index k for Γ

$D_{h,k} f$ is form of index h.

operator

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - zik\, y \frac{\partial}{\partial x}$$

carries form of index k into another of same index, can see that if f is eigenfunction of Δ_k then $D_{h,k} f$ eigenfunction of Δ_h with same eigenvalue. (but $D_{h,k} f$ may vanish identically).

look at spectrum of eigenfunctions of Δ_k which are (Γ, k) forms and for which $\int_{\mathcal{D}_\Gamma} |f|^2 d\omega_z < \infty$

write eigenvalue in form $\frac{1}{4} + r^2$ whether r real or imaginary. analysis of spectrum for general integral k essentially same as for $k = 0$, the only changes being by analytic forms; for $k > 0$ have finite nr of eigenvalues of form $\frac{1}{4} + r^2 = \ell(1-\ell)$ where ℓ is integral and $0 \leq \ell \leq k$ multiplicity of each equals nr of regular analytic forms of weight ℓ for Γ. Similarly for $k < 0$ anti-analytic...

Eigenfunction expansion of ...

(2) $r_i \quad u_i^k(z)$

$$D_{k,h}\, u_i^k(z) = \lambda\, u_i^h(z)$$

$$|\lambda|^2 = \frac{\Gamma(k+\frac{1}{2}+ir)\,\Gamma(k+\frac{1}{2}-ir)}{\Gamma(h+\frac{1}{2}+ir)\,\Gamma(h+\frac{1}{2}-ir)}$$

if u_i^k diss annihilated by $D_{k,h}$ $\lambda = 0$ by this formula.

for ~~[illegible]~~ ~~define~~ $h \geq k$ define

$$L^{k,h}(z,\xi;s)$$

$$= \sum_{\gamma\in\Gamma} u_1(\gamma z,\xi)^{-s}\left(1-\frac{1}{u_1(\gamma z,\xi)}\right)^{\frac{h-k}{2}} \cdot e^{ih\arg\frac{\gamma z-\xi}{\gamma z-\bar\xi}+ik\arg\frac{\bar z-\gamma^{-1}\bar\xi}{z-\gamma^{-1}\xi}}$$

$$= \sum_{\gamma\in\Gamma} \varepsilon_\gamma^{-k}(z)\,(u_1(\gamma z,\xi))^{-s}\,\frac{(\gamma z-\xi)^{h-k}\,|\gamma z-\bar\xi|^{2k}}{(\gamma z-\bar\xi)^{h+k}}$$

(type (Γ, k) in z, $(\Gamma,-h)$ in ξ.

for $h \leq k$

$$L^{k,h}(z,\xi;s) = \overline{L^{h,k}(\xi,z;\bar s)}$$

caution (simplicity suppose compact Γ)

$|\varepsilon| = 1$.

NB: for $h \neq k$ wrong real expression more complicated

$$L^{k,h}(z,\xi;s) = 4\pi \sum_{r_i} \varepsilon \left|\frac{\Gamma(h+\frac{1}{2}+ir)}{\Gamma(k+\frac{1}{2}+ir)}\right| \frac{\Gamma(s-\frac{1}{2}-ir)\,\Gamma(s-\frac{1}{2}+ir)}{\Gamma(s-k)\,\Gamma(s+h)}\, u_i^k(z)\,\overline{u_i^h(\xi)}$$

(3)

Constant eigenfunction $\frac{1}{\sqrt{A(\mathcal{D})}}$ only occurs for $h = k = 0$

which then gives a ~~pole~~ $\frac{4\pi}{A(\mathcal{D})} \frac{1}{(s-1)}$ only pole

in $\sigma \geq 1$

assume no eigenvalues between $0 <$ $< \frac{1}{4}$

$$\sum_{\mathcal{K}_1(z,\zeta) \leq X} e^{ih \arg \frac{\gamma z - \zeta}{\gamma z - \bar{\zeta}} + ik \arg \frac{\bar{z} - \gamma^{-1}\bar{\zeta}}{z - \gamma^{-1}\bar{\zeta}}}$$

$$= \frac{4\pi}{A(D)} X + O(X^{\frac{2}{3}}) \quad \text{for } h = k = 0$$

for $h^2 + k^2 > 0$

$$= O\left(X^{\frac{2}{3}} + (1 + |k|)(1 + |h|) X^{\frac{1}{2}} \right).$$

$$\sum_{|r_i| < R} |u^k_{r_i}(z)|^2 = O\left((1 + |k| + R)^2 \right)$$ unif. in $\underline{\underline{k}}$, z in compact region.

(4)

Lattice points. carry out for hyperbolic plane mention general result.

$$k_X'(z,\xi) = 1 \quad \text{for} \quad \frac{|z-\xi|^2}{4\eta} \le X, \quad 0 \text{ otherwise}$$

$$k_X(z,\xi) = X - \frac{|z-\xi|^2}{4\eta} \quad \text{for} \quad '' \qquad 0 \text{ otherwise}$$

$$A(X) = \sum_{\gamma\in\Gamma} k_X'(z,\gamma\xi)$$ no of lattice points in circle of area πX

$$A_1(X) = \sum_{\gamma\in\Gamma} k_X(z,\gamma\xi)$$

$$= \int_0^X A(u)\,du$$

have

$$A_1(X) = \sum h_1(r_i)\, u_i(z)\, \overline{u_i(\xi)}$$

$$h\left(\tfrac{i}{2}\right) = \int_0^X \pi t\,dt = \pi\frac{X^2}{2}$$

with $= h_1(r)$.

have $Q(w) =$ $\quad x = e^{\rho} + e^{-\rho} - 2$

$\rho \sim \log x$

$$= Q(w) = e^{(x-t)}(x-t)^{\frac{3}{2}}$$

$$h(r) = e\int_{-\rho}^{\rho} \left(e^{\rho}+e^{-\rho}-e^{u}-e^{-u}\right)^{\frac{3}{2}} e^{iur}\,du$$

r real

estimate $|r|<1$ $\quad h_X(r) \le O\, X^{\frac{3}{2}} \log X$;

$|r|\ge 1$ $\quad h_X(r) = O\left(\frac{X^{\frac{3}{2}}}{|r|^{\frac{5}{2}}}\right)$

5

for $|\xi| \le \frac{X}{2}$; $|r| > 1$

have $h_{X+\xi}(r) - h_X(r) = O\left(\xi \frac{X^{\frac{1}{2}}}{|r|^{\frac{3}{2}}}\right)$

$\xi X^{\frac{1}{2}} \le X$; $|r| \le \phi$

also

$$\sum_{|r_i| \le R} |u_i(z)|^2 = O(R^2)$$

want to estimate $A(X)$

have $\frac{1}{\xi}\int_{X-\xi}^{X} A(t)\,dt \le A(X) \le \frac{1}{\xi}\int_{X}^{X+\xi} A(t)\,dt$

$$A(X) \le \frac{A_1(X+\xi) - A_1(\cancel{X})}{\xi}$$

$$= \bar{u}\,X + O(\xi) + \sum_i \frac{h_{X+\xi}(r_i) - h_X(r_i)}{\xi}\, u_i(z)\overline{u_i(\xi)}$$

$$= O\left(X^{\frac{1}{2}}\xi X + \sum_{1 \le |r_i| \le R} \frac{\cancel{X}^{\frac{1}{2}}}{|r|^{\frac{3}{2}}}\left(|u_i(z)|^2 + |u_i(\xi)|^2\right)\right)$$

$$+ O\left(\frac{1}{\xi}\sum_{R \le r_i} \frac{X^{\frac{3}{2}}}{|r|^{\frac{5}{2}}}\left(\qquad\right)\right)$$

$$X^{\frac{1}{2}} R^{\frac{1}{2}} + \frac{1}{\xi}\frac{X^{\frac{3}{2}}}{\sqrt{R}} \quad ; \quad R = \frac{X}{\xi}$$

$$\xi + \frac{X^{\frac{3}{2}}}{\sqrt{\xi}} \quad ; \quad \xi = X^{\frac{2}{3}}.$$

$\bar{u}\,x + O(X^{\frac{2}{3}})$; general $cX + O\left(X^{1-\frac{1}{n+2}}\right)$