Euler:

$$(1) \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

$$(2) \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

generalizations:

Dirichlet:

$$\underset{\substack{0<t_i \\ \sum t_i<1}}{\int\cdots\int} t_1^{x_1-1}\cdots t_n^{x_n-1}(1-t_1-\cdots-t_n)^{x_{n+1}-1}\, dt_1\cdots dt_n$$

or more general:

$$\underset{\substack{0<t_i \\ \sum t_i<1}}{\int\cdots\int} t_1^{x_1-1}\cdots t_n^{x_n-1}(1-t_1)^{y_1-1}\cdots(1-t_1-\cdots-t_n)^{y_n-1}\, dt_1\cdots dt_n$$

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p prime ; $\varepsilon = e^{\frac{2\pi i}{p}}$, $\chi \neq \chi_0$

$$(1') \quad \tau_\chi = \sum_{h \,(\mathrm{mod}\, p)} \chi(h)\varepsilon^h \quad ; \quad |\tau_\chi|^2 = p$$

$$(2') \quad \sum_{h \,(\mathrm{mod}\, p)} \chi_1(h)\chi_2(1-h) = \frac{\tau_{\chi_1}\tau_{\chi_2}}{\tau_{\chi_1\chi_2}}$$

if $\chi_1 \neq \chi_0$, $\chi_2 \neq \chi_0$, $\chi_1\chi_2 \neq \chi_0$

Similar analogies to Dirichlets or more general multiple integrals.

$$\Delta(t) = \prod_{i<j} (t_j - t_i)$$

(3) $$\int_0^1 \cdots \int_0^1 (t_1 \cdots t_m)^{x-1} \left((1-t_1)\cdots(1-t_m)\right)^{y-1} |\Delta(t)|^{2z} dt_1 \cdots dt_n$$

$$= \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)\,\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(1+z)\,\Gamma(x+y+(n+\nu-2)z)}.$$

valid for real part of all $1+\nu z$, $x+(\nu-1)z$ and $y+(\nu-1)z$ positive $1 \le \nu \le n$.

(4) $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{|\Delta(t)|^{2z}\, dt_1 \cdots dt_m}{\left((\frac{1}{2}+it_1)\cdots(\frac{1}{2}+it_m)\right)^{x} \left((\frac{1}{2}-it_1)\cdots(\frac{1}{2}-it)\right)^{y}}$$

$$= (2\pi)^m \prod_{\nu=1}^{m} \frac{\Gamma(1+\nu z)\,\Gamma(x+y-1-(m+\nu-2)z)}{\Gamma(1+z)\,\Gamma(x-(\nu-1)z)\,\Gamma(y-(\nu-1)z)}$$

valid if real parts of $1+\nu z$, $x+y-1-(m+\nu-2)z$ are positive for $1 \le \nu \le m$

(5) $$\underset{\substack{t_i>0 \\ \sum t_i<1}}{\int \cdots \int} (t_1 \cdots t_m)^{x-1} (1-t_1-\cdots-t_m)^{y-1} |\Delta(t)|^{2z} dt_1 \cdots dt_n$$

$$= \frac{\Gamma(y)}{\Gamma(y+mx+m(m-1)z)} \prod_{\nu=1}^{m} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \Gamma(x+(\nu-1)z)$$

Limiting cases of (3)

$$(6)\quad \int_0^\infty \cdots \int_0^\infty (t_1 \cdots t_n)^{x-1} e^{-t_1-\cdots-t_n} |\Delta(t)|^{2z}\, dt_1 \cdots dt_n$$

$$= \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)}\, \Gamma(x+(\nu-1)z),$$

$$(7)\quad \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty e^{-t_1^2-\cdots-t_n^2} |\Delta(t)|^{2z}\, dt_1 \cdots dt_n$$

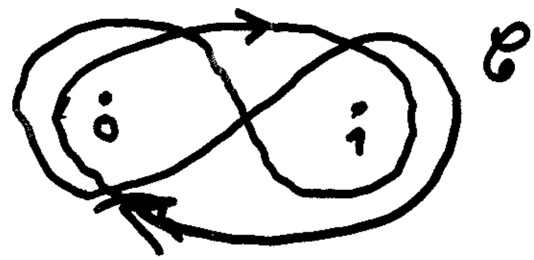
$$= \pi^{\frac{n}{2}} \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} .$$

Also versions for unit circle (or interval $-\pi, \pi$) involving $|\Delta(e^{i\theta})|^{2z}$ f.ex.

$$(8)\quad \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left(\prod_{j=1}^{n} (1+e^{i\theta_j}) \right)^{x+y-2(n-1)z-2} \cos\left(\frac{x-y}{2} \sum \theta_i\right) |\Delta(e^{i\theta})|^{2z}\, d\theta_1 \cdots d\theta_n$$

$$= (2\pi)^n \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)\ \Gamma(x+y-1-(n+\nu-2)z)}{\Gamma(1+z)\ \Gamma(x-(\nu-1)z)\,\Gamma(y-(\nu-1)z)},$$

and for $z \geq 0$, integral, version of (3) involving complex integration path and with no restrictions on x and y.



$$(9)\ \int_{\mathcal{C}}\cdots\int_{\mathcal{C}} (t_1\cdots t_n)^{x-1}((1-t_1)\cdots(1-t_n))^{y-1}|\Delta(t)|^{2z}dt_1\cdots dt_n$$

$$=(4\sin\pi x\sin\pi y)^n\prod_{\nu=1}^{n}\frac{\Gamma(1+\nu z)\,\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(1+z)\,\Gamma(x+y+(n+\nu-2)z)}$$

(1941)

$$(10)\ \sum_{h_1,h_2\,(\mathrm{mod}\ p)} \chi_1(h_1h_2)\,\chi_2((1-h_1)(1-h_2))\,\chi_3^2(h_1-h_2)$$

$$=\frac{\tau_{\chi_3^2}\ \tau_{\chi_1}\ \tau_{\chi_1\chi_3}\ \tau_{\chi_2}\ \tau_{\chi_2\chi_3}}{\tau_{\chi_3}\ \tau_{\chi_1\chi_2\chi_3}\ \tau_{\chi_1\chi_2\chi_3^2}}+$$

$$+\ (\chi_3\to\chi_3\psi)$$

where ψ is quadratic character. exceptions where certain of the τ symbols have principal character as subscript.

Form of (10) indicates correct analogue of (3) is:

Conjecture:

Let P_m run over all polynomials $x^m + a_1 x^{m-1} + \dots + a_m$ mod p and write $D(P_m)$ for the discriminant of P_m, then

$$(11)\quad \sum_{P_m} \chi_1\left((-1)^m P_m(0)\right) \chi_2\left(P_m(1)\right) \chi_3\phi\left(D(P_m)\right) = \prod_{\nu=1}^{n} \frac{\tau\chi_3^{\nu}}{\tau\chi_3} \, \frac{\tau\chi_1\chi_3^{\nu-1}\;\tau\chi_2\chi_3^{\nu-1}}{\tau\chi_1\chi_2\chi_m^{m+\nu-2}} .$$

Analogues of say (6), (7) and (5) can similarly be written down.

Proved for $n=2$ in 1941 by myself analog of (5) proved also for $n=3$ by myself and Ronald Evans ca 1980; Finally general form proved 1990, Greg Anderson

Some other Beta-type integrals:
Let $X_1, \dots, X_n$ be m dimensional vectors and e vector of unit length $|e| = 1$, then

$$\int\cdots\int \frac{dX_1\, dX_2 \cdots dX_n}{|X_1|^{\alpha_1}|X_2|^{\alpha_2}\cdots|X_n|^{\alpha_n}|e-X_1-\cdots-X_n|^{\alpha_{n+1}}}$$

$$= \pi^{\frac{mn}{2}} \frac{\Gamma\left(\frac{\Sigma\alpha - mn}{2}\right)}{\Gamma\left(\frac{(n+1)m-\Sigma\alpha}{2}\right)} \prod_{i=1}^{n+1} \frac{\Gamma\left(\frac{m-\alpha_i}{2}\right)}{\Gamma\left(\frac{\alpha_i}{2}\right)}.$$

Valid when all arguments of Γ's in numerator have pos. real part.

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let the Y denote symmetric n by n matrices and write $dY = \frac{\prod dy_{i,j}}{|Y|^{\frac{n+1}{2}}}$ and let E denote the unit matrix, then

$$\int\cdots\int_{\substack{Y_i>0 \\ Y_1+\cdots+Y_m<E}} |Y_1|^{x_1-1}|Y_2|^{x_2-1}\cdots|Y_m|^{x_m-1}|E-Y_1|^{z_1-1}\cdots|E-Y_1-\cdots-Y_m|^{z_m-1}\, dY_1\cdots dY_m$$

expressible in terms of a product of Γ-functions divided by another product of Γ-functions.

Also if we write

$$Y(s_1, \ldots, s_n) = \prod_{\nu=1}^{n} |Y^{(\nu)}|^{s_\nu},$$

where

$$Y^{(\nu)} = (y_{i,j})_{i,j \le \nu}.$$

Then

$$\int \cdots \int_{Y_i > 0} Y_1(s_{1,1}, \ldots, s_{1,n}) \cdots Y_m(s_{m,1}, \ldots, s_{m,n}) \cdot$$

$$Y_{m+1}(s_{m+1,1}, \ldots, s_{m+1,n})\, dY_1 \cdots dY_m$$

where

$$Y_{m+1} = E + Y_1 + \cdots + Y_m$$

is again so expressible. Here the integral exists if say the $s_{i,j}$ with $1 \le i \le m$ have nonnegative real part while the $s_{m+1,j}$ have sufficiently large negative real part. character analogues.