Before 1743.

Euler:

$$(1) \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

$$(2) \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

generalizations:

Dirichlet:

$$(3) \quad \int \cdots \int_{\substack{0<t_i \\ \sum t_i < 1}} t_1^{x_1-1} \cdots t_n^{x_n-1} (1-t_1-\cdots-t_n)^{x_{n+1}-1} dt_1 \cdots dt_n,$$

or, more general:

$$(4) \int \cdots \int_{\substack{0<t_i \\ \sum t_i<1}} t_1^{x_1-1} \cdots t_n^{x_n-1} (1-t_1)^{y_1-1} \cdots (1-t_1-t_2-\cdots-t_n)^{y_n-1} dt_1 \cdots dt_n$$

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p prime ; $\varepsilon = e^{\frac{2\pi i}{p}}$, $\chi \neq \chi_0$, Jacobi ca. 1837

$$(1)' \quad \tau_\chi = \sum_{h \,(\mathrm{mod}\, p)} \chi(h)\varepsilon^h \quad ; \ |\tau_\chi|^2 = p,$$

$$(2') \quad \sum_{h \,(\mathrm{mod}\, p)} \chi_1(h_1)\chi_2(1-h_1) = \frac{\tau_{\chi_1}\tau_{\chi_2}}{\tau_{\chi_1\chi_2}}$$

if $\chi_1 \neq \chi_0$, $\chi_2 \neq \chi_0$, $\chi_1\chi_2 \neq \chi_0$.

Similar analogies for (3) and (4).

$$\Delta(t) = \prod_{i<j}^{2} (t_j - t_i),$$

(5) $$\int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{x-1} ((1-t_1)\cdots(1-t_n))^{y-1} |\Delta(t)|^{2z} dt_1 \cdots dt_n =$$

$$= \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)\,\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(1+z)\,\Gamma(x+y+(n+\nu-2)z)},$$

valid for real parts of $1+\nu z$, $x+(\nu-1)z$ and $y+(\nu-1)z$ are positive for $1 \le \nu \le n$.

(6) $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{|\Delta(t)|^{2z} dt_1 \cdots dt_n}{((\tfrac12 + it_1)\cdots(\tfrac12 + it_n))^{x} (\tfrac12 - it_1)\cdots(\tfrac12 - it_n))^{y}} =$$

$$=(2\pi)^n \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)\,\Gamma(x+y-1-(n+\nu-2)z)}{\Gamma(1+z)\,\Gamma(x-(\nu-1)z)\,\Gamma(y-(\nu-1)z)},$$

valid of real parts of $1+\nu z$, $x+y-1-(n+\nu-2)z$ are positive for $1 \le \nu \le n$.

(7) $$\int_{\substack{t_i>0 \\ \sum t_i<1}} \cdots \int (t_1 \cdots t_n)^{x-1} (1-t_1-\cdots-t_n)^{y-1} |\Delta(t)|^{2z} dt_1 \cdots dt_n$$

$$= \frac{\Gamma(y)}{\Gamma(y+nx+n(n-1)z)} \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)}\,\Gamma(x+(\nu-1)z),$$

valid for real parts of y, $1+\nu z$ and $x+(\nu-1)z$ positive when $1 \le \nu \le n$.

Limiting cases of (5)

$$(8)\quad \int_0^\infty\cdots\int_0^\infty (t_1\cdots t_n)^{x-1} e^{-t_1-\cdots-t_n} |\Delta(t)|^{2z} dt_1\cdots dt_n$$

$$= \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \Gamma(x+(\nu-1)z),$$

$$(9)\quad \int_{-\infty}^\infty\cdots\int_{-\infty}^\infty e^{-t_1^2-\cdots-t_n^2} |\Delta(t)|^{2z} dt_1\cdots dt_n =$$

$$= \pi^{\frac{n}{2}} \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)},$$

Both valid when Γ-functions in numerator have arguments with positive real parts.

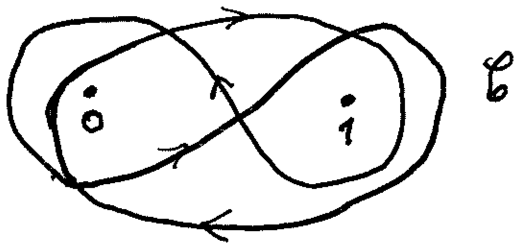
Also versions for unitcircle (or interval $-\pi, \pi$) involving $|\Delta(e^{i\theta})|^{2z}$, f. ex.

$$(10)\quad \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\Big(\prod_{j=1}^{n} |1+e^{i\theta_j}|\Big)^{x+y-2(n-1)z-2} \cos\Big(\frac{x-y}{2}\sum_1^n \theta_j\Big) |\Delta(e^{i\theta})|^{2z} d\theta_1\cdots d\theta_n$$

$$=(2\pi)^n \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \frac{\Gamma(x+y-1-(n+\nu-2)z)}{\Gamma(x-(\nu-1)z)\Gamma(y-(\nu-1)z)},$$

which is in reality only a modified form of (6), an analogous form of (5) can also be given.

For $z \geq 0$ and integral, there is a version of (5) involving complex integration path and with no restrictions on x and y,



$$(11)\quad \int_{\mathscr{C}} \cdots \int_{\mathscr{C}} (t_1 \cdots t_n)^{x-1} \left((1-t_1)\cdots(1-t_n)\right)^{y-1} \left(\Delta(t)\right)^{2z} dt_1 \cdots dt_n =$$

$$= (4 \sin \pi x \sin \pi y)^n \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \frac{\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(n+\nu-2)z)} .$$

(1941)

$$(12)\quad \sum_{h_1, h_2 \,(\mathrm{mod}\, p)} \chi_1(h_1 h_2)\, \chi_2((1-h_1)(1-h_2))\, \chi_3^2(h_1 - h_2) =$$

$$= \frac{\tau_{\chi_3^2}\; \tau_{\chi_1}\; \tau_{\chi_1\chi_3}\; \tau_{\chi_2}\; \tau_{\chi_1\chi_2}}{\tau_{\chi_3}\; \tau_{\chi_1\chi_2\chi_3}\; \tau_{\chi_1\chi_2\chi_3^2}} + (\chi_3 \to \chi_3 \psi)$$

where ψ is a quadratic character, exceptions when any of the τ have principal character χ_0 as subscript (some of these exceptions can be avoided if we interprete τ_{χ_0} as -1).

Proof of (12) indicated that correct analogue of (5) should be:

Conjecture: let P_n run over all polynomials $x^n + a_1 x^{n-1} + \cdots + a_n$ mod p of degree n and write $D(P_n)$ for the discriminant of P_n, then

$$(13) \quad \sum_{P_n} \chi_1((-1)^n P_n(0))\, \chi_2(P_n(1))\, \chi_3\psi(D(P_n))$$

$$= \prod_{\nu=1}^{n} \frac{\tau_{\chi_3^\nu}\ \tau_{\chi_1\chi_3^{\nu-1}}\ \tau_{\chi_2\chi_3^{\nu-1}}}{\tau_{\chi_3}\ \tau_{\chi_1\chi_2\chi_3^{n+\nu-2}}}.$$

Analogues of say, (8), (9), (7) can similarly be written down (the analogue of (6) turns out to be essentially identical with (13)). (13) proved by myself for $n = 2$ in 1941, as well as analogue of (8). Analogue of (9) proved independently by myself and Ronald Evans ca 1980 for $n = 3$, about same time Evans also independently found proof of (13) and analogue of (8) for $n = 2$. Finally general form of (13) proved 1990 by Greg Anderson.

Some other Beta-type integrals:
Let $X_1, \dots, X_n$ be m dimensional vectors and e vector of unit length $|e| = 1$, then:

$$\int \cdots \int \frac{dX_1 \cdots dX_n}{|X_1|^{\alpha_1} \cdots |X_n|^{\alpha_n} \, |e - X_1 - X_2 - \cdots - X_n|^{\alpha_{n+1}}} =$$

$$= \pi^{\frac{mn}{2}} \frac{\Gamma\left(\frac{\sum \alpha_i - mn}{2}\right)}{\Gamma\left(\frac{(n+1)m - \sum \alpha_i}{2}\right)} \prod_{i=1}^{n+1} \frac{\Gamma\left(\frac{m - \alpha_i}{2}\right)}{\Gamma\left(\frac{\alpha_i}{2}\right)},$$

valid when all arguments of Γ's in numerator have positive real part.
- A character analogue is fairly easy to state and prove.

Let the Y denote symmetric n by n real matrices and write

$$dY = |Y|^{-\frac{n+1}{2}} \prod dy_{ij}.$$

Let E denote the unit matrix, then:

$$\int \cdots \int_{\substack{Y_i > 0 \\ Y_1 + \cdots + Y_m < E}} |Y_1|^{x_1 - 1} \cdots |Y_m|^{x_m - 1} |E - Y_1|^{z_1 - 1} \cdots |E - Y_1 - \cdots - Y_m|^{z_m - 1} \prod dY_i$$

is again expressible as a quotient of products of Γ-functions whose arguments are linear expressions in the x_i and z_i.

If we write

$$Y(s_1, \dots, s_n) = \prod_{\nu=1}^{n} |Y^{(\nu)}|^{s_\nu},$$

where

$$Y^{(\nu)} = (y_{i,j})_{i,j \le \nu},$$

then

$$\int_{Y_i > 0} \cdots \int Y_1(s_{1,1}, \dots, s_{1,n}) \cdots Y_m(s_{m,1}, \dots, s_{m,n}) \cdot Y^*(s_{m+1,1}, \dots, s_{m+1,n})\, dY_1 \cdots dY_m$$

where $Y^* = (E + Y_1 + \cdots + Y_m)^{-1}$,

is again expressible as a quotient of products of Γ-functions whose arguments are linear expressions in the $s_{i,j}$. Integral exists if the $s_{i,j}$ with $1 \le i \le m$, have non-negative real parts and the $s_{m+1,j}$ have sufficiently large real parts.

Character analogues undoubtedly exist of these last two Beta-type integrals, but I have not tried to establish them.

q-analogue of (5) conjectured by Richard Askey for general n, proved for $n=2$.

Write
$$(x;q)_m = (1-x)(1-xq)\cdots(1-xq^{m-1}),$$
and define for $|q|<1$
$$\int_0^1 f(t)\,d_qt = (1-q)\sum_{m=0}^{\infty} f(q^m)\,q^m,$$
and
$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}.$$

Then:
$$\int_0^1\cdots\int_0^1 \prod_{i=1}^{n} t_i^{x-1}\frac{(t_iq;q)_\infty}{(t_iq^y;q)_\infty}\cdot\prod_{1\le i<j\le n} t_i^{2k}\left(\frac{t_j q^{1-k}}{t_i};q\right)_{2k}\cdot\prod_{i=1}^{n} d_qt_i =$$
$$= q^{xk\binom{n}{2}+k^2\binom{n}{3}}\prod_{\nu=1}^{n}\frac{\Gamma_q(1+\nu k)\,\Gamma_q(x+(\nu-1)k)\,\Gamma_q(y+(\nu-1)k)}{\Gamma_q(1+k)\,\Gamma_q(x+y+(n+\nu-2)k)}.$$

If we let $q\to1$ we recover formula (5).

The formula was proved for general n 1988 published independently by K.W.J. Kadell and Laurent Habsieger. It is not clear whether Greg Andersons ideas can be modified so as to work for this case also.

Own Proof: 1 1940-41

Consider first case that z is a positive integer, then

$$|\Delta(t)|^{2z} = \sum C_{\alpha_1,\dots,\alpha_n} t_1^{\alpha_1}\cdots t_n^{\alpha_n}$$

with integer coefficients C. Thus the integral in (5) is a linear combination of terms

$$\prod_{\nu=1}^{n} \frac{\Gamma(x+\alpha_\nu)\Gamma(y)}{\Gamma(x+y+\alpha_\nu)},$$

where without loss of generality we may assume $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$.

Since obviously

$$\sum_1^n \alpha_\nu = n(n-1)z,$$

we have

$$\alpha_n \geq (n-1)z.$$

In the same way, since $\Delta(t_1,\dots,t_n)$ is divisible by $\Delta(t_1,\dots,t_\nu)$ for each $1 \leq \nu \leq n$, we have generally

$$\alpha_\nu \geq (\nu-1)z.$$

Also, since

$$|\Delta(t)|^{2z} = (t_1\cdots t_n)^{2(n-1)z}\left|\Delta\left(\tfrac{1}{t}\right)\right|^{2z},$$

we find also

$$\alpha_\nu \leq 2(n-1)z-(n-\nu)z = (n+\nu-2)z.$$

This means that

$$\frac{\Gamma(x+\alpha_\nu)}{\Gamma(x+y+\alpha_\nu)} = \frac{\Gamma(x+(\nu-1)z)}{\Gamma(x+y+(n+\nu-2)z)}\, q_{\alpha_\nu}(x,y),$$

where $q_{\alpha_\nu}(x,y)$ is a polynomial in x and y of degree $(n+\nu-2)z-\alpha_\nu$ in y, and

$$\prod_{\nu=1}^{n} \frac{\Gamma(x+\alpha_\nu)\Gamma(y)}{\Gamma(x+y+\alpha_\nu)} = Q_\alpha(x,y)\prod_{\nu=1}^{n}\frac{\Gamma(x+(\nu-1)z)\Gamma(y)}{\Gamma(x+y+(n+\nu-2)z)}$$

where $Q_\alpha(x,y)$ is a polynomial in x and y of degree $\frac{1}{2}n(n-1)z$ in y.

Since I, the integral in (5), is a linear combination of such terms, we have

$$I = Q(x,y)\prod_{\nu=1}^{n}\frac{\Gamma(x+(\nu-1)z)\Gamma(y)}{\Gamma(x+y+(n+\nu-2)z)} =$$

$$= \frac{Q(x,y)}{P(y)}\prod_{\nu=1}^{n}\frac{\Gamma(x+(\nu-1)z)\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(n+\nu-2)z)},$$

where $Q(x,y)$ is a polynomial of degree at most $\frac{1}{2}n(n-1)z$ in y, and

$$P(y) = \prod_{\nu=1}^{n}\frac{\Gamma(y+(\nu-1)z)}{\Gamma(y)},$$

a polynomial of degree $\frac{1}{2}n(n-1)z$ in y. Since I is symmetric in x and y, we must have

$$\frac{Q(x,y)}{P(y)} = \frac{Q(y,x)}{P(x)}.$$

It follows that this quotient is independent of x and y, so that

$$I = c_n(z) \prod_{\nu=1}^{n} \frac{\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(n+\nu-2)z)}.$$

To determine $c_n(z)$, we take $x=y=1$ in $I = I_n(x,y;z)$. A simple transformation of variables gives

$$I_n(1,1;z) = \frac{1}{(n-1)z+1}\, I_{n-1}(1,2z+1;z),$$

which reduces to

$$c_n(z) = \frac{\Gamma(1+nz)}{\Gamma(1+z)}\, c_{n-1}(z),$$

or

$$c_n(z) = \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)},$$

since $c_1(z) = 1$.

This proves (5) for z a positive integer. A standard interpolation argument extends this to all z with positive real part, and then by analytic continuation to all complex x, y, z for which I is well defined.

Greg Andersens 1 1990

Proof: Let

$$S_n(x,y;z) = \frac{1}{n!} I_n(x,y;z) =$$

$$\underset{0<t_1<\cdots<t_n<1}{\int\cdots\int} (t_1\cdots t_n)^{x-1}\left((1-t_1)\cdots(1-t_n)\right)^{y-1} |\Delta(t)|^{2z} dt_1\cdots dt_n,$$

then

$$S_n(x,y;z) = \int_{(F)} |F(0)|^{x-1} |F(1)|^{y-1} |D(F)|^{z-\frac{1}{2}} dF_0\cdots dF_{n-1}$$

where

$$F(t) = (t-\theta_1)\cdots(t-\theta_n) = \sum_{i=0}^{n} F_i t^{i},$$

and $0<\theta_1<\theta_2\cdots<\theta_n<1$, $D(F)$ is the discriminant of F, and (F) denotes the integration domain in $F_0, F_1, \ldots, F_{n-1}$ implied by the conditions on the θ_i.

Lemma. Let $\tau_0<\theta_1<\tau_1<\theta_2<\cdots<\theta_n<\tau_n$, $F(t)=\prod_{i=1}^{n}(t-\theta_i)$, $T(t)=\prod_{i=0}^{n}(t-\tau_i)$; then

$$\int_{(F)} \prod_{i=0}^{n} |F(\tau_i)|^{s_i-1} dF_0\cdots dF_{n-1} =$$

$$= \frac{\Gamma(s_0)\cdots\Gamma(s_n)}{\Gamma(s_0+\cdots+s_n)} \prod_{i=0}^{n} |T'(\tau_i)|^{s_i-\frac{1}{2}}$$

This is proved by writing $\frac{F(t)}{T(t)} = \sum_{i=0}^{n} \frac{\rho_i}{t-\tau_i}$, then $\rho_i = \frac{F(\tau_i)}{T'(\tau_i)}$, and we have $\rho_i>0$, $\sum_{i=0}^{n}\rho_i = 1$. Also, for every set of ρ_i fullfilling the last two conditions, there corresponds a polynomial F satisfying the conditions of the lemma. If we write the integral of the lemma with the new variables $\rho_1, \dots, \rho_n$ it becomes:

$$\prod_{i=0}^{n} |T'(\tau_i)|^{\delta_i - \frac{1}{2}} \int_{\substack{\rho_i > 0 \\ \Sigma \rho_i < 1}} \cdots \int \rho_1^{\delta_1 - 1} \cdots \rho_n^{\delta_n - 1} (1 - \rho_1 - \cdots - \rho_n)^{\delta_0 - 1} d\rho_1 \cdots d\rho_n$$

which proves the lemma.

Now let: $0 < \vartheta_1 < \theta_1 < \vartheta_2 < \dots < \theta_{n-1} < \vartheta_n < 1$, write

$$F(t) = \prod_{i=1}^{n-1} (t - \theta_i) = \sum_{i=0}^{n-1} F_i t^i ; \; G(t) = \prod_{i=1}^{n} (t - \vartheta_i) = \sum_{i=0}^{n} G_i t^i .$$

Conditions define integration domain (F, G) in the integral

$$J = \int_{(F,G)} |G(0)|^{x-1} |G(1)|^{y-1} |R(F,G)|^{z-1} dF_0 \cdots dF_{n-2} dG_0 \cdots dG_{n-1}$$

where $|R(F,G)|$ is the abs. value of the resultant of ~~the~~ F and G.

$$|R| = \prod_{i=1}^{n} |F(\vartheta_i)| = \prod_{i=1}^{n-1} |G(\theta_i)| .$$

If we integrate first over F (and let G play the role of T in the lemma) we get

$$J = S_n(x, y; z) \frac{(\Gamma(z))^n}{\Gamma(nz)} .$$

If we integrate first over G (and let $t(t-1)F$ play the role of T in the lemma), we get

$$J = S_{m-1}(x+z, y+z; z)\,\frac{\Gamma(z)^{m-1}\Gamma(x)\Gamma(y)}{\Gamma(x+y+(m-1)z)}.$$

Thus

$$S_m(x,y;z) = S_{m-1}(x+z, y+z; z)\,\frac{\Gamma(mz)\,\Gamma(x)\Gamma(y)}{\Gamma(z)\,\Gamma(x+y+(m-1)z)},$$

which at once gives the evaluation of $S_m(x,y;z)$ since $S_1(x,y;z) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and so proves (5).

One can modify Greg Andersons lemma to get direct proofs of the two formulas (8) and (9) which were earlier only obtained as limiting cases of (5). This is important since in the character case the limiting process has no analogue.