$\Re x > 0, \Re y > 0 \quad \Re z > -\min\{\frac{1}{n}, \frac{\Re x}{n-1}, \frac{\Re y}{n-1}\}$

(1) $$\int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{x-1} ((1-t_1)\cdots(1-t_n))^{y-1} |\Delta(t)|^{2z} dt_1 \cdots dt_n$$

$$= \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \frac{\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(n+\nu-2)z)}$$

(1') $$\int_{-1}^{1} \cdots \int ((1-t_1)\cdots(1-t_n))^{x-1}((1+t_1)\cdots(1+t_n))^{y-1} |\Delta(t)|^{2z} dt_1 \cdots dt_n$$

$$= 2^{n(x+y+(n-1)z-1)} \prod_{\nu=}^{n} (\cdot \; - \; .$$

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(2) $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{|\Delta(t)|^{2z} \, dt_1 \cdots dt_n}{((\omega+it_1)\cdots(\omega+it_n))^{x}((\omega'-it_1)\cdots(\omega'-it_n))^{y}}$$

$$= \frac{(2\pi)^n}{(\omega+\omega')^{n(x+y-(n-1)z-n)}}$$

$$\prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \frac{\Gamma(x+y-1-(n+\nu-2)z)}{\Gamma(x-(\nu-1)z)\,\Gamma(y-(\nu-1)z)}$$

(3) $\left(2^n \prod_{j=1}^{n} \cos\frac{\theta_j}{2}\right)^{x+y-2(n-1)z-2}$ $\cos\left[\frac{x+y}{2} - 2(n-1)z - 1\right]\sum\theta$

$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(\prod_{j=1}^{n}(1+e^{i\theta_j})\right)^{x+y-2(n-1)z-2} \cos\frac{x-y}{2}\sum\theta_j \cdot |\Delta(e^{i\theta_j})|^{2z} \, d\theta_1\cdots d\theta_n$$

$$= (2\pi)^n \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \frac{\Gamma(x+y-1-(n+\nu-2)z)}{\Gamma(x-(\nu-1)z)\Gamma(y-(\nu-1)z)}$$

(4) $$\int_0^\infty\cdots\int_0^\infty (t_1\cdots t_n)^{x-1} e^{-y(t_1+\cdots t_n)} |\Delta(t)|^{2z} dt_1\cdots dt_n$$

$$= y^{-n(x+(n-1)z)} \prod_{i=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \Gamma(x+(\nu-1)z)$$

(5) $$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\beta(t_1^2+\cdots t_n^2)} |\Delta(t)|^{2z} dt_1\cdots dt_n$$

$$= \frac{\pi^{\frac{n}{2}}}{\beta^{\frac{n(n-1)}{2}z+\frac{n}{2}}} \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)}.$$

$$\sum_{r,s} \chi_1(r)\chi_2(1+s+r)\chi_3(s^2-4r)\psi(s^2-4r) \qquad \psi^2 = \chi_0$$

$$= \frac{\tau_{\chi_3}^2}{\tau_{\chi_3}} \frac{\tau_{\chi_1}\tau_{\chi_1\chi_3}\tau_{\chi_2}\tau_{\chi_2\chi_3}}{\tau_{\chi_1\chi_2\chi_3}\tau_{\chi_1\chi_2\chi_3^2}} \quad \Big|\ \text{exceptions.}$$

1940-41 found in connection with an interpolation problem a multidimensional generalisation of (x·x). which I stated in a paper

give here in notation:

$$\Delta(t) = \prod_{i<j} (t_i - t_j)$$

$$\int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{x-1} \big((1-t_1)\cdots(1-t_n)\big)^{y-1} |\Delta(t)|^{2z} dt_1 \cdots dt_n$$

(1)
$$= \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \frac{\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(n+\nu-2)z)}$$

valid if ~~real part of all arg in~~ numerator ~~are positive~~
$Rx>0, R(y)>0 \quad Rz> -\frac{1}{n}$
$R(x+(n-1)z) > 0$
$R(y+(n-1)z) \geq 0$

dual

(2)
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\Delta(t)|^{2z} \, dt_1 \ldots dt_n}{\big((\omega+it_1)\cdots(\omega+it_n)\big)^{x} \big(\omega'-it_1) \cdots (\omega'-it_n)\big)^{y}}$$

$$= \frac{(2\pi)^n}{(\omega+\omega')^{n(x+y+(n-1)z-1)}} \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \cdot \frac{\Gamma(x+y-1+(n+\nu-2)z)}{\Gamma(x-(\nu-1)z)\,\Gamma(y-(\nu-1)z)}$$

also form relating to circle

(3)
$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \Big(\prod_{\nu=1}^{n} (1+e^{i\theta_\nu})\Big)^{x+y-2(n-1)z-2} e^{i\frac{x-y}{2}\sum\theta_\nu} \, |\Delta(e^{i\theta})|^{2z} d\theta_1 \ldots d\theta_n$$

$$= (2\pi)^n \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \frac{\Gamma(x+y-1-(n+\nu-2)z)}{\Gamma(x-(\nu-1)z)\,\Gamma(y-(\nu-1)z)}$$

also limit cases
(4) & (5)

indicate proof of (1). assume first z is ~~an natural integer~~ integer $m \geq 0$

$$|\Delta(t)|^{2m} = \sum c_{\alpha_1 \cdots \alpha_n} t_1^{\alpha_1} \cdots t_n^{\alpha_n}$$

assume $\alpha_1 \leq \alpha_2 \cdots \leq \alpha_n$; find easily that

$$(\nu-1)m \leq \alpha_\nu \leq (n+\nu-2)m \qquad |\Delta(\tfrac{1}{t})|^{2m} = (t_1 \cdots t_n)^{2(n-1)m} |\Delta(\tfrac{1}{t})|^{2m}$$

term gives
$$\frac{P(x+\alpha_1)\cdots P(x+\alpha_n)P(y)^n}{P(x+y+\alpha_1)\cdots P(x+y+\alpha_n)}$$

can be written:
$$\frac{Q_{\alpha_1 \cdots \alpha_n}(x,y)}{\prod_{\nu=1}^{n} y(y+1)\cdots(y+(\nu-1)m-1)} \prod \frac{P(x+(\nu-1)m)\,P(y+(\nu-1)m)}{P(x+y+(n+\nu-2)m)}$$

where Q is polynomial in x and y of degree $\frac{n(n-1)}{2}m$ in y.

Thus
$$\mathcal{J} = \frac{Q_{n,m}(x,y)}{\prod_{\nu=1}^{n} y(y+1)\cdots(y+(\nu-1)m-1)} \prod \cdots$$

where $Q_{n,m}$ of degree $\frac{n(n-1)}{2}m$ at most in y.

Interchanging x and y $\mathcal{J}$ is unchanged thus.

thus
$$\frac{Q_{n,m}(x,y)}{\prod y(y+1)\cdots(y+(\nu-1)m-1)} = \frac{Q_{n,m}(y,x)}{\prod x(x+1)\cdots(x+(\nu-1)m-1)}$$

since denominators rel. prime they must divide numerators, which ~~of ratio~~ is obviously of degree zero in both x and y thus a constant $c_n(m)$ –

By looking at case $x = y = 1$ and putting $t_1 \cdots t_{n-1} \leq t_n$ and $t_\nu = t_\nu' t_n$ for $\nu < n$

get recursion formula

$$c_n(m) = \frac{\Gamma(1+\nu m)}{\Gamma(1+m)}\, c_{n-1}(m)$$

~~or since $c_1(m) = 1$~~ this proves (1) for z nonneg. integer

Both sides in (1) bounded functions in z in right half plane, difference vanishes for z = integer m, thus easily seen to vanish identically. Also easily seen to be valid up to to the right of nearest singularity of numerators.

limit cases: also complex. if

m = integer.

Discrete sums $n = 2$ kind.

$$\sum_{h_1, h_2} \chi_1(h_1 h_2)\, \chi_2((1-h_1)(1-h_2))\, \chi_3^2(h_1 - h_2)$$

found. ψ quadratic character

$$= \frac{\tau(\chi_3^2)}{\tau(\chi_3)} \frac{\tau(\chi_1)\,\tau(\chi_1\chi_3)\,\tau(\chi_2)\,\tau(\chi_2\chi_3)}{\tau(\chi_1\chi_2\chi_3)\,\tau(\chi_1\chi_2\chi_3^2)}$$

$$+ \frac{\tau(\chi_3^2)}{\tau(\chi_3\psi)} \frac{\tau(\chi_1)\,\tau(\chi_1\chi_3\psi)\,\tau(\chi_2)\,\tau(\chi_2\chi_3\psi)}{\tau(\chi_1\chi_2\chi_3\psi)\,\tau(\chi_1\chi_2\chi_3^2)}$$

$h_1 h_2 = b$, $h_1 + h_2 = a$; $(h_1 - h_2)^2 = a^2 - 4b$

$$\chi_1(b)\, \chi_2(1+a+b)\, \chi_3(a^2-4b)\left(1 + \psi(a^2-4b)\right)$$

$I_n(x,y,z)$

$(\nu-1)z \le \alpha_\nu \le (n+\nu-2)z$

$$\sum c_{\alpha_1\ldots\alpha_n} \prod_{\nu=1}^{n} \frac{\Gamma(x+\alpha_\nu)\Gamma(y)}{\Gamma(x+y+\alpha_\nu)}$$

$$= \sum c_{\alpha_1\ldots\alpha_n} Q_\alpha(x,y) \prod_{\nu=1}^{n} \frac{\Gamma(x+(\nu-1)z)\Gamma(y)}{\Gamma(x+y+(n+\nu-2)z)}.$$

$Q_\alpha(x,y)$ degree in y $\sum_{\nu=1}^{n}((n+\nu-2)z - \alpha_\nu)$

$= \frac{n(n-1)}{2} z$

$$I = \frac{Q(x,y)}{R(y)\prod_{\nu=1}^{n} y(y+1)\cdots(y+(\nu-1)z-1)} \prod_{\nu=1}^{n} \frac{\Gamma(x+(\nu-1)z)\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(n+\nu-2)z)}$$

Q degree in y at most $\frac{n(n-1)}{2} z$. $\Delta(t) = \Delta(1-t)$

so I symmetric in y and x

$$\frac{Q(x,y)}{R(y)} = \frac{Q(y,x)}{R(x)} = C_n(z)$$

$$I_n(1,1,z) = \frac{1}{1+(n-1)z} I_{n-1}(1,2z+1,z)$$

get $C_n(z) = \frac{\Gamma(1+nz)}{\Gamma(1+z)} C_{n-1}(z)$; $C_1(z) = 1$

$$C_n(z) = \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)}{\Gamma(1+z)}$$

$R(\omega)>0\ ;\ R(\omega')>0$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{|\Delta(t)|^{2z}}{\prod(\omega+it_j)^{x+1}(\omega'-it_j)^{y+1}}\,dt_1\ldots dt_k$$

$$= C_k(z)\,\frac{(2\pi)^k}{(\omega+\omega')^{k(1+x+y+(k-1)z)}}\prod_{\nu=1}^{k}\frac{\Gamma(1+x+y+(k+\nu-1)z)}{\Gamma(1+x-(\nu-1)z)\,\Gamma(1+y-(\nu-1)z)}$$

determine $C_k(z)=\prod_{\nu=1}^{k}\frac{\Gamma(1+\nu z)}{\Gamma(1+z)}$ (prove - best from limiting case

start from

$$\int_{-\infty}^{\infty}\frac{dt}{(\omega+it)^{1+x}(\omega'-it)^{1+y}}=2\pi\,\frac{\Gamma(1+x+y)}{\Gamma(1+x)\Gamma(1+y)}\,\frac{1}{(\omega+\omega')^{1+x+y}}$$

Circle ; take $\omega=\omega'=1$, put

$$t_j=\operatorname{tg}\frac{\theta_j}{2}$$

$$1+i\operatorname{tg}\frac{\theta}{2}=\frac{e^{i\frac{\theta}{2}}}{\cos\frac{\theta}{2}}\ ;\quad 1-i\operatorname{tg}\frac{\theta}{2}=\frac{e^{-i\frac{\theta}{2}}}{\cos\frac{\theta}{2}}$$

$$t_i-t_j=\operatorname{tg}\frac{\theta_i}{2}-\operatorname{tg}\frac{\theta_j}{2}=\frac{\sin\frac{\theta_i-\theta_j}{2}}{\cos\frac{\theta_i}{2}\cos\frac{\theta_j}{2}}$$

$$|e^{i\theta_i}-e^{i\theta_j}|=2\left|\sin\frac{\theta_i-\theta_j}{2}\right|$$

$$|t_i-t_j|=\frac{|e^{i\theta_i}-e^{i\theta_j}|}{2\left|\cos\frac{\theta_i}{2}\cos\frac{\theta_j}{2}\right|}$$

$$\sum_{v} \chi_1(v)\, \chi_2(1-v) = \frac{\tau_{\chi_1} \tau_{\chi_2}}{\tau_{\chi_1 \chi_2}} .$$

$$\cancel{\chi_2}(\cancel{a}) = \bar{\chi}(a) \underline{\sum \quad \bar{\chi}(ah)\, \varepsilon^{ah}} =$$

$$\frac{1}{\tau_{\bar{\chi}}} \sum \bar{\chi}(h)\, \varepsilon^{ah}$$

$$\frac{1}{\tau_{\chi_2} \tau_{\chi_3}} \sum \bar{\chi}_1(h)\, \bar{\chi}_2(k)\, \varepsilon^{h(1-s+r) + k(s^2-4r)}$$

$$\sum_{h,k} \bar{\chi}_1(h)\, \chi_2(k)\, \varepsilon^{h} \sum_{\cancel{s},\cancel{r}} \varepsilon^{ks^2 - hs} \cdot \sum_{\chi_1(r)} \varepsilon^{r(h-4k)}$$

$r = 0$

$$\bar{\chi}_1(h-4k)\, \tau_{\chi_1}$$

$$\bar{\chi}_2(\ell)\, \chi_1(1-\ell)\, \varepsilon^{-\ell}$$

$$\varepsilon^{k\left(s - \frac{h}{2k}\right)^2 - \frac{h}{4} \bar{k}}$$

$$\frac{h}{k} = \frac{\cancel{h}}{4}\ell \ldots$$

$$\tau \left(\frac{k}{p}\right) \quad \cancel{\chi}_1(h)\, \chi_2(k)\, \chi_1(h-4k)\, \varepsilon^{h - \frac{h}{4}\bar{k}}$$

$$\chi_1^2 \chi_2(\cancel{4}\ \ell h) \quad \chi_2(\ell)\, \chi_1(1-\cancel{\bar{\ell}})\, \varepsilon^{h - \bar{\ell}}$$

$$\chi_2(\ell)\, \chi_1(1-\bar{\ell})\, \varepsilon^{-\bar{\ell}}$$

$$\sum_{r,s} \chi_1(r)\,\chi_2(1+r+s)\,\chi_3(s^2-4r)\,\psi(s^2-4r)$$

$$= \frac{\tau_{\chi_3^2}}{\tau_{\chi_3}} \; \frac{\tau_{\chi_1}\,\tau_{\chi_2}\,\tau_{\chi_1\chi_3}\,\tau_{\chi_2\chi_3}}{\tau_{\chi_1\chi_2\chi_3}\;\tau_{\chi_1\chi_2\chi_3^2}}$$

$u-v \qquad u+v = s$

$\qquad\quad uv = r \qquad\qquad \chi_3^2(u-v)$

$$\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \end{vmatrix} = |u-v| = \sqrt{s^2-4r}$$

twice.

$$\frac{D(s,r)}{D(u,v)} \quad 2\int \frac{ds\,dr}{\sqrt{s^2-4r}} \qquad \boxed{\rho^3 = 1}$$

invariant.

$$\iint_{\cancel{s>}2\sqrt{r}} r^{x-1}(1+r+s)^{y-1}(s^2-4r)^{z-\frac{1}{2}}\,ds\,dr$$

$\chi_3(\,|u-v|$ ③

$\sum_{\rho} \chi((1-\rho u)(\dots$

$\chi_1(\rho^3)$

$$\sum \chi_1(\alpha\beta)\, \chi_2((1-\alpha)(1-\beta))\, \chi_3\psi((\alpha-\beta)^2)$$

get (*)

May be led to assume that

$$P_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

$$D_n = \prod_{i<j} (\alpha_i - \alpha_j)^2$$

$$\sum \chi_1((-1)^n P_n(0))\, \chi_2(P_n(1))\, \chi_3\psi(D_n))$$

$$= \prod_{\nu=1}^{n} \frac{\tau_{\chi_3^\nu}}{\tau_{\chi_3}} \frac{\tau_{\chi_1\chi_3^{\nu-1}}\ \tau_{\chi_2\chi_3^{\nu-1}}}{\tau_{\chi_1\chi_2\chi_3^{n+\nu-2}}}$$

Also for other simpler sums.

Lecture in San Diego last year
Ronald Evans; Independently arrived at gen. conj after my lecture & computed two sides for some smaller primes & $n = 3$ or 4. Agreed

Thus now much more likely
But general proof would require

some quite new ideas analog of ( ), proved for $n=3$. Independently by myself and by Evans.