1. Hejhal: On the distribution of $\log|\xi'(\frac{1}{2}+it)|$. Report of Oslo Symposium 1987 pp 343–370

2. Selberg: Contributions to the theory of the Riemann Zeta function (1946), Collected papers vol 1, pp 214–279

3. Selberg: Old and new conjectures and results about a class of Dirichlet series. Report of Amalfi meeting 1989 pp 367–385. Collected papers vol 2, pp 47–63

In [1] Hejhal considers the expression

$$(1)\quad \varkappa'(t) = \frac{\log \dfrac{|\xi'(\frac{1}{2}+it)|}{\log t}}{\sqrt{\pi \log\log t}}, \quad t \geq 9$$

and proves that if $m'_{a,b}(T)$ denotes the measure of the set for which $a < \varkappa'(t) < b$ in $9 < t < T$, then

$$(2)\quad m'_{a,b}(T) \sim T \int_a^b e^{-\pi u^2}\,du$$

as $T \to \infty$. His proof, which assumes R.H., is based on [2] (where the relevant part dealt with $\arg \xi(\frac{1}{2}+it)$) as well as on old, but at the time unpublished, results concerning the expression

$$(3)\quad \varkappa(t) = \frac{\log|\xi(\frac{1}{2}+it)|}{\sqrt{\pi}\,\log\log t},$$

for which if we define $m_{a,b}(T)$ in a corresponding way, we have

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(4) $m_{a,b}(T) = T \int_a^b e^{-\pi u^2}\,du + O\left(\frac{T(\log\log\log T)^2}{\sqrt{\log\log T}}\right),$

which was proved without any hypothesis. In a form concerned with more general dirichlet series (4) occurs in [3], where it is proved assuming a certain density-hypothesis, which can be proved for some fairly large groups of dirichlet series with Eulerproduct and functional equation, roughly when $\Lambda \le 1$ in the class considered in [3].

I shall sketch a proof of (2) in the stronger form:

(5) $m'_{a,b}(T) = T \int_a^b e^{-\pi u^2}\,du + O\left(\frac{T(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$

The proof is considerably shorter than Hejhals and uses no hypothesis.

We begin by quoting some definitions and formulas from [2].

Let $4 \le x \le t^2$, $t > 0$ and define:

(6) $\sigma_{x,t} = \frac{1}{2} + 2 \max_{\rho}\left(\beta - \frac{1}{2}, \frac{2}{\log x}\right),$

when $\rho = \beta + i\gamma$ runs over the zeros of $\zeta(s)$ for which

$$(7) \quad |t-\gamma| \le \frac{x^{3(\beta-\frac{1}{2})}}{\log x}.$$

Also let

$$(8) \quad \Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1 \le n \le x, \\ \Lambda(n)\, \dfrac{\log^2 \frac{x^3}{n} - 2\log^2 \frac{x^2}{n}}{2\log^2 x} & \text{for } x \le n \le x^2, \\ \Lambda(n)\, \dfrac{\log^2 \frac{x^3}{n}}{2\log^2 x} & \text{for } x^2 \le n \le x^3. \end{cases}$$

Put $s = \frac{1}{2} + it$ and $s_x = \sigma_{x,t} + it$, then

$$(9) \quad \sum_{\rho} \frac{\sigma_{x,t} - \frac{1}{2}}{|s_x - \rho|^2} = O\left(\left| \sum_{n < x^3} \frac{\Lambda_x(n)}{n^{s_x}} \right| + \log t \right),$$

and

$$(10) \quad \frac{\zeta'}{\zeta}(s_x) = O\left(\left| \sum_{n < x^3} \frac{\Lambda_x(n)}{n^{s_x}} \right| + \log t \right),$$

$$\left(\frac{\zeta'}{\zeta}\right)'(s_x) = \sum_{\rho} \frac{1}{(s_x - \rho)^2} + O\left(\frac{1}{t}\right) \qquad (11)$$

$$= O\left(\frac{1}{\sigma_{x,t} - \frac{1}{2}} \left(\left| \sum_{n < x^3} \frac{\Lambda_x(n)}{n^{s_x}} \right| + \log t \right) \right),$$

Finally we have

$$\cancel{\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(s_x) + (s - s_x)\left(\frac{\zeta'}{\zeta}\right)'(s_x)}$$

$$(12)\quad \frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(s_x) + (s-s_x)\left(\frac{\zeta'}{\zeta}\right)'(s_x) + $$

$$+ \sum_\rho \frac{(s-s_x)^2}{(s_x-\rho)^2(s-\rho)} + O\left(\frac{1}{t^2}\right) =$$

$$= O\left(\left|\sum_{n<x^3} \frac{\Lambda_x(n)}{n^{s_x}}\right| + \log t\right)$$

$$+ \sum_\rho \frac{(s-s_x)^2}{(s_x-\rho)^2(s-\rho)}.$$

From the expression $\sum_\rho$ we subtract for $t<T$ the expression

$$\sum_{|t-\gamma|<\frac{1}{\log T}} \frac{1}{s-\rho}$$

and obtain after a little manipulation, using (9), that

$$(13)\quad \sum_\rho \frac{(s-s_x)^2}{(s_x-\rho)^2(s-\rho)} = \sum_{|t-\gamma|<\frac{1}{\log T}} \frac{1}{s-\rho} +$$

$$+ O\left((\sigma_{x,t}-\tfrac{1}{2})\log T\left(\left|\sum_{n<x^3}\frac{\Lambda_x(n)}{n^{s_x}}\right| + \log t\right)\right),$$

or inserting this in (12)

$$(14)\quad \frac{\zeta'}{\zeta}(s) = \Sigma_1(t) + \Sigma_2(t),$$

where

$$(15) \quad \sum\nolimits_1(t) = \sum_{|t-\gamma| < \frac{1}{\log T}} \frac{1}{s-\rho}$$

and

$$(16) \quad \sum\nolimits_2(t) = O\left(\left(\sigma_{x,t}-\tfrac{1}{2}\right)\log T\left(\left|\sum_{n<x^3}\frac{\Lambda_x(n)}{n^{\sigma_x}}\right| + \log t\right)\right).$$

Using Lemma 12 of [2] we can now easily show by choosing x as a sufficiently small power of T (with positive exponent depending on k), that for any integer k

$$(17) \quad \int_9^T \left|\frac{1}{\log t}\sum\nolimits_2(t)\right|^k dt = O_k(T),$$

while for Σ_2 we easily obtain that for $0<\theta<1$, uniformly

$$(18) \quad \int_9^T \left|\frac{1}{\log t}\sum\nolimits_1(t)\right|^\theta dt = O\left(\frac{T}{1-\theta}\right).$$

Thus in particular

$$(19) \quad \int_9^T \left|\frac{1}{\log t}\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+it\right)\right|^\theta dt = O\left(\frac{T}{1-\theta}\right).$$

Since, as we shall shortly see,

$$(20) \quad \frac{1}{\log t}\left|\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+it\right)\right| > c > 0,$$

with some positive constant c, (19) lets us derive (5) from (4) with little effort and without degrading the remainder-term.

If we write

(20) $\xi(\tfrac{1}{2}+it) = e^{-i\vartheta(t)}\, X(t)$

where

(21) $\vartheta(t) = \frac{t}{2}\log\pi - \arg\Gamma(\tfrac{1}{4}+\tfrac{it}{2})$,

so that

(22) $\vartheta'(t) = \tfrac{1}{2}\log\frac{t}{2\pi} + O(\tfrac{1}{t})$,

we see that

(23) $\frac{\xi'}{\xi}(\tfrac{1}{2}+it) = -\vartheta'(t) - i\,\frac{X'}{X}(t)$,

or

$$\operatorname{Re}\frac{1}{\vartheta'(t)}\,\frac{\xi'}{\xi}(\tfrac{1}{2}+it) = -1.$$

From this (20) follows at once.

Hejhal also considers the distribution of

(24) $$\frac{\log\left|\frac{1}{\vartheta'(t)}\,X'(t)\right|}{\sqrt{\pi\log\log t}}$$

and proves a result similar to (2) on R.H.

While we obviously have

$$(25)\quad \int_9^T \left| \frac{1}{\vartheta'(t)} \cdot \frac{X'}{X}(t) \right|^{\theta} = O\left(\frac{T}{1-\theta}\right),$$

from (23) and (19), the analog of (20) does not hold, since $X'(t)$ has many real zeros for $9 < t < T$ (actually $> AT\log T$).

Using a theorem of Littlewood we can prove

$$\int_9^T \log\left| \frac{1}{\vartheta'(t)} X'(t) \right| dt \geq -c\log T,$$

since also

$$\int_9^T \log|X(t)|\,dt = O(T),$$

it follows that

$$(26)\quad \int_9^T \log\left| \frac{1}{\vartheta'(t)} \frac{X'}{X}(t) \right| dt = O(T),$$

and using (25) that

$$(27)\quad \int_9^T \left| \log\left| \frac{1}{\vartheta'(t)} \frac{X'}{X}(t) \right| \right| dt = O(T).$$

From this we can deduse that if

$$(28)\quad x_d^*(t) = \frac{\log\left(\frac{1}{\vartheta'(t)} X'(t)\right)}{\sqrt{\pi \log\log t}}$$

and

$m^*_{a,b}(T)$

denotes the measure of the subset of $(9, T)$ for which

$$a < x_d^*(t) < b,$$

then

$$(29)\quad m^*_{a,b}(T) = T\int_a^b e^{-\pi u^2}\,du + O\left(\frac{T}{(\log\log T)^{\frac{1}{4}}}\right).$$

As we see the remainder term is degraded compared to the earlier cases (on R.H. we can actually do somewhat better).

More generally one can show if

$$(30)\quad f(t) = \sum_{i=0}^{N} c_i(\vartheta'(t))^{-i}\,\zeta^{(i)}(\tfrac{1}{2}+it),$$

$N > 0$, then

$$(31) \quad \int_9^T \left| \frac{f(t)}{\zeta(\frac{1}{2}+it)} \right|^{\frac{\theta}{N}} = O\left(\frac{T}{1-\theta} \right),$$

for $0 < \theta < 1$, and that

$$(32) \quad \int_9^T \left| \log \left| \frac{f(t)}{\zeta(\frac{1}{2}+it)} \right| \right| dt = O(T),$$

so that again if we define

$$(33) \quad x^f(t) = \frac{\log |f(t)|}{\sqrt{\pi \log\log t}}$$

and $m^f_{a,b}(T)$ as before, then

$$(34) \quad m^f_{a,b}(T) = T \int_a^b e^{-\pi u^2} du + O\left(\frac{T}{(\log\log T)^{\frac{1}{4}}} \right).$$

Similar results to those given above hold for the general class defined in [3] if one assumes that for the function in question we have

$$N(\sigma, T) = O\left(T^{1-\alpha(\sigma-\frac{1}{2})} \log T\right),$$

with some positive constant α. Essentially this can be proved to hold in those cases for which (in the notation of that paper) $\Lambda \leq 1$.

In [3] I have indicated how one can determine the distribution also of $\dfrac{\log |F(\frac{1}{2}+it)|}{\sqrt{\log\log t}}$ where

F is a finite linear combination of the type of functions considered there.
$F(s) = \sum_{i=1}^{n} c_i \, F_i(s)$, with distinct $F_i(s)$.

One can show in general that

$$\frac{\log \left| \frac{1}{\log t} F'(\tfrac{1}{2}+it) \right|}{\sqrt{\log\log t}}$$

has the same distribution.
Similar results hold also for expressions involving the higher derivatives in analogy with (30).