1

<u>Heuristic!</u>

Consider a Dirichlet L-function

$$L(s,\chi) = \sum_n \frac{\chi(n)}{n^s} = \prod_p \left(1-\chi(p)p^{-s}\right)^{-1}, \quad s=\sigma+it$$

for $\sigma > \frac{1}{2}$, fixed, and let t vary.

If there exists a distribution function $\psi_\sigma(u)$ such that if $m_{a,b}(\sigma,T)$ denote the measure of the set of t with $0 < t < T$, $a < \log|L(\sigma+it,\chi)| < b$, we have

$$\lim_{T\to\infty} \frac{m_{a,b}(T)}{T} = \int_a^b \psi_\sigma(u)\,du,$$

then the fourier transform $\hat{\psi}_\sigma$ would be given by

$$(1)\quad \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(L(\sigma+it,\chi)\, L(\sigma-it,\bar{\chi})\right)^{\pi i u} dt$$

Here

$$\left(\qquad\qquad\right)^{\pi i u} = \prod_p \left(1-\chi(p)p^{-\sigma-it}\right)^{-\pi i u} \cdot \left(1-\bar{\chi}(p)p^{-\sigma+it}\right)^{-\pi i u}$$

$$= \prod_p \left(\sum_\nu (-1)^\nu \chi^\nu(p) \binom{-\pi i u}{\nu} p^{-\nu\sigma - i\nu t}\right)\left(\sum_\nu (-1)^\nu \bar{\chi}^\nu(p) \binom{-\pi i u}{\nu} p^{-\nu\sigma + i\nu t}\right).$$

Here the $t \log p$ act much like independent variables mod 2π, so we can expect that the expression (1) may be equal to

$$\prod_p \int_0^1 \left(\sum_{\nu=1}^{\infty} (-1)^{\nu} \binom{-\pi i u}{\nu} \chi^{\nu}(p) p^{-\nu\sigma} e^{2\pi i \nu \theta_p} \right) \cdot \left(\sum_{\nu=1}^{\infty} (-1)^{\nu} \binom{-\pi i u}{\nu} \bar{\chi}^{\nu}(p) p^{-\nu\sigma} e^{-2\pi i \nu \theta_p} \right) d\theta_p$$

$$= \prod_p \left(\sum_{\nu=1}^{\infty} \binom{-\pi i u}{\nu}^2 p^{-2\nu\sigma} \right),$$ a product

which is convergent for $\sigma > \frac{1}{2}$:

$$\left(1 - \frac{\pi^2 u^2}{p^{2\sigma}} + \cdots \right)$$

For small $\frac{u}{p^{\sigma}}$ this is close to

$$e^{-\frac{\pi^2 u^2}{p^{2\sigma}}}$$, so the product

should be approximated by

$$e^{-\pi^2 u^2 \sum_p p^{-2\sigma}}.$$

As $\sigma \to \frac{1}{2}$ the expression $\sum_p p^{-2\sigma}$ grows large, this suggests using

$$\frac{1}{\sqrt{\pi \sum_p p^{-2\sigma}}}$$ as a scaling factor, and

looking at the distribution of

$$\frac{\log |L(\sigma+it, \chi)|}{\sqrt{\pi \sum_p p^{-2\sigma}}} \tag{2}$$

instead, we see that the fourier-transform of the distribution function of (2) approaches $e^{-\pi u^2}$ as $\sigma \to \frac{1}{2}$, and so the distribution tends to a normal Gaussian distribution on the line.

Not difficult to make this rigorous. It does suggest looking at the behaviour and distribution of similar functions in a closer neighbourhood of $\sigma = \frac{1}{2}$ say when σ is of the form $\frac{1}{2} + \eta(T)$. where $\eta(T) \to 0$ as $T \to \infty$, and even on the line $\sigma = \frac{1}{2}$ itself.

General class of Dirichlet series

$$F(s) = \sum_n \frac{a_n}{n^s} \;;\; a_1 = 1 \;;\; a_n = O(n^\delta)$$

for all $\delta > 0$. $F(s)$ analytic, except possibly for a pole at $s = 1$. $(s-1)^m F(s)$ integral of finite order for m large enough.

There should exist constants

$|\varepsilon| = 1$; $Q > 0$; $\lambda_i > 0$, $R(\mu_i) \geq 0$; $i = 1, \ldots, r$

such that if we put

$$\phi(s) = \varepsilon Q^s \prod_{i=1}^{r} \Gamma(\lambda_i s + \mu_i) \; F(s),$$

then

$$\phi(s) = \overline{\phi(1 - \bar{s})},$$

(so $\phi(\frac{1}{2} + it)$ is real).

We also assume that

$$\log F(s) = \sum_n \frac{b_n}{n^s},$$

where $b_n = 0$ ~~except when~~ if n is not of the form $n = p^r$, where p is a prime and r a positive integer. We have clearly $b_p = a_p$ and assume that in general $b_n = O(n^\theta)$ with some fixed $\theta < \frac{1}{2}$.

These last conditions imply that

$$\sum_{n<x} \frac{|b_n|^2}{n} = \sum_{p<x} \frac{|a_p|^2}{p} + O(1).$$

In order to investigate the value distribution of, say, $\log F(\frac{1}{2} + it)$ (with a suitable scaling factor), we shall need some assumptions:

$$\sum_{p<x} \frac{|a_p|^2}{p} = n_F \log\log x + O(1),$$

where n_F is a constant dependent on F (actually in all cases where it can be determined it is an integer, and if F can not be factorized into other functions of the same general class, n_F equals 1 in all such cases.

If $N(T)$ denotes the zeros $\rho = \beta + i\gamma$ of $F(s)$ which lie in the region $0 < \beta <$ $0 < \gamma < T$, then

$$N(T) = \frac{\Lambda}{\pi} T(\log T + C) + O(\log T)$$

where $\Lambda = \sum_{i=1}^{n} \lambda_i$; C a constant.

We shall need to assume either:

<u>R.H</u> that is, all zeros in the strip $0 \leq \sigma \leq 1$ lie on the line $\sigma = \frac{1}{2}$ $(\beta = \frac{1}{2})$

(This has not been proved for any function $F(s)$ in the class so far), or:

<u>D.H.</u> $N(\sigma, T) = O(T^{1-\alpha(\sigma-\frac{1}{2})} \log T)$,

Here α is a positive constant, and $N(\sigma, T)$ denotes the number of zeros ~~in the region~~ with $\beta > \sigma$, $|\gamma| < T$. D.H. has been proved for some extensive classes of $F(s)$, for instance $\zeta(s)$, all dirichlet L-functions, and the dirichlet series with Euler-product that are associated with analytic cusp forms for congruence subgroups of the modular group.

We shall need some approximation formulas for $\log F(\sigma+it)$ for $\sigma \geq \frac{1}{2}$

I. Assuming R.H.

For $x \geq 2$ define

$$\Theta_x(n) = \begin{cases} 1 & \text{for } 1 \leq n \leq x \\ 2 - \frac{\log n}{\log x} & \text{for } x \leq n \leq x^2 \\ 0 & \text{for } n \geq x^2. \end{cases}$$

We write $b_x(n) = b_n \Theta_x(n)$, let $2 \leq x \leq t^2$ and $\sigma_x = \frac{1}{2} + \frac{1}{\log x}$. Then, for $\sigma_x \leq \sigma^* \leq \sigma$

$$\log F(\sigma+it) = \sum_{n<x^2} \frac{b_x(n)}{n^{\sigma+it}} +$$

$$+ O\left(\frac{x^{\frac{1}{2}-\sigma}}{(\sigma^* - \frac{1}{2})\log^2 x} \left(\left| \sum_{n<x^2} \frac{b_x(n)\log n}{n^{\sigma^*+it}} \right| + \log|t| \right)\right).$$

Also, for $\frac{1}{2} \leq \sigma \leq \sigma_x$, if we write

$$\eta_t = \min_{\rho} |t-\gamma|,$$

$$\log F(\sigma+it) = \sum_{n<x^2} \frac{b_x(n)}{n^{\sigma_x+it}} +$$

$$+ O\left(\frac{1+\log^+ \frac{1}{\eta_t \log x}}{\log x} \left(\left| \sum_{n<x^2} \frac{b_x(n)\log n}{n^{\sigma_x+it}} \right| + \log|t| \right)\right).$$

The constants in O symbols depend on F only.

Also if we take the imaginary part of the last formula, we can drop the term $\log^+ \frac{1}{\eta_t \log x}$.

II Assuming D.H.

We now define a new $\theta_x(n)$ as follows

$$\theta_x(n) = \begin{cases} 1 \text{ for } 1 \le n \le x \\ 1 - 2\left(\frac{\log \frac{n}{x}}{\log x}\right)^2 \text{ for } x \le n \le x^{\frac{3}{2}} \\ 2\left(\frac{\log \frac{x^2}{n}}{\log x}\right)^2 \text{ for } x^{\frac{3}{2}} \le n \le x^2 \\ 0 \quad \text{for } n \ge x^2. \end{cases}$$

Also if ρ runs over the zeros $\rho = \beta + i\gamma$ with $|t - \gamma| \le \frac{x^{2\beta - 1}}{\log x}$,

we define

$$\sigma_{x,t} = \frac{1}{2} + 2 \max_\rho \left(\beta - \frac{1}{2}, \frac{2}{\log x}\right).$$

We then have for $\sigma_{x,t} \le \sigma^* \le \sigma$, if we again write $b_x(n) = b_n \theta_x(n)$

$$\log F(\sigma + it) = \sum_{n < x^2} \frac{b_x(n)}{n^{\sigma + it}} +$$

$$+ O\left(\frac{x^{\frac{1}{2} - \sigma}}{(\sigma^* - \frac{1}{2}) \log^2 x}\left(\left|\sum_{n < x^2} \frac{b_x(n) \log n}{n^{\sigma^* + it}}\right| + \log|t|\right)\right),$$

and for $\frac{1}{2} \le \sigma \le \sigma_{x,t}$

$$\log F(\sigma+it) = \sum_{n \le x^2} \frac{\Lambda_x(n)}{n^{\sigma_{x,t}+it}} +$$

$$+O\left(\left(\sigma_{x,t}-\tfrac{1}{2}\right)\left(1+\log^{+}\frac{1}{\eta_t \log x}\right)\left(\left|\sum_{n<x^2}\frac{\Lambda_x(n)\log n}{n^{\sigma_{x,t}+it}}\right|+\log|t|\right)\right).$$

If $(\xi x)^2 < T^{\alpha'}$ and $x > T^{\frac{\alpha''}{k}}$, where $0<\alpha''<\frac{1}{2}$; $0<\alpha'<\alpha$ are positive constants (the α is the one occurring in the D.H.), then

$$\int_0^T \left(\sigma_{x,t}-\tfrac{1}{2}\right)^k \xi^{2\sigma_{x,t}-\frac{1}{2}}\, dt = O\left(\frac{k^k e^{Ak}\, T}{(\log T)^k}\right),$$

where A depends on α, α' and α''. This shows that on the average $\sigma_{x,t}-\frac{1}{2}$ is quite small.

We can now prove assuming either R.H. or D.H.:

Th. 1. Let k be a positive integer $0<a<1$ and $T^{\frac{a}{k}} \le x \le T^{\frac{1}{k}}$, then for $\sigma \ge \frac{1}{2}$ we have

$$\frac{1}{T}\int_0^T \left|\log F(\sigma+it) - \sum_{p<x}\frac{a_p}{p^{\sigma+it}}\right|^{2k} dt = O\left(k^{4k} e^{Ak}\right),$$

here the constant A depends on a and F. If we replace the expression in the $|\quad|$ symbol by its imaginary part, we can replace the k^{4k} in the O symbol by k^{2k}.

From this one can deduce

Th 2. For $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log T)^{-\delta}$ where $\delta > 0$ is fixed, the function

$$x(\sigma,t) = x_F(\sigma,t) = \frac{\log F(\sigma+it)}{\sqrt{\pi \sum_{p<t} \frac{|a_p|^2}{p^{2\sigma}}}},$$

has a normal Gaussian distribution in the complex plane. Also the real and imaginary part of $x(\sigma,t)$ have a normal Gaussian distribution on the real line.

More precisely: Let $\chi_{a,b}(u)$ denote the characteristic function of the interval (a,b), then

$$\int_0^T \chi_{a,b}(\operatorname{Re} x(\sigma,t))\,dt = T\int_a^b e^{-\pi u^2}\,du + O\left(T\,\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right),$$

and

$$\int_0^T \chi_{a,b}(\mathrm{Im}\, x(\sigma,t)) = T\int_a^b e^{-\pi u^2}\,du + O\left(T\,\frac{\log\log\log T}{\sqrt{\log\log T}}\right).$$

~~Since~~ In the region for σ considered in Th 2, we have

$$\sum_{p<T}\frac{|a_p|^2}{p^{2\sigma}} = n_F \log\left(\min\left(\frac{1}{\sigma-\frac{1}{2}}, \log T\right)\right) + O(1).$$

The proofs of the more precise results depend on approximating functions, in this case the $\chi_{a,b}(x)$, by functions whose fourier transform has compact support. We can show ~~that~~ there is a bounded function g ~~such~~ that

$$\left|\chi_{a,b}(x) - \int_{-M}^{M}\frac{\sin\pi(b-a)u}{u}\, e^{2\pi i\left(x-\frac{a+b}{2}\right)u}\, g_1\!\left(\frac{u}{M}\right)du\right|$$

$$\leq \frac{1}{2}\left(\frac{\sin^2\pi M(x-a)}{(M\pi(x-a))^2} + \frac{\sin^2\pi M(b-x)}{(M\pi(b-x))^2}\right),$$

for all x, and $M>0$. This can be used to find polynomial approximations

10'

$\exists$ $g(u)$ bounded on $(-1, 1)$ such that for $M > 0$

$$|x| \geq \int_{-M}^{M} \frac{1-\cos 2\pi x u}{u^2}\, g\left(\frac{u}{M}\right) du,$$

$$|x| \leq \int_{-M}^{M} \frac{1-\cos 2\pi x u}{u^2}\, g\left(\frac{u}{M}\right) du + \frac{1}{2}\, \frac{\sin^2 \pi M x}{M^3 (\pi x)^2}.$$

$$\int_0^T \sum_m a_m m^{it}\, \overline{\sum_n b_n n^{it}}\, dt = T \sum_m a_m \overline{b_m} + O\left(\left(\sum_n n|a_n|^2 \cdot \sum_m m|b_m|^2\right)^{\frac{1}{2}}\right).$$

$$\left(\sum_p a_p p^{it}\right)^h = \sum_m A_m^{(h)} m^{it},$$

$$\int_0^T \left(\sum_p a_p p^{it}\right)^h \left(\overline{\sum_p b_p p^{it}}\right)^k dt$$

$$= \delta_{h,k}\, T \sum_m A_m^{(h)} B_m^{(k)} +$$

$$+ O\left((h!\,k!)^{\frac{1}{2}} \left(\sum_p p|a_p|^2\right)^{\frac{h}{2}} \left(\sum_p p|b_p|^2\right)^{\frac{k}{2}}\right).$$

to $\chi_{a,b}(x)$ whose error can be easily estimated by observing that

$$\left| e^{2\pi i x u} - \sum_{n<N} \frac{(2\pi i x u)^n}{n!} \right| \le \frac{(2\pi |x u|)^N}{N!},$$

We also appeal to the results

$$\int_0^T \left| \log|F(\sigma+it)| - \mathrm{Re}\sum_{p<x} \frac{a_p}{p^{\sigma+it}} \right|^{2k} dt = O(T k^{4k} e^{Ak})$$

for $T^{\frac{a}{k}} \le x \le T^{\frac{1}{k}}$, and

$$\int_0^T \left| \mathrm{Im} \log F(\sigma+it) - \mathrm{Im} \sum_{p<x} \frac{a_p}{p^{\sigma+it}} \right|^{2k} dt = O(T k^{2k} e^{Ak})$$

Choosing M, N and x judiciously, we obtain our result.

We can improve the remainder terms in Th 2. somewhat if σ is of the form $\sigma = \frac{1}{2} + \frac{\varphi(T)}{\log T}$, where $\varphi(T)$ tends to infinity with T (but slower than $(\log T)^{1-\delta}$ for some $\delta > 0$).

Slides wrapped in this

2

Om man ser på fordelingen av $\overbrace{F(s)}^{\text{nullp.}} - a$ ser man at om $a \neq a_0$ i det vesentlige samme fordeling for a-punkter i strip $-A < \sigma < A$ hvor A tilstr. stor. (en annen fam ligger nær trivielle nullp.).

Bevis ved argument rundt rekt.