HODGE THEORY AND REPRESENTATION THEORY

(APPLICATION OF SOME WORK OF CARAYOL)

PHILLIP GRIFFITHS

Talk given at Duke University on October 21, 2010. Based on joint work and correspondence with Mark Green and Matt Kerr.
The speaker and his collaborators would like to thank Wushi Goldring for bringing the papers of Carayol to our attention.

Outline

I. Summary

It is classical that an arithmetic automorphic form takes arithmetic values on moduli points of complex-multiplication (CM) elliptic curves. Here, arithmetic means in $\bar{\mathbb{Q}}$. This is a wonderful result that says that a highly transcendental function takes arithmetic values at arithmetically defined points. Through deep work of Shimura and others it has been extended to arithmetic automorphic forms on Shimura varieties. These results may formulated purely Hodge-theoretically using Mumford-Tate domains parameterizing polarized Hodge structures of weight one whose generic point has a given Mumford-Tate group. The purpose of this talk is to discuss how these results might be extended to a non-classical, higher weight case.

It is now well known that automorphic forms should be interpreted in terms of representation theory. Very roughly speaking they correspond to irreducible summands V_π of a representation of $M(\mathbb{A})$ in $L^2(M(\mathbb{Q})\backslash M(\mathbb{A}))$ whose infinite component is a discrete series representation of the real Lie group $M(\mathbb{R})$. These V_π are the "cuspidal" automorphic representations. Here, M is a reductive algebraic group defined over $\mathbb{Q}$ and $\mathbb{A} = \mathbb{R} \oplus \prod_p{}' \mathbb{Q}_p$ are the adeles. Those M that give rise to cuspidal automorphic representations constitute a special class of reductive $\mathbb{Q}$-algebraic groups such that $M(\mathbb{R})$ has a discrete series representation in $L^2(M(\mathbb{R}))$.

On the Hodge-theoretic side, the $\mathbb{Q}$-algebraic groups that are Mumford-Tate groups G of polarized Hodge structures, and the different ways in which a given G may appear as such, have recently been classified (cf. the reference below to the joint work with Mark Green and Matt Kerr). A consequence of that work is

(∗) **The Mumford-Tate groups coincide exactly with the reductive $\mathbb{Q}$-algebraic groups G such that $L^2(G(\mathbb{R}))$ contains discrete series representations.**

These G then have cuspidal automorphic representations and a natural question is

What is the Hodge-theoretic meaning of the representation V_π?

It has been known for some 40 years that the infinite component of V_π "should" correspond to automorphic cohomology $H^d(X, \mathcal{L}_\lambda)$ where $X = \Gamma\backslash D$ with Γ an arithmetic subgroup of G and where

- $D = G(\mathbb{R})/H$ is a Mumford-Tate domain;
- $\mathcal{L}_\lambda \to D$ is a homogenous vector bundle corresponding to the weight associated to the Harish-Chandra parameter associated to V_π;[1]
- $d = \dim_{\mathbb{C}} K/H$ where $K \subset G(\mathbb{R})$ is a maximal compact subgroup, and where $d = 0$ (automorphic forms) occur only in the very special case when D fibres holomorphically over an Hermitian symmetric domain.

[1]In this talk we shall be mainly concerned with the case when $H = T$ is a compact maximal torus and $\mathcal{L}_\lambda \to X$ is a line bundle.

However, the geometric and arithmetic meaning of cuspidal automorphic representations, realized as automorphic cohomology when $d > 0$, has remained mysterious, to say the least. Recently, motivated by the result (*) and work of H. Carayol, it has been possible to

- define some conditions under which an automorphic cohomology class α in $H^d(X_U, \widetilde{\mathcal{L}}_\lambda)$ may be said to be "arithmetic". Here, $U \subset G(\mathbb{A}_f)$ is a compact open subgroup of $G(\mathbb{A}_f)$ where $\mathbb{A}_f$ are the finite adeles, $X_U = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/U$ is the "adelification" of X and $\widetilde{\mathcal{L}}_\lambda \to X_U$ is the natural extension of $\mathcal{L}_\lambda \to \Gamma\backslash D$;
- in the case of $D = \mathrm{SU}(2,1)/T$, realized as a Mumford-Tate domain for polarized Hodge structures of weight three and Hodge numbers $1, 2, 2, 1,^*$ define an arithmetic cohomology class $\alpha \in H^1(X_U, \widetilde{\mathcal{L}}_\lambda)$ where λ corresponds to a degenerate limit of discrete series;
- define a Stein manifold $W \subset D \times D$ and a line bundle $\mathcal{E}_\lambda \to W$ such that for each point $p \in W$ there is an "evaluation"

$$\alpha(p) \in \mathcal{E}_{\lambda,p} \otimes T_p^* W;$$

- if $p = (p', p'') \in W \subset D \times D$ corresponds to a a pair of CM polarized Hodge structures, then

$$\begin{cases} \mathcal{E}_{\lambda,p} \cong \bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \\ T_p^* W \cong \bar{\mathbb{Q}}^m \otimes_{\mathbb{Q}} \mathbb{C} \end{cases}$$

  have arithmetic structures and main result is the

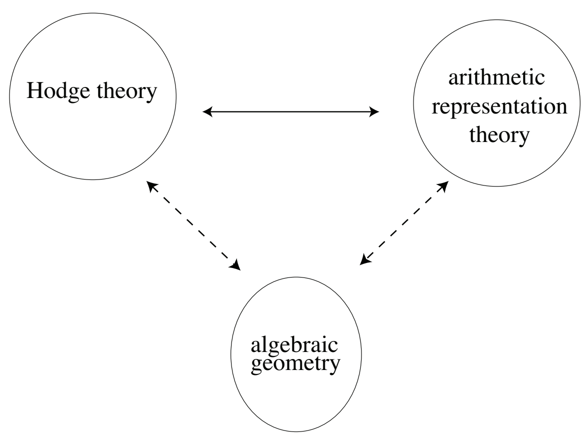
---

*Here $\mathrm{SU}(2,1)$ is the simple real Lie group associated to the $\mathbb{Q}$-algebraic group associated to a central simple algebra of dimension 9 over an imaginary quadratic field in which Γ is a co-compact subgroup.

Theorem. *$\alpha(p) \epsilon \bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}}^m$ is arithmetic.*

The proof of this result is work in progress, completed except for checking some details. It consists of interpreting some quite deep calculations of Carayol (Compositio Math, 121: 305-335, 2000) in the Hodge-theoretic context of Mumford-Tate domains as discussed in a joint work [GGK]. (Available at http://www.math.wustl.edu/~matkerr/MTgroups.pdf)

Conclusion: In this special case, the solid arrow gives a connection between



In the classical case the dotted arrows may be filled in leading to an extensive, deep and rich story. In the non-classical case the direct extension of this story seems not to be possible. But, at least in some cases, it seems feasible that the connection given by the top arrow may be done.

II. Hodge theoretic background

- $V_{\mathbb{Z}} \cong \mathbb{Z}^b$ is a lattice, $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, $V_{\mathbb{C}} = \mathbb{V} \otimes_{\mathbb{Q}} \mathbb{C}$
- $Q : V \otimes V \to \mathbb{Q}$, $Q(u, v) = (-1)^n Q(v, u)$
- $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C}^*$; a real Lie group isomorphic to $\mathbb{R}^{>0} \times S^1$, $z = re^{i\theta} \in \mathbb{S}$.

Definition. A *polarized Hodge structure* of weight n is given by

$$\varphi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

such that $\varphi(r) = r^n \mathrm{Id}_{V_{\mathbb{R}}}$, φ preserves Q up to scaling, and such that Hodge-Riemann bilinear relations are satisfied.

Over $\mathbb{C}$ we have the eigenspace decomposition of $\varphi(\mathbb{S})$

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \qquad V^{q,p} = \bar{V}^{p,q}$$

where

$$V^{p,q} = \{v \in V_{\mathbb{C}} : \varphi(z)v = z^p \bar{z}^q \cdot v\}.$$

Then the Hodge-Riemann bilinear relations are

$$\begin{cases} \text{(HRI)} & Q(V^{p,q}, V^{p',q'}) = 0 \qquad p' \neq n - p \\ \text{(HRII)} & i^{p-q} Q(V^{p,q}, \bar{V}^{p,q}) > 0. \end{cases}$$

The *Hodge filtration* is $F^p = \bigoplus_{p' \geq p} V^{p',q'}$. Then (HRI) is $Q(F^p, F^{n-p+1}) = 0$. We shall sometimes write V_φ for the pair (V, φ).

Definition. The *Mumford-Tate group* $\widetilde{M}_\varphi \subset \mathrm{GL}(V)$ is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ such that $\varphi(\mathbb{S}) \subset \widetilde{M}_\varphi(\mathbb{R})$.

Mumford-Tate groups are the basic symmetry groups of Hodge theory, encoding the $\mathbb{Q}$-structure and the possible Hodge structures. We will work primarily with $M_\varphi := \widetilde{M}_\varphi \cap \operatorname{Aut}(V,Q)$, given up to a finite group by modding out the scalings $r^n \operatorname{Id}_V$.

Definition. (i) The *period domain* D is the set of all polarized Hodge structures with given Hodge numbers $h^{p,q} = \dim V^{p,q}$.

(ii) The *compact dual* $\check{D}$ is the set of Hodge filtrations $\{F^p\}$ satisfying (HRI) and where $\dim F^p = \sum_{p' \geq p} h^{p',q'}$.

(iii) The *Mumford-Tate domain* $D_M \subset D$ is the $M_\varphi(\mathbb{R})$ orbit of $\varphi \in D$. We also have $D_M \subset \check{D}_M$.

Remark. If φ is generic, then $M_\varphi = \operatorname{Aut}(V,Q)$. The other extreme is when φ is a *complex-multiplication* (CM) Hodge structure. When V_φ is simple this means that $\operatorname{End}(V_\varphi, Q) := L$ is a purely imaginary quadratic extension of a totally real field with $\dim_{\mathbb{Q}} L = \dim V$. Then $\widetilde{M}_\varphi \cong L^*$ and M_φ are the elements of norm one in the multiplicative group L^*.

Running example. $V_{\mathbb{Z}} = \mathbb{Z}^2 =$ column vectors and $Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then $V_{\mathbb{C}} = \mathbb{C}^2$ and

$$\begin{array}{ccccl} \check{D} & = & \mathbb{P}^1 & & \\ \cup & & \cup & & \\ D & = & \mathcal{H} & = & \left\{ \left[\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right] : \operatorname{Im} \tau > 0 \right\}. \end{array}$$

We note that

$$\mathcal{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2)$$

where for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$

$$g \begin{bmatrix} \tau \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{a\tau+b}{c\tau+d} \\ 1 \end{bmatrix}.$$

Moreover,

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\} = \text{isotropy group of } \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right].$$

Then

$$\widetilde{M_\tau} = \begin{cases} \mathrm{GL}_2 & \tau \neq \text{ quadratic imaginary} \\ \mathbb{Q}(\tau)^* & \text{if } \tau \text{ is quadratic imaginary} \end{cases}$$

A Mumford-Tate domain

$$D_M = M(\mathbb{R})/H$$

is a homogeneous complex manifold. We have

$$\begin{array}{ccc} D_M & = & M(\mathbb{R})/H \\ \cap & & \\ \check{D}_M & = & M(\mathbb{C})/P \end{array}$$

where P is a parabolic group. In the standard example

$$P = \left\{ g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : b + c + i(a-d) = 0 \right\}$$

is conjugate to $\left\{ \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \right\}$. Given a character

$$\lambda : P \to \mathbb{C}^*$$

there is a holomorphic, homogeneous line bundle

$$\check{\mathcal{L}}_\lambda = M(\mathbb{C}) \times_P \mathbb{C}$$

over $\check{D}$, and we denote by $\mathcal{L}_\lambda \to D$ its restriction to D. In the standard example a character is given by

$$\lambda_k(g) = \alpha^k$$

and we denote by

$$\mathcal{L}_k \to \mathcal{H}$$

the corresponding homogeneous line bundle. From

$$d\left(\frac{a\tau + d}{c\tau + d}\right) = \frac{d\tau}{(c\tau + d)^2}$$

we infer that the canonical bundle

$$\omega_{\mathcal{H}} = \mathcal{L}_2.$$

Over a Mumford-Tate domain D_M one has the Hodge bundles

$$\begin{cases} \mathcal{F}^P \to D_M \\ \mathcal{V}^{p,q} = \mathcal{F}^p/\mathcal{F}^{p+1}. \end{cases}$$

These are Hermitian vector bundles, using the Hodge-Riemann bilinear relations to define metrics in the fibres. They are also homogeneous vector bundles arising from representations of P. In the standard example

$$\mathcal{V}^{1,0} = \mathcal{L}_1,$$

so that

$$\omega_{\mathcal{H}} = (\mathcal{V}^{1,0})^{\otimes 2}.$$

The reductive $\mathbb{Q}$-algebraic groups M that can be realized as Mumford-Tate groups, and the different ways in which a given M can be realized, have recently been classified (cf. the reference in the summary). As noted in the summary, a consequence is, assuming M is semi-simple:

($*$) *M can be realized as a Mumford-Tate group if, and only if, $L^2(M(\mathbb{R}))$ contains discrete series representations.*

We will discuss this below. For the moment we wish to note a major difference between the classical case of weight one and the general higher weight case. Namely, in the latter case a Mumford-Tate domain

$$D = M(\mathbb{R})/H$$

will contain positive dimensional maximal compact subvarieties. One of these is given by

$$Y_0 = K/H$$

where $K \subset M(\mathbb{R})$ is a maximal compact subgroup. One set of such is given by

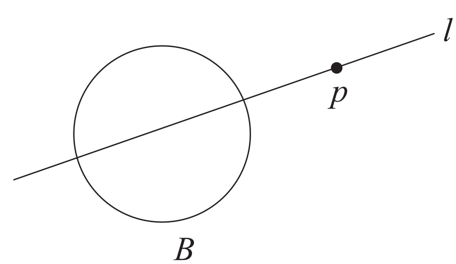
$$\mathcal{U} = \left\{ \begin{array}{c} Y = gY_0 \ \text{ where } \ g \in M(\mathbb{C}) \\ \text{and } \ gY_0 \subset D \end{array} \right\}.$$

It is a Stein manifold; we shall return to it later.

Except in the case when D fibres holomorphically over an Hermitian symmetric domain B and a few other exceptional but interesting cases, $\mathcal{U}$ is the whole set of deformations of Y in D and

$$\mathcal{U} \subset G(\mathbb{C})/K(\mathbb{C}).$$

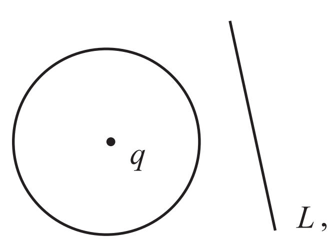
Example. $\mathrm{SU}(2,1)/T$ is a Mumford-Tate domain for polarized Hodge structures of weight three and with Hodge numbers $h^{3,0} = 1$, $h^{2,1} = 2$, $h^{1,2} = 2$, $h^{0,3} = 1$.[2] The form Q gives an isomorphism $\mathbb{C}^3 \cong \check{\mathbb{C}}^3$, and using this we may identify $\check{D}$ with the incidence correspondence $\{(p,l) : p \in l\}$ in $\mathbb{P}^2 \times \check{\mathbb{P}}^2$. Moreover, Q defines a unit ball $B \subset \mathbb{C}^2 \subset \mathbb{P}^2$, and when this is done the picture of D is



That is

$$D = \{(p,l) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 : p \in l,\ l \cap B \neq \phi,\ p \in \mathbb{P}^2 \backslash B^c\}.$$ [3]

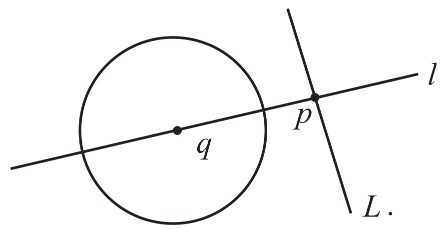
Then $\mathcal{U} \cong B \times B^c$ is given by the picture



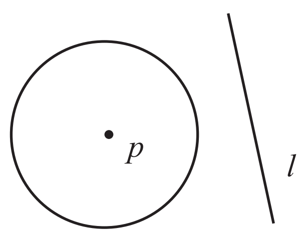
[2]The Mumford-Tate group $\widetilde{M_\varphi}$ at a generic point φ is $U(2,1)$.

[3]Here, B^c is the ball B with the conjugate complex structure, realized geometrically as the lines L such that $L \cap B^c = \phi$.

and the corresponding maximal compact subvariety $Y(q,L) \cong \mathbb{P}^1$ is given by the set of $(p,l) \in D$ in the picture



There are three open orbits of $\mathrm{SU}(2,1)$ acting on $\check{D}$ (think of the two open orbits of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{P}^1$). In addition to the non-classical one D pictured above, there are two more D', $D'' = D'^c$ that fibre over the ball and its conjugate. For example D'' has the picture



and the map $D' \to B$ is given by $(p,l) \to p$.

III. Representation theory

Let $M_{\mathbb{R}}$ be a real semi-simple Lie group that contains a compact maximal torus T. Then the weight lattice Λ, set of roots Φ and Weyl group W are defined. Given a choice Φ^+ of positive roots we set

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Theorem (Harish-Chandra)**.** *Let $\lambda \in \Lambda$ with $\lambda + \rho$ regular. Then there exists a unique discrete series representation*

$$\pi : M_{\mathbb{R}} \to H_\pi \subset L^2(M_{\mathbb{R}})$$

whose character $\Theta_\pi = \Theta_{\lambda+\rho}$ is given by

$$\Theta_{\lambda+\rho} \mid_T = (-1)^e \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}}{\prod_{\left\{ \substack{\alpha \in \Phi \\ (\lambda+\rho,\alpha)>0} \right.} (e^{\alpha/2} - e^{-\alpha/2})}$$

where $e = \dim_{\mathbb{C}} M_{\mathbb{R}}/K$. Moreover, λ_1 and λ_2 determine the same discrete series representation if, and only if,

$$w(\lambda_1 + \rho) = \lambda_2 + \rho$$

for some $w \in W_K$.

Theorem (Schmid)**.** *Let $D = M(\mathbb{R})/T$. If $(\lambda + \rho, \alpha) < 0$ for all $\alpha \in \Phi^+$,*[4] *then the L^2-cohomology groups*

$$H^q_{(2)}(D, \mathcal{L}_\lambda) = 0 \ \ for \ \ q \neq d := \dim_{\mathbb{C}} K/T$$

and $H^d_{(2)}(D, \mathcal{L}_\lambda)$ is a discrete series representation with character $\Theta_{\lambda+\rho}$.

Standard example. In this case $d = 0$. Then for $k \geqq 2$

$$H^0_{(2)}(\mathcal{H}, \mathcal{L}_k) = \left\{ \begin{array}{c} f(\tau) \text{ holomorphic in } \mathcal{H} \text{ and for } \tau = x + iy \\ \iint_{\mathcal{H}} |f(x+iy)|^2 y^k \frac{dxdy}{y^2} < \infty \end{array} \right\}.$$

[4]One says that $\lambda + \rho$ is *anti-dominant*. This can always been achieved by a suitable choice of positive Weyl chambers.

The corresponding section of $\mathcal{L}_k$ is

$$\psi_f = f(\tau)(d\tau)^{k/2}.$$

It is a general fact that sections of the bundles $\mathcal{L}_\lambda \to D$ lift to functions on $M(\mathbb{R})$. In this case, the function, usually denoted by φ_f, corresponding to ψ_f is given by

$$\varphi_f(g) = (ci+d)^{-k} f(g \cdot i)$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

For $k = 1$ there is also a *limit of discrete series* given by $f(\tau)$ holomorphic in $\mathcal{H}$ and with norm $\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 dx$. The results of Carayol we shall use below will be concerned with a degenerate limit of discrete series, and not with a proper discrete series representation. But these form a context for this.

In general, for λ as above Schmid proved that $H^q(D, \mathcal{F}) = 0$ for $q > d$ and any coherent sheaf $\mathcal{F}$, that

$$H^d_{(2)}(D, \mathcal{L}_\lambda) \to H^d(D, \mathcal{L}_\lambda)$$

is injective with dense image, and that for a maximal compact subvariety $Y \subset D$ with ideal sheaf $\mathcal{I}$ and normal sheaf $\mathcal{N}$, $\check{\mathcal{N}} \cong \mathcal{I}/\mathcal{I}^2$, the restriction maps

$$H^d(D, \mathcal{L}_\lambda) \to H^d(\mathcal{O}_Y \otimes \mathcal{L}_\lambda) \to 0$$

$$H^d(D, \mathcal{I} \otimes \mathcal{L}_\lambda) \to H^d(\mathcal{L}_\lambda \otimes \check{\mathcal{N}}) \to 0$$

$$H^d(D, \mathcal{I}^2 \otimes \mathcal{L}_\lambda) \to H^d(\mathcal{L}_\lambda \otimes \mathrm{Sym}^2 \check{\mathcal{N}}) \to 0$$

$$\vdots$$

give an "expansion" of $H^d(D, \mathcal{L}_\lambda)$ around Y. When restricted to $H^d_{(2)}(D, \mathcal{L}_\lambda)$ this gives the "K-expansion" of the restriction of the unitary representation π to the maximal compact subgroup K.[5]

Let now $\Gamma \subset M$ be an arithmetic group. Then a classical and very important question is the decomposition of the unitary $M(\mathbb{R})$ module $L^2(\Gamma\backslash M(\mathbb{R}))$. For arithmetic reasons to be explained below, one is interested in the part $L^2_0(\Gamma\backslash M(\mathbb{R}))$ that is given by cuspidal L^2-functions on $\Gamma\backslash M(\mathbb{R})$, which will be seen to correspond to the main object of arithmetic interest, viz the *cuspidal automorphic representations*. It is known that $L^2_0(\Gamma\backslash M(\mathbb{R}))$ is contained in the discrete part of the specturm of $L^2(\Gamma\backslash M(\mathbb{R}))$; for the purposes of this lecture one may think of it as equal to the discrete part.

As noted above, the discrete series in $L^2(M(\mathbb{R}))$ are realized by L^2-cohomology. This suggests that a central object of interest should be the *automorphic cohomology*

$$H^d_{(2)}(X, \mathcal{L}_\lambda)$$

where $X = \Gamma\backslash D$.

This suggestion does *not* mean that automorphic cohomology should appear directly in $L^2(\Gamma\backslash M(\mathbb{R}))$ when $\mathcal{L}_\lambda$ and $T(\Gamma\backslash D)$ is trivialized up on $M(\mathbb{R})$. Rather, when say Γ is co-compact there is a formula

$$H^q(\Gamma\backslash D, \mathcal{L}_\lambda) = \sum_{\pi\in\check{M}(\mathbb{R})} m_\pi(\Gamma) H^q(\mathfrak{n}, H_\pi)_{-\lambda}$$

[5]For the standard example, the K-type is the direct sum of the irreducible SO(2) modules with weights $k+2$, $k+3, \dots$.

expressing automorphic cohomology in terms of Lie algebra cohomology where $\mathfrak{n} = \oplus$ (negative root spaces) and π runs over the irreducible unitary representations of $M(\mathbb{R})$. It is then a theorem that only the discrete series H_π's contribute to the above sum (cf. the references cited in Carayol).

Classical example. When $X = \Gamma\backslash\mathcal{H}$ where Γ is commensurable with $\mathrm{SL}_2(\mathbb{Z})$, then

$$H^0(X, \mathcal{L}_k) \cong \left\{ \begin{array}{c} \text{holomorphic functions } f(\tau) \text{ such that} \\ f(\gamma\cdot\tau) = (c\tau + d)^k f(\tau) \text{ for all} \\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \subset \Gamma \end{array} \right\}.$$

Usually one puts in the condition that "$f(\tau)$ is bounded at the cusps" to obtain a classical automorphic form. The condition to be in $H^0_{(2)}(X, \mathcal{L}_k)$ means that $f(\tau)$ *is a cusp form*. Note the function φ_f on $\mathrm{SL}_2(\mathbb{R})$ now drops to a function on $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$. It is cuspidal if $f(\tau)$ is a cusp form.

For arithmetic reasons, it is better to replace SL_2 by GL_2, so that for a suitable $\Gamma' \subset \mathrm{GL}_2(\mathbb{Z})$

$$\Gamma\backslash\mathcal{H} = Z(\mathrm{GL}_2(\mathbb{R}))\cdot\Gamma'\backslash \mathrm{GL}_2(\mathbb{R})/O(2).$$

In fact, in general it is better to consider the Mumford-Tate groups $\widetilde{M}_\varphi$, so that the groups of interest are reductive instead of being semi-simple. We will finesse this point.

In the classical case, it known that there is an "adelification"

$$(**) \qquad Z(\mathrm{GL}_2(\mathbb{A}))\cdot \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/U$$

where $U = O(2) \times U_f$ with

$$U_f \subset \mathrm{GL}_2(\mathbb{A}_f)$$

being a compact open subgroup. Here $\mathbb{A}$ are the adeles and $\mathbb{A}_f$ are the finite adeles. The algebra of Hecke operators acts on the space of automorphic forms defined on (∗∗), and the deep arithmetic properties of classical automorphic forms are revealed through this action.

In general, there is an adelification

$$X_U = Z(\widetilde{M}(\mathbb{A}))\widetilde{M}(\mathbb{Q})\backslash\widetilde{M}(\mathbb{A})/U$$

of X, an extension $\widetilde{\mathcal{L}}_\lambda \to X_U$ of $\mathcal{L}_\lambda \to X$, and automorphic cohomology groups

$$H^d_{(2)}(X_U, \widetilde{\mathcal{L}}_\lambda)$$

in the adelic setting may be defined. Until the recent work of Carayol, so far as I know nobody had ever actually "seen" this group in a non-classical case. We will give Carayol's explicit formulae for this in the case of $H^1(X, \mathcal{L}_\lambda)$ where $\Gamma \subset U(2,1)$ is co-compact.

If $\Gamma \subset M$ is co-compact and acts freely on D, which may be assumed by passing to a finite index subgroup of Γ, then for λ sufficiently large

$$(\natural) \qquad \dim H^q(X, \mathcal{L}_\lambda) = \begin{cases} 0 & q \neq d \\ (\operatorname{vol} X)C_\lambda & q = d \end{cases}$$

where $C_\lambda = |\lambda|^{\dim X} + \cdots$ is like the leading coefficient in a Hilbert polynomial. Thus, there is a lot of automorphic cohomology — the issue is: *What does it mean? What is its arithmetic and/or algebro-geometric significance?*

If Γ is arithmetic but not assume to be co-compact, then so far as I am aware it is not known if in general

$$\dim H^q_{(2)}(X, \mathcal{L}_\lambda) < \infty,$$

or that if $|\lambda| \gg 0$ the analogue of ($\natural$) is true (it is known that $\mathrm{vol}(X) < \infty$).

The results of Carayol deal with the opposite extreme to when $|\lambda| \gg 0$; namely to a degenerate limit of discrete series. But in his case there are no holomorphic or anti-holomorphic forms, so it is a truly non-classical case.

IV. Arithmetic automorphic cohomology

Can one give meaning to the statement that "*an automorphic cohomology class* $\alpha \in H^p_{(2)}(X, \mathcal{L}_\lambda)$ *is arithmetic*". Here arithmetic has the following meaning:

We are to be given a complex vector space $E_{\mathbb{C}}$ with a "natural" $\bar{\mathbb{Q}}$-structure; i.e., there should be a $\bar{\mathbb{Q}}$-vector space E such that

$$E_{\mathbb{C}} = \mathbb{C} \otimes_{\bar{\mathbb{Q}}} E.$$

Then $e \in E_{\mathbb{C}}$ is algebraic if it is in $E \subset E_{\mathbb{C}}$. The meaning of "natural" will hopefully be clear from the context in which it is used below.

Classically, it is well known that one may define arithmetic automorphic forms $f(\tau)$, and then $f(\tau)$ is arithmetic if τ gives a CM polarized Hodge structure. There are deep extensions of this, due especially to Shimura, to certain Mumford-Tate domains parametrizing weight one polarized Hodge structures. Here, we note a general result about CM polarized Hodge structures.

If $(V, Q, F^p_{\mathbb{C}})$ *gives a CM polarized Hodge structure, then the vector spaces* $F^p_{\mathbb{C}}$ *and* $V^{p,q}_{\mathbb{C}} = F^p_{\mathbb{C}} \cap \overline{F}^q_{\mathbb{C}}$ *have natural arithmetic structures.*[6]

The reason is that there is a finite algebraic extension L' of the reflex field of the CM field such that

$$F^p_{\mathbb{C}} = \mathbb{C} \otimes_{L'} (F^p_{\mathbb{C}} \cap V_{L'})$$

is defined over L'. Thus it makes sense to say that a section of a Hodge bundle has an arithmetic value at a CM point.

Note. There is a slightly subtle point here. For a classical automorphic form $f(\tau)$ of weight k to be arithmetic (up to scaling) may be defined as saying that it is a simultaneous eigenfunction for the Hecke operators, or that the coefficients of its Fourier expansions at the cusps be in a fixed algebraic number field. Usually one takes the latter definition, the former implies it. Then $f(\tau)$ has values in the CM field at a CM point $\tau \in \mathcal{H}$. The fibre $\mathcal{L}_{k,\tau}$ then also has an arithmetic structure, and $f(\tau)(d\tau)^{k/2}$ is arithmetic in the above "Hodge-theoretic" sense.

So what to do with automorphic cohomology in higher degrees?[7]

Step A. The result stated in the summary is based on four steps. For the first there is a general integral-geometric method due to Eastwood-Gindikin-Wong

[6] The compact dual $\check{D}$ is a homogeneous, rational projective variety defined over $\mathbb{Q}$. It is a subvariety of a product of Grasmannians, and to say that $F^p_{\mathbb{C}}$ is arithmetic means that its Plücker coordinates are in $\overline{\mathbb{Q}}$.

[7] The obvious difficulty is that we cannot "evaluate" a higher degree cohomology class α at a point. In the present situation, as noted above we may evaluate α at a point $Y \in \mathcal{U}$, the space of maximal compact subvarieties of D. But except in the case when D fibres over an Hermitian symmetric domain, it does not seem possible to put a natural arithmetic structure on the fibre $\mathcal{E}_{\lambda,E}$. So one is forced to adopt a more subtle process, to be described below in the special case of $\mathrm{SU}(2,1)$.

([EGW]), cited in both of the references given in the summary, that converts sheaf cohomology on a complex manifold in a global, holomorphic object.[8] Briefly, given a complex manifold N and holomorphic vector bundle

$$\mathcal{E} \to N,$$

assume given a Stein manifold Z and holomorphic submersion

$$Z \to N$$

with contractible Stein fibres. Then there is an isomorphism

$$(\natural\natural) \qquad H^q(N, \mathcal{E}) \cong H^q_{DR}(\Gamma(\Omega^\bullet_\pi \otimes \mathcal{E}_\pi), d_\pi)$$

where $(\Gamma(\Omega^\bullet_\pi \otimes \mathcal{E}_\pi), d_\pi)$ is the complex of global, relative holomorphic differential forms with values in $\mathcal{E}_\pi = \pi^{-1}(\mathcal{E})$ and d_π is the induced exterior derivative.

Step B. In the paper [EGW], this result is applied to the picture

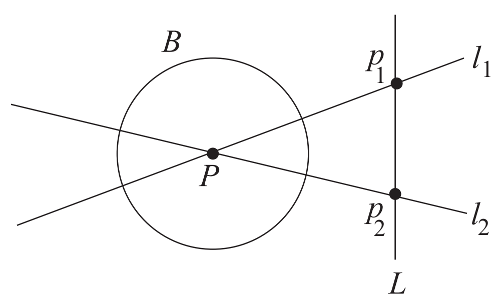


FIGURE 1

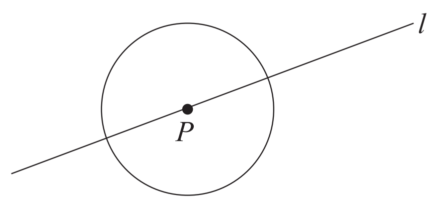
[8]This is a variant of the Penrose transform.

where $Z \subset D \times D$ consists of all pairs of points $(p_1, l_1; p_2, l_2) \in D \times D$ such that $p_1 \neq p_2$ and

$$\begin{cases} L \cap \bar{B}^c = \phi \\ l_1 \cap l_2 \in B. \end{cases}$$

Applying (♮♮) to the projection $\pi_1 : Z \to B$ converts $H^1(D, \mathcal{L}_\lambda)$ into the global holomorphic object on the RHS of (♮♮). In fact, [EGW] go further in that, in the particular case at hand and for certain λ's, there is a "holomorphic harmonic theory" so that the de Rham cohomology classes on the RHS of (♮♮) have natural unique holomorphic representatives.

Step C. As previously noted, the picture



defines another open SU(2,1)-orbit $D' \subset \check{D}$, and the map

$$\begin{array}{rcl} \pi_B : D' & \to & B \\ \cup & & \cup \\ (P, l) & \to & P \end{array}$$

gives a holomorphic fibration over the ball, which is an Hermitian symmetric domain, with $\mathbb{P}^1$'s as fibres. Setting

$$X' = \Gamma \backslash D'$$

$$W = \Gamma \backslash Z,$$

Carayol considers the picture

$$\begin{array}{ccccc} & & W & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi'} & \\ X & & & & X' \\ & & & & \downarrow{\scriptstyle \pi_B} \\ & & & & \Gamma\backslash B \end{array}$$

FIGURE 2

where π' and π_B are defined by

$$\begin{cases} \pi' \text{ (Fig. 1)} = (P, l_2) \\ \quad \pi_B(P, l_2) = P. \end{cases}$$

He then shows, through a detailed and intricate calculation, that an analogue of $(\natural\natural)$ applied to the right side of the picture leads to an isomorphism

$$(\natural\natural\natural) \qquad \boxed{H^1(X, \mathcal{L}_\lambda) \cong H^0(X', \mathcal{L}_{\lambda'}).}$$

Here, the RHS are essentially classical *Picard automorphic forms*, and thus the vector space $H^0(X', \mathcal{L}_{\lambda'})$ has an arithmetic structure.

It seems that it was not this arithmetic structure that Carayol was interested in; his perspective was from representation theory and the Langlands program. For a particular value of λ, the archimedean component of the adelic object $H^1(X_U, \widetilde{\mathcal{L}}_\lambda)$ is related to a degenerate limit of discrete series, which is a particular representation of $\mathrm{SU}(2,1)$ that is inaccessible to the standard Shimura-variety approaches to the program. The fact that D, D' and B are Mumford-Tate domains did not seem to be relevant to his work.

Carayol's proof of ($\natural\natural\natural$) was by ingenious representation-theoretic and cohomological calculations. At the end of the day, one finds the very explicit formula

$$\boxed{\omega_\alpha = f(P, l_2) l_1(p_2)^{-a} d_{\pi_1} x(P)}$$

where for p, p_0 distinct points we have for the x-coordinate of p

$$x(p) = \frac{\det'(p, p_0)}{\det'(p, p_2)\det'(p_0, p_2)},$$

and where for a point q with $l_2(q) \neq 0$ we set

$$\det'(\bullet, \bullet) = l_2^{-1}(q)\det(\bullet, \bullet, q).$$

Here, everything is expressed in a particular homogeneous coordinate system but the end result is independent of the choice. The integer "a" is due to

$$\mathcal{L}_\lambda \cong \mathcal{O}_{\mathbb{P}^2}(a) \boxtimes \mathcal{O}_{\check{\mathbb{P}}^2}(b).$$

Aside from the to me extraordinary formula for ω_α — the first time one has "held an automorphic cohomology class in ones' hand" — the point is that the observation:

If $f(P, l_2)$, $l_1(p_2)$, P are arithmetic, then ω_α is arithmetic.

Here, $d_{\pi_1} x(P)$ is in $T_{(P,l_2)}D'$, which has an arithmetic structure if D' is realized as a Mumford-Tate domain and $(P, l_2) \in D'$ is a CM point. Moreover, $f(P, l_2)$ is the value at $(P, l_2) \in D'$ of the section in $H^0(X', \mathcal{L}_{\lambda'})$ of the line bundle $\mathcal{L}_{\lambda'} \cong \mathcal{O}_{\mathbb{P}^2}(a') \boxtimes \mathcal{O}_{\check{\mathbb{P}}^2}(b')$. From the Leray spectral sequence,

$$H^0(X', \mathcal{L}_{\lambda'}) \cong H^0(\Gamma\backslash B, R^0_{\pi_B}\mathcal{L}_{\lambda'})$$

together with the discussion above about the arithmetic structures in the fibres at CM points of bundles constructed by linear algebra from the Hodge bundles, once we realize B as a Mumford-Tate domain we will have an arithmetic structure in the fibre of $\mathcal{L}_{\lambda'}$ at (P, l_2).

Step D. The final step if Hodge-theoretic. Namely, it is the result

> *There exist realizations of D, D' and B as Mumford-Tate domains, where B parametrizes polarized Hodge structures of weight one (abelian varieties), and a pair of CM points $(\varphi_1, \varphi_2) \in Z \subset D \times D$ such that in Fig. 2*
>
> $$\begin{cases} \pi'(\varphi_1, \varphi_2) \in D' \\ \pi_B \pi'(\varphi_1, \varphi_2) \in B \end{cases}$$
>
> *are CM points.*

This result is somewhat subtle, in that there are many realizations of D and D' as Mumford-Tate domains, two realizations of B as a Mumford-Tate domain for weight one polarized Hodge structures, and it is non-trivial to make choices so as to have CM points "line up".

From the results in [GGK], chapter V, it follows that the coordinates in $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ of any lift of φ_1, φ_2 and the point $P \in B$ has arithmetic coordinates if $f(P, l_2)$ does.

Conclusion: *In the special case of a non-classical Mumford-Tate domain* $D = \mathrm{SU}(2,1)/T$*, we may define an arithmetic structure in the automorphic cohomology group* $H^1(X, \mathcal{L}_\lambda)$ *where* $X = \Gamma\backslash D$*. Moreover, in the manner described above we may evaluate an automorphic cohomology class at points* $(\varphi_1, \varphi_2) \in Z \subset D \times D$*. When this is done, an arithmetic cohomology class takes arithmetic values at a pair of* CM *points.*