HODGE-THEORETIC INVARIANTS FOR ALGEBRAIC CYCLES

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Abstract. In this paper we use Hodge theory to define a filtration on the Chow groups of a smooth, projective algebraic variety. Assuming the generalized Hodge conjecture and a conjecture of Bloch-Beilinson, we show that this filtration terminates at the codimension of the algebraic cycle class, thus providing a complete set of period-type invariants for a rational equivalence class of algebraic cycles.

Outline

1. Introduction

Some years ago, inspired by earlier work of Bloch, Beilinson (cf. [R]) proposed a series of conjectures whose affirmative resolution would have far reaching consequences on our understanding of the Chow groups of a smooth projective algebraic variety X. For any abelian group G, denoting by $G_{\mathbb{Q}}$ the image of G in $G \otimes_{\mathbb{Z}} \mathbb{Q}$, these conjectures would have the following implications for the Chow group $CH^P(X)_{\mathbb{Q}}$:

(I) *There is a filtration*

$$(1.1) \qquad \begin{aligned} CH^p(X)_{\mathbb{Q}} &= F^0CH^p(X)_{\mathbb{Q}} \supset F^1CH^p(X)_{\mathbb{Q}} \\ &\supset \cdots \supset F^pCH^p(X)_{\mathbb{Q}} \supset F^{p+1}CH^p(X)_{\mathbb{Q}} = 0 \end{aligned}$$

whose successive quotients

$$(1.2) \qquad Gr^mCH^p(X)_{\mathbb{Q}} = F^mCH^p(X)_{\mathbb{Q}}/F^{m+1}CH^p(X)_{\mathbb{Q}}$$

may be described Hodge-theoretically.[1]

The first two steps in the conjectural filtration (**??**) are defined classically: If

$$\psi_0 : CH^p(X)_{\mathbb{Q}} \to H^{2p}(X, \mathbb{Q})$$

is the cycle class map, then

$$F^1CH^p(X)_{\mathbb{Q}} = \ker \psi_0 \ .$$

Setting in general

$$F^mCH^p(X) = CH^p(X) \cap F^mCH^p(X)_{\mathbb{Q}} \ ,$$

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[1] In fact, assuming the existence of the conjectural category of mixed motives Beilinson proposed such a description — cf. (3.9) below. In (**??**) we use the index range $m = 0, 1, \ldots, p$.

then if

$$\psi_1 : F^1CH^p(X) \to J^p(X)$$

is the Abel-Jacobi map we have that

$$F^2CH^p(X)_{\mathbb{Q}} = \text{image of } \{\ker \psi_1 \to CH^p(X)_{\mathbb{Q}}\} .$$

These two constructions are usually aggregated by saying that:

$F^2CH^p(X)_{\mathbb{Q}}$ is the kernel of the Deligne class mapping

$$CH^p(X)_{\mathbb{Q}} \to \mathbb{H}^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)) ,$$

which we shall denote by

$$Z \to [Z]_{\mathcal{D}}$$

where $Z \in Z^p(X)$.

The second of the implications of Beilinson's conjectures is

(II) *If X is defined over $\bar{\mathbb{Q}}$, then*

$$F^2CH^2(X(\bar{\mathbb{Q}}))_{\mathbb{Q}} = 0 . \tag{1.3}$$

More precisely, if X is defined over a number field k and we set

$$F^mCH^p(X(k))_{\mathbb{Q}} = \left\{ \begin{array}{l} \text{filtration induced from } F^mCH^p(X)_{\mathbb{Q}} \text{ by} \\ \text{the natural map } CH^p(X(k)) \to CH^p(X) \end{array} \right\}$$

then

$$F^2CH^p(X(k))_{\mathbb{Q}} = 0 .$$

There have been several proposed definitions of the filtration (**??**), in particular by H. Saito-Jannsen and by Murre (cf. [S], [J] and [M]). In an earlier work [G-G1] on the tangent space to the space of cycles, we have proposed a definition of the induced filtration on $TCH^p(X)_{\mathbb{Q}}$ that is compatible with the infinitesimal version of each of previous two proposed definitions; it has the additional properties that the associated graded

$$Gr^mTCH^p(X) = F^mTCH^p(X)_{\mathbb{Q}}/F^{m+1}TCH^p(X)_{\mathbb{Q}}$$

has a Hodge-theoretic description compatible with what is expected in (**??**), and that the infinitesimal version of (**??**) is valid.

In this paper we shall propose a definition of a filtration (**??**) that has the property (**??**) and whose infinitesimal version is consistent with that in our earlier work. Although the construction of the $F^mCH^p(X)$ for $0 \leqq m \leqq p$ makes no assumptions, the proof that the construction "stops" at $p = m$ and that rational equivalence is captured as in (**??**) depends on the generalized Hodge conjecture (GHC) and the Beilinson conjecture (**??**).

The construction will be in terms of Hodge theory; we will define Hodge-theoretic objects $\mathcal{H}_m$, $0 \leqq m \leqq 2p$ together with maps from subspaces $\mathcal{K}_m$ of $CH^p(X)_{\mathbb{Q}}$

$$\varphi_m : \mathcal{K}_m \to \mathcal{H}_m \tag{1.4}$$

such that

$$\ker \{\varphi_0, \dots, \varphi_{2m-2}\} = F^mCH^p(X)_{\mathbb{Q}}$$

for $m = 1, \dots, p$. The maps φ_0 and $\{\varphi_1, \varphi_2\}$ will capture the usual fundamental class and Abel-Jacobi map. The assertion that

$$\varphi_{2p} : F^pCH^p(X)_{\mathbb{Q}} \to \mathcal{H}_p$$

be injective will be a consequence of the GHC and Beilinson's conjecture (**??**).

The approach we have taken in this paper is geometric and concrete — in effect, we propose a Hodge-theoretic definition of $F^m CH^p(X)$ and an algorithmic test to determine if a class $[Z] \in F^m CH^p(X)$. We are grateful to the referee for suggesting a reformulation of our construction in terms of the derived category and M. Saito's theory of mixed Hodge modules (cf. [M,S] and [A].) The outline of this approach appears in the appendix.

We shall restrict our discussion to the situation when X is defined over $\bar{\mathbb{Q}}$. The essential geometric ideas appear already in this case. There is work by M. Saito, J. Lewis and others dealing with related constructions in the general case (cf. [R-M.S], [L] and the references cited therein).

For a codimension-p cycle $Z \in Z^p(X)$, we denote by $[Z] \in CH^p(X)$ the corresponding rational equivalence class.

The material below was presented by the first author in his lectures at Banff [G] and at the conference [Conference AG2000 held in Azumino, Japan – July 2000].

2. Spreads; explanation of the idea

Before describing our construction we begin with some heuristic remarks. The first is:

> *Even if one is only interested in the complex geometry of X, in higher codimension the field of definition of X and its subvarieties must enter the picture.*

This is certainly suggested by the Beilinson conjecture (**??**), and it is clearly evident from the infinitesimal study of the space of cycles and Chow groups in [G-G1]. Other than the complex numbers, the fields we shall consider will always be finitely generated over $\mathbb{Q}$. Because we are working modulo torsion, and because of the elementary fact:

> If X and a cycle Z on X are defined over a field k, and if L/k is a finite extension such that $Z \equiv_{\text{rat}} 0$ over L, then for some non-zero integer m there is a rational equivalence $mZ \equiv_{\text{rat}} 0$ over k,

it is really the transcendence level of k that is critical (this is again consistent with (**??**)).

For any field k there is a smooth projective variety S, defined over $\mathbb{Q}$ and unique up to birational equivalence, with

$$\mathbb{Q}(S) \cong k\ .$$

If both X and an algebraic cycle $Z \subset X$ are defined over k, then the critical notion is that of the *spread* given by a picture

$$\begin{array}{ccc} \mathcal{Z} & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ S & = & S \end{array} \tag{2.1}$$

Here both $\mathcal{Z}$ and $\mathcal{X}$ are defined over $\mathbb{Q}$, and we may take $\mathcal{X}$ to be smooth and projective. The diagram (**??**) is not unique; it is only defined up to *ambiguities* as discussed below. Roughly speaking, if X and Z are defined by a set of equations

$$F_\lambda(a, x) = \sum_I a_{\lambda I} x^I = 0$$

where the coefficients $a_{\lambda I} \in k$, and if

$$G_i(a) = 0$$

are a set of equations with $\mathbb{Q}$ coefficients that define the relations over $\mathbb{Q}$ that the $a_{\lambda I}$ satisfy, then S is defined by

$$G_i(s) = 0 \qquad s = (\cdots, s_{\lambda I}, \cdots)$$

and the spread (**??**) by

$$F_\lambda(s, x) = 0, \qquad s \in S\ .$$

We shall sometimes write (**??**) as a family

$$\{Z_s \subset X_s\}_{s \in S}$$

where our original $Z \subset X$ is $Z_{s_0} \subset X_{s_0}$ with s_0 being a generic point of S. Although the spread (**??**) is not unique, its infinitesimal structure over the generic point is specified and has a very natural description (cf. [G-G1]).

The basic idea is to organize the Hodge-theoretic invariants of the spread (**??**), with the ambiguities in these invariants arising from the ambiguities in the spread construction factored out, and assign these to the cycle $Z \subset X$. As will be explained below, assuming the GHC and the Beilinson conjecture (**??**), the Leray filtration and the VHS associated to (**??**) will lead to a filtration $F^m CH^p(X)_{\mathbb{Q}}$ where, when for example X is defined over $\bar{\mathbb{Q}}$, the filtration level of a class $[Z] \in CHY^p(X)_{\mathbb{Q}}$ will be related to the minimal transcendence degree of cycles in that class. Moreover, again from properties of the VHS associated to (**??**) the successive quotients $Gr^m CH^p(X)_{\mathbb{Q}}$ will have a Hodge theoretic description and, assuming (**??**), taken together will capture rational equivalence modulo torsion.

Before turning to specifics, we offer three remarks. The first is that this whole structure is quite visible and has rich geometric content at the infinitesimal level (loc. cit). The second is that the use of spreads has been "in the air" for some time (e.g., cf. Schoen [S] where he refers to a construction of Nori); our main point is to systematize their use as described below. The third remark is that our construction is not yet satisfactory in several respects, among them that the "final" Hodge theoretic object has yet to be constructed (see section 5 below).

3. Construction of the filtration on $CH^p(X)_{\mathbb{Q}}$

Now we assume that X is defined over $\mathbb{Q}$ (or over a number field); for any finitely generated subfield $k \subset \mathbb{C}$ as usual we denote by $X(k)$ the points of X defined over k. Any cycle $Z \in Z^p(X(k))$ then has a spread

$$\mathcal{Z} \subset X \times S$$

defined over $\mathbb{Q}$, where as above $\mathbb{Q}(S) \cong k$. We have

$$\mathcal{Z} \in Z^p((X \times S)(\mathbb{Q}))\ ;$$

this cycle is well-defined modulo cycles $\mathcal{W} \in Z^p((X \times S)(\mathbb{Q}))$ such that $\pi_S(\mathcal{W})$ is supported on a proper subvariety of S that is also defined over $\mathbb{Q}$. In general, these ambiguities in our constructions will occur over proper subvarieties of S that are defined over $\mathbb{Q}$. We are also working modulo finite field extensions; i.e., modulo finite coverings $\tilde{S} \to S$ defined over $\mathbb{Q}$.

If $Y \subset X$ is a subvariety of codimension $p-1$ and defined over k, and if $f \in k(Y)$, then the pair (Y, f) spreads over S to give

$$\begin{cases} \mathcal{Y} \subset X \times S \\ F \in \mathbb{Q}(\mathcal{Y}) \ , \end{cases}$$

and $\operatorname{div} F$ is well-defined modulo ambiguities as above. It follows that:

$$Z \equiv_{\text{rat}} 0 \ \textit{on} \ X(k) \Leftrightarrow \mathcal{Z} \equiv_{\text{rat}} \mathcal{W} \ \textit{on} \ (X \times S)(\mathbb{Q})$$

where $\mathcal{W}$ *is as above*,

and

the Deligne cycle class

$$[\mathcal{Z}]_{\mathcal{D}} \in \mathbb{H}^{2p}_{\mathcal{D}}(X \times S)$$

and is specified by Z *modulo the ambiguities*

$$\{[\mathcal{W}]_{\mathcal{D}} : \mathcal{W} \ \textit{as above}\} \ .$$

The Deligne class $[\mathcal{Z}]_{\mathcal{D}}$ may be decomposed into

$$\psi_0(\mathcal{Z}) \in Hg^p(X) \subset H^{2p}(X \times S)_{\mathbb{Q}}$$

and, if this vanishes, the Abel-Jacobi image

$$AJ^p_{X \times S}(\mathcal{Z}) \in J^p(X \times S) \ .$$

The ambiguities are

$$\{\psi_0(\mathcal{W}) : \mathcal{W} \ \textit{as above}\} \subset Hg^p(X \times S)$$

and

$$\left\{ AJ^p_{X \times S}(\mathcal{W}) : \mathcal{W} \ \textit{as above and} \ \mathcal{W} \equiv_{\text{hom}} 0 \right\} \ .$$

We may write

$$\psi_0(\mathcal{Z}) = [\mathcal{Z}]_0 + \cdots + [\mathcal{Z}]_{2p}$$

where

$$[\mathcal{Z}]_m \in Hg^p(X \times S) \cap \left(H^{2p-m}(X)_{\mathbb{Q}} \otimes H^m(S)_{\mathbb{Q}} \right) .$$

Similarly, we may decompose the Abel-Jacobi image into the pieces (here cohomology is with $\mathbb{C}$ coefficients)

$$AJ^p_{X \times S}(\mathcal{Z})_m \in \frac{H^{2p-1-m}(X) \otimes H^m(S)}{F^p(\text{num}) + \text{integral}} =: J^p(X \times S)_m$$

where $F^p(\text{num})$ is the p^{th} Hodge filtration of the numerator and "integral" is the image of

$$H^{2p-1-m}(X, \mathbb{Z}) \otimes H^m(S, \mathbb{Z}) \to H^{2p-1-m}(X) \otimes H^m(S) \ .$$

In the following all equalities are modulo torsion and modulo ambiguities. We will define a filtration (**??**) which, assuming the GHC and (**??**), will have exactly p steps. Let $Z \in Z^p(X(k))$. The first two steps in the filtration (**??**) have been defined, and it is an observation that

$$[\mathcal{Z}]_0 = 0 \Leftrightarrow [Z] \in F^1 CH^p(X(k)) \ , \tag{3.1}$$

and

$$[\mathcal{Z}]_0 = [\mathcal{Z}]_1 = 0 \ \text{ and } \ AJ^p_{X \times S}(\mathcal{Z})_0 = 0 \Leftrightarrow [Z] \in F^2 CH^p(X(k)) \ . \tag{3.2}$$

The geometric interpretations of (**??**) and (**??**) will be given at the beginning of section 4 below.

Suppose now that inductively $F^0CH^p(X)$, $F^1CH^p(X), \dots, F^mCH^p(X)$ have been defined and have the properties

$$[\mathcal{Z}] \in F^iCH^p(X) \Leftrightarrow \begin{cases} [\mathcal{Z}]_0 = \cdots = [\mathcal{Z}]_i = 0 \\ AJ^p_{X\times S}(\mathcal{Z})_0 = \cdots = AJ^p_{X\times S}(\mathcal{Z})_{i-1} = 0 \ . \end{cases} \tag{3.3}$$

A second observation is

$$\left. \begin{array}{l} [\mathcal{Z}]_0 = \cdots = [\mathcal{Z}]_{m+1} = 0 \\ AJ^p_{X\times S}(\mathcal{Z})_0 = \cdots = AJ^p_{X\times S}(\mathcal{Z})_{m-1} = 0 \end{array} \right\} \Rightarrow AJ^p_{X\times S}(\mathcal{Z})_m \text{ is defined .} \tag{3.4}$$

We may then define

(3.5)

$$F^{m+1}CH^p(X)_{\mathbb{Q}} = \left\{ [Z] : [Z] \in F^mCH^p(X)_{\mathbb{Q}} \text{ and } [\mathcal{Z}]_m = 0, AJ^p_{X\times S}(\mathcal{Z})_{m-1} = 0 \right\} .$$

In order to justify this definition we have the result

Assume the GHC. Then the $[\mathcal{Z}]_m$ are well-defined for $m \leqq p$ and

(3.6)

$$[\mathcal{Z}]_0 = \cdots = [\mathcal{Z}]_p \equiv 0 \ \textit{modulo ambiguities} \ \Rightarrow [\mathcal{Z}]_i \in \ \textit{ambiguities for } i \geqq p+1 \ .$$

*Similarly, if (**??**) holds then the $AJ^p_{X\times S}(\mathcal{Z})_m$ are well-defined for $m \leqq p-1$ and*

$$\begin{aligned} AJ^p_{X\times S}(\mathcal{Z})_0 = \cdots = AJ^p_{X\times S}(\mathcal{Z})_{p-1} \equiv 0 \ \textit{modulo ambiguities} \\ \Rightarrow A^p_{X\times S}(\mathcal{Z})_i \in \ \textit{ambiguities for } i \geqq p \ . \end{aligned} \tag{3.7}$$

A corollary is

*If $[Z] \in F^{p+1}CH^p(X)_{\mathbb{Q}}$, then assuming (**??**) we have that $[Z] = 0$.*

The reason is that, by (**??**) and (**??**) we may choose $\mathcal{W}$ belonging to the ambiguities as above so that

$$[\mathcal{Z} + \mathcal{W}]_{\mathcal{D}} = 0 \ .$$

The relationship between the transcendence level and the filtration we have defined on $CH^p(X)_{\mathbb{Q}}$ is expressed by:

(3.8) *If $Z \in Z^p(X(k))$ where $\operatorname{tr}\deg k \leqq m$, then assuming the* GHC *and* (1.3) *and working modulo torsion we have that*

$$[Z] \in F^{m+1}CH^p(X) \Rightarrow [Z] = 0 \ .$$

In other words, if $\operatorname{tr}\deg k \leqq m$ then the complete set of Hodge-theoretic invariants of Z is captured by $[\mathcal{Z}]_0, \dots, [\mathcal{Z}]_m$, $AJ^p_{X\times S}(\mathcal{Z})_0, \dots, AJ^p_{X\times S}(\mathcal{Z})_{m-1}$.

If in addition to (**??**) we assume Beilinson's proposed description

$$Gr^mCH^p(X) = \operatorname{Ext}^m_{\mathcal{MM}}\left(\mathbb{Z}, H^{2p-m}(X)(p)\right) \tag{3.9}$$

of the associated graded to the filtration on the Chow groups,[2] then the following converse to (3.8) may be established:

(3.10) *Every cycle $[Z]$ in $F^mCH^p(X)$ is a sum of cycles defined over fields of* $\operatorname{tr}\deg \leqq m$. *We need to use* $\operatorname{tr}\deg m$ *if, and only if, $[\mathcal{Z}]_m \neq 0$.*

This discussion may be summarized by the picture of $CH^p(X)$ in Figure 1.

[2]Here, $\mathcal{MM}$ stands for the conjectural category of mixed motives.

FIGURE 1

Here we have dropped the superscript p on $AJ^p_{X\times S}(\mathcal{Z})_i$. The regions to the left of the vertical lines define the $F^m CH^p(X)_{\mathbb{Q}}$, with the region between successive vertical lines being the associated graded. The region to the left of the slanted line under the i^{th} bracket represents where the invariants of the class in $CH^p(X)$ represented by a cycle of transcendence level i lies. Here we recall our assumption that X is defined over $\mathbb{Q}$ or over a number field. In general there will be a similar but more complicated picture.

The Hodge-theoretic maps (**??**) of the introduction are defined as follows: For each field k we use the spread construction to construct maps as indicated by the following companion diagram to Figure 1

$$\begin{array}{ccccc} \varphi_0 & \varphi_1 & \varphi_3 & \cdots & \varphi_{2p-1} \\ 0 & \varphi_2 & \varphi_4 & & \varphi_{2p} \end{array}$$

If $k \subset k'$ is a subfield, then $Z^p(X(k)) \subset Z^p(X(k'))$ and there are dominant maps $S' \to S$, diagrams

$$\begin{array}{ccc} \mathcal{Z}' & \subset & X \times S' \\ \downarrow & & \downarrow \\ \mathcal{Z} & \subset & X \times S \end{array}$$

and maps

$$H^*(X \times S) \to H^*(X \times S') \, .$$

These maps are injective since the map

$$H^*(S) \to H^*(S')$$

is injective by virtue of $S' \to S$ being dominant. Thus, none of the information in our invariants is lost when we consider a cycle as being defined over a larger field.

If Z is defined over fields k_1 and k_2, then by a standard construction k_1 and k_2 generate a field k and the invariants of $Z \in Z^p(X(k_i))$ for $i = 1, 2$ are injected into those of $Z \in Z^p(X(k))$. Thus the condition

$$[Z] \in F^m CH^p(X)_{\mathbb{Q}}$$

is independent of the field over which X is defined. The subspaces $\mathcal{K}_m$ are defined by the vanishing of $\varphi_0, \cdots, \varphi_{m-1}$, and the Hodge-theoretic objects $\mathcal{H}_m$ are the pieces of $H^{2p}(X \times S)$ and $J^p(X \times S)$ depicted above. As mentioned in the introduction, we do not yet have a satisfactory construction of a final Hodge-theoretic object.

If $Z \in Z^p(X(k))$ is a cycle with $[Z] \in F^m CH^p(X)$, then the successive invariants

$$\begin{cases} [\mathcal{Z}]_m, & \text{and if this vanishes} \\ AJ^p_{X \times S}(\mathcal{Z})_{m-1} \end{cases}$$

have Hodge-theoretic descriptions as generalized "periods." The first few cases of this will be worked out in the next section. There we will also give the corresponding arguments that, assuming the GHC, the ambiguities can be removed. We shall also give the argument justifying the above statement.

4. Interpretations and proofs

In this section we shall give the geometric interpretation of our constructions in the first few special cases. We shall also give the proofs of (3.6)–(3.8) in the first few cases, especially the case just beyond the classical one of the usual fundamental class and Abel-Jacobi map. The essential ideas already appear in these cases, and we feel that presenting the argument in this way allows one to better isolate the essential geometric/Hodge-theoretic content without having complicated indicial notations obscure the basic geometric points.

$p = 1$. Let $Z \subset X(k)$ be a divisor with spread

$$\mathcal{Z} \subset X \times S\ .$$

According to our general construction, the condition that $\psi_0(Z) = 0$ is that

$$[\mathcal{Z}]_0 = 0 \text{ in } H^2(X) \otimes H^0(S)\ .$$

When S is connected, this is just the condition that the usual fundamental class be zero (mod torsion). If we think of $X(k)$ as an abstract variety defined over k, this is independent of the embedding $k \hookrightarrow \mathbb{C}$.

If $\psi_0(Z) = 0$, then the next invariants are given by

$$[\mathcal{Z}]_1 \in H^1(X) \otimes H^1(S)\ ,$$

and if this is zero then by

$$AJ^1_{X \times S}(\mathcal{Z})_0 \in J^1(X \times S)_0\ .$$

As always, working modulo torsion the divisors $Z_s \in Z^1(X_s)$ are homologous to zero, and thus there is defined a map

$$\varphi_1 : S \to J^1(X)\ . \tag{4.1}$$

The induced map

$$(\varphi_1)_* : H_1(S)_{\mathbb{Q}} \to H_1(J^1(X))_{\mathbb{Q}}$$

may be identified with $[\mathcal{Z}]_1$, and thus the condition $[\mathcal{Z}]_1 = 0$ is that (**??**) be constant on each component of S.[3] If this is the case, then assuming that S is connected

$$AJ^1_{X \times S}(\mathcal{Z})_0 \in J^1(X \times S)_0 \cong J^1(X)$$

[3]If S is not connected, then $J^1(X \times S)_0 \cong \oplus J^1(X)$ where the direct sum is over the components of S. It will simplify this illustrative discussion if we henceforth make the blanket assumption that S be connected.

is given by $AJ^1_X(Z_s) = AJ^1_X(Z)$. The vanishing of

$$[\mathcal{Z}]_0, [\mathcal{Z}]_1, \text{ and } AJ^1_{X\times S}(\mathcal{Z})_0$$

is then equivalent to the vanishing of the usual Deligne class

$$[Z]_{\mathcal{D}} \in \mathbb{H}^2_{\mathcal{D}}(X, \mathbb{Q}(1)) \ ,$$

for every complex embedding of the field over which Z is defined.

Remark that in this case the ambiguities are given by

$$[\mathcal{Z}]_2 \in Hg^1(X\times S) \cap \left(H^0(X) \otimes H^2(S)\right) \cong Hg^1(S) \ ,$$

and

$$AJ^1_{X\times S}(\mathcal{Z})_1 \in J^1(X\times S)_1 \cong J^1(S) \ .$$

Since it is known that $Z \to [Z]_{\mathcal{D}}$ captures rational equivalence, they play no essential role in this case.

$p = 2$. First we observe that the above discussion giving the interpretation of the first three invariants in the case $p = 1$ extends to an analogous interpretation of

$$[\mathcal{Z}]_0, [\mathcal{Z}]_1, \text{ and } AJ^p_{X\times S}(\mathcal{Z})_0$$

for all p. If $p = 2$, and if these all vanish then the first non-classical invariant is

$$[\mathcal{Z}]_2 \in H^2(X) \otimes H^2(S) \qquad \text{mod ambiguities.}$$

Assuming that X is a surface and setting

$$H^2(X)_{\mathbb{Q}} = H^2_{\mathrm{tr}}(X) \oplus Hg^1(X)$$

where $H^2_{\mathrm{tr}}(X)$ is the transcendental part of $H^2(X)$, the ambiguities are in $Hg^1(X) \otimes Hg^1(S)$, and so keeping the same notation we consider

$$[\mathcal{Z}]_2 \in H^2_{\mathrm{tr}}(X) \otimes H^2(S) \ .$$

Only the $(2,2)$ part of this is relevant, and the piece in $H^{2,0}(X) \otimes H^{0,2}(S)$ is conjugate to the piece

$$[\mathcal{Z}]_2^{(0,2)} \in H^{0,2}(X) \otimes H^{2,0}(S) \cong \mathrm{Hom}\left(H^0\left(\Omega^2_{X/\mathbb{C}}\right), H^0\left(\Omega^2_{S/\mathbb{C}}\right)\right) \ . \tag{4.2}$$

In fact, $[\mathcal{Z}]_2^{(0,2)}$ is just the globalization of the *trace mapping*

$$\mathrm{Tr}_{\mathcal{Z}} : \Omega^2_{X/\mathbb{C}} \to \Omega^2_{S/\mathbb{C}} \ .$$

Suppose now that $[\mathcal{Z}]_0 = [\mathcal{Z}]_1 = [\mathcal{Z}]_2 = 0$ and that $AJ^2_{X\times S}(\mathcal{Z})_0 = 0$. In particular, $[\mathcal{Z}]_2^{(0,2)} = 0$. Then our final invariant is

$$AJ^2_{X\times S}(\mathcal{Z})_1 \in J^2(X\times S)_1 \qquad \text{mod ambiguities.}$$

Now

$$J^2(X\times S)_1 = \frac{H^2(X)\otimes H^1(S)}{F^2(\text{num}) + \text{integral}} \ .$$

The part of $J^2(X\times S)_1$ coming from $Hg^1(X) \otimes H^1(S)$ is contained in the ambiguities. Assuming still that X is a surface we consider the piece

$$\frac{H^{0,2}(X)\otimes H^1(S)}{\text{integral}} \cong \frac{\mathrm{Hom}\left(H^0\left(\Omega^2_{X/\mathbb{C}}\right), H^1(S)\right)}{\text{integral}} \tag{4.3}$$

coming from the unambiguous part of $J^2(X\times S)_1$. We denote by

$$AJ^2_{X\times S}(\mathcal{Z})_1^{(0,2)}$$

the part of $AJ^2_{X\times S}(\mathcal{Z})$ corresponding to (**??**). We shall give a geometric description of $AJ^2_{X\times S}(\mathcal{Z})^{(0,2)}_1$.

Let λ be a closed curve (loop) in S parametrized by $0 \leqq s \leqq 2\pi$. For each $s \in \lambda$ we may choose a 1-chain γ_s in X with

$$\partial\gamma_s = Z_s \ .$$

As s turns around λ we have from $Z_{2\pi} = Z_0$ that

$$\partial\left(\gamma_{2\pi} - \gamma_0\right) = 0 \ ,$$

and since $[\mathcal{Z}]_1 = AJ^2_{X\times S}(\mathcal{Z})_0 = 0$ we will have

$$\gamma_{2\pi} - \gamma_0 = \partial\Delta$$

for some 2-chain Δ in X. Setting

$$\Gamma = \bigcup_{s\in\lambda} \gamma_s - \Delta$$

we have by construction that

$$\partial\Gamma = \mathcal{Z}_\lambda$$

is that part of $\mathcal{Z}$ lying over λ.

Let now $\omega \in H^0(\Omega^2_X)$ and consider the integral

$$\int_\Gamma \omega \ .$$

If $\lambda = \partial\Lambda$ is a boundary, then an argument using Stokes' theorem and the fact that $\omega\big|_{\mathcal{Z}_\Lambda} = 0$ gives

$$\begin{aligned} \int_\Gamma \omega &= \int_\Lambda \mathrm{Tr}_{\mathcal{Z}}\omega \\ &= 0 \ . \end{aligned}$$

Thus we may define an element $AJ^2_{X\times S}(\mathcal{Z})^{(0,2)}_1$ of (**??**) by

$$\left\langle AJ^2_{X\times S}(\mathcal{Z})^{(0,2)}_1(\omega), \lambda\right\rangle = \int_\Gamma \omega$$

where $\lambda \in H_1(S,\mathbb{Z})$.

If we have a different choice $\tilde{\gamma}_s$ with $\partial\tilde{\gamma}_s = Z_s$, then

$$\gamma'_s = \gamma_s + \delta_s \ .$$

Since $\delta_{2\pi} - \delta_0 = \partial\Delta'$ for some 2-chain Δ', setting

$$\Gamma' = \bigcup_{S\in\lambda} \delta_s - \Delta' \ ,$$

we have for $\tilde{\Gamma} = \bigcup_{s\in\lambda} \tilde{\gamma}_s - \Delta - \Delta'$ that

$$\int_{\tilde{\Gamma}} \omega = \int_\Gamma \omega + \int_{\Gamma'} \omega$$

where in (4.3)

$$\int_{\Gamma'} \omega \in \ \text{“integral”}$$

since $\partial\Gamma' = 0$.

As explained in [G-G1], we may consider this construction as "integrating" the differential equations that define the subspace $TZ^2_{\mathrm{rat}}(X) \subset TZ^2(X)$.

Dealing with the ambiguities in the case $p = 2$ requires non-trivial Hodge-theoretic considerations and makes essential use of the fact that $\mathcal{Z} \subset X \times S$ is a spread. Assuming still that X is a surface, the first ambiguity is

$$[\mathcal{Z}]_3 \in Hg\left(H^1(X) \otimes H^3(S)\right) \ .$$

Suppose that X and Z are given by

$$\begin{cases} F_i(x) = 0 & \text{(defines } X\text{)} \\ G_\lambda(x,a) = 0 & \text{(defines } Z \subset X\text{)} \end{cases}$$

where the $F_i(x) \in \mathbb{Q}[x_1, \ldots, x_N]$, $G_\lambda(x,a) \in \mathbb{Q}[x_1, \ldots, x_N, \ldots, a_{\lambda I}, \ldots]$ and where the $a_{\lambda I} \in k$ satisfy

$$H_\alpha(a) = 0, \qquad H_\alpha \in \mathbb{Q}[\ldots, a_{\lambda I}, \ldots] \ .$$

Then $X \times S$ and $\mathcal{Z}$ are given in (x,s)-space by

$$\begin{cases} F_i(x) = 0 \\ G_\lambda(x,s) = 0 \\ H_\alpha(s) = 0 \ . \end{cases} \ .$$

We adjoin to these the equation of a linear space Λ defined over $\mathbb{Q}$

$$\sum_i \lambda_i^\nu x_i = 0 \qquad \lambda_i^\nu \in \mathbb{Q} \ .$$

We set $X_\Lambda = X \cap \Lambda$, and $\mathcal{Z}_\Lambda = \mathcal{Z} \cap \Lambda$. Assume $\operatorname{Dim} X_\Omega = 1$; then the spread of X_Λ is $X_\Lambda \times S$. Choosing Λ generically, we may assume that the projection of $\mathcal{Z}_\Lambda$ to S is contained in a proper subvariety $W \subset S$ because $\eta(\mathcal{Z}_\Lambda) < \operatorname{Dim}(S)$; W is defined over $\mathbb{Q}$, and for simplicity of exposition we assume it to be smooth.

By the Lefschetz theorem, $H^1(X)$ injects into $H^1(X_\Lambda)$ and we thus have a diagram

$$\begin{array}{ccc} [\mathcal{Z}]_3 & \in & Hg\left(H^1(X) \otimes H^3(S)\right) \\ \downarrow & & \hookdownarrow \\ [\mathcal{Z}_\Lambda]_{\mathcal{Z}} & \in & Hg\left(H^1(X_\Lambda) \otimes H^3(S)\right) \ . \end{array}$$

In the diagram

$$\begin{array}{ccc} H^1(X) \otimes H^1(W) & \longrightarrow & H^1(X) \otimes H^3(S) \\ \hookdownarrow & & \hookdownarrow \\ H^1(X_\Lambda) \otimes H^1(W) & \longrightarrow & H^1(X_\Lambda) \otimes H^3(S) \end{array}$$

we may lift $[\mathcal{Z}_\Lambda]_{\mathcal{Z}}$ to a Hodge class in $H^1(X) \otimes H^1(W)$, which by the HC is represented by a cycle $\mathcal{W} \in Z^1(X \times W)$.[4] Denoting by $i : X \times W \to X \times S$ the inclusion, we see first that

$$[i(\mathcal{W})]_0 = [i(\mathcal{W})]_1 = [i(\mathcal{W})]_{2,\mathrm{tr}} = 0$$

where $[i(\mathcal{W})]_{2,\mathrm{tr}}$ is the part of $[i(\mathcal{W})]_2$ in $H^2_{\mathrm{tr}}(X) \otimes H^2(S)$, and then that by construction

$$[i(\mathcal{W})]_3 = [\mathcal{Z}]_3 \ .$$

Modifying $\mathcal{Z}$ by minus $i(\mathcal{W})$ thus kills the ambiguity $[\mathcal{Z}]_3$. The ambiguity

$$[\mathcal{Z}]_4 \in Hg\left(H^0(X) \otimes H^4(S)\right) \cong H^0(X) \otimes Hg^2(S)$$

[4]In this case we do not need the HC; the usual Lefschetz $(1,1)$ theorem applies.

may be treated similarly, using the HC for $Hg^2(S)$.

In these constructions there is the issue of the field of definition of the cycles, such as $\mathcal{W}$, that are produced by the GHC. This requires a separate argument, and we shall illustrate the type of idea that is involved. Suppose that X is defined over k and L/k is a finite extension. Then the norm map $N_{L/k}$ induces a commutative diagram

$$\begin{array}{ccc} CH^p(X(L)) & \longrightarrow & H^p\left(\Omega^p_{X(L)/k}\right) \\ \downarrow N_{L/k} & & \downarrow N_{L/k} \\ CH^p(X(k)) & \longrightarrow & H^p\left(\Omega^p_{X(k)/k}\right) \end{array}$$

where on the RHS the norm map acts on the coefficients of the differential forms. Suppose now that $\gamma \in H^p\left(\Omega^p_{X(k)/k}\right)$ gives a Hodge class in $H^p\left(\Omega^p_{X(k)/k}\right) \otimes \mathbb{C} \cong H^p\left(\Omega^p_{X/\mathbb{C}}\right)$ that is represented by an algebraic cycle Z. By using the spread construction, Z will be algebraically equivalent to a cycle Z' defined over a finite extension L of k. Then $N_{L/k}(Z')$ is defined over k and has fundamental class equal to $m\gamma$ where $m = \deg[L : k]$.

Turning to the Abel-Jacobi part of the ambiguities, still assuming that $p = 2$ and that X is a surface, by the preceding argument we may assume that $Z \subset X$ satisfies

$$[\mathcal{Z}]_0 = \cdots = [\mathcal{Z}]_4 = 0 \ .$$

The first ambiguous part of $AJ^2_{X\times S}(\mathcal{Z})$ is

$$AJ^2_{X\times S}(\mathcal{Z})_2 \in J^2(X\times S)_2 = \frac{H^1(X)\otimes H^2(S)}{F^2(\text{num}) + \text{integral}} \ .$$

As above, we may choose a linear space Λ such that (up to isogeny)

We choose Ω so $\mathrm{Dim}(X_\Omega) = 1$. Writing

$$H^1(X_\Lambda) = H^1(X_\Lambda)_{\text{inv}} \oplus H^1(X_\Lambda)_{\text{ev}}$$

where $H^1(X) \to H^1(X_\Lambda)_{\text{inv}}$ is an isomorphism, the induced map

$$H^1(X_\Lambda) \to H^1(X)$$

is in $\mathrm{Hom}_{Hg}(H^1(X_\Lambda), H^1(X))$ and by the Hodge conjecture is represented by a cycle

$$Y \in CH^1\left(X_\Lambda \times X\right) \ .$$

Let $W \subset S$ be a smooth hypersurface with $\pi_S(\mathcal{Z}_\Lambda) \subset W$, again because $\mathrm{Dim}(\mathcal{Z}_\Lambda) < \mathrm{Dim}(S)$. In the diagram

$$\begin{array}{ccc} & X_\Lambda \times X \times W & \\ {\scriptstyle \pi_{13}} \swarrow & \downarrow {\scriptstyle \pi_{12}} & \searrow {\scriptstyle \pi_{23}} \\ X_\Lambda \times W & X_\Lambda \times X & X \times W \end{array}$$

we set

$$\mathcal{W} = \pi_{23*}\left(\pi_{13}^*(\mathcal{Z}_\Lambda)\cdot\pi_{12}^*Y\right) \in Z^1(X\times W)$$

whose fundamental class belongs to $H^1(X)\otimes H^0(W)$. Then

$$i(\mathcal{W}) \in Z^2(X\times S)$$

has

$$AJ^2_{X\times S}\left(i(\mathcal{W})\right)_2 = AJ^2_{X\times S}(\mathcal{Z})_2$$

and no other non-zero Abel-Jacobi components. We may thus replace $\mathcal{Z}$ by $\mathcal{Z}-i(\mathcal{W})$ to remove this ambiguity. A similar argument kills off $AJ^2_{X\times S}(\mathcal{Z})_3$.

We now turn to (3.8), and shall give the argument of the first non-trivial case of the following stronger statement:

If $\operatorname{tr}\deg k = m$*, then we may modify* $\mathcal{Z}$ *by an ambiguity* $\mathcal{W}$ *with* $\operatorname{codim}\pi_S(\mathcal{W})$
$\geqq 1$ *such that*

$$[\mathcal{Z}+\mathcal{W}]_{m+1} = \cdots = [\mathcal{Z}+\mathcal{W}]_{2p} = 0\ .$$

Moreover, if all $[\mathcal{Z}]_i = 0$ *we may modify* $\mathcal{Z}$ *by an ambiguity* $\mathcal{W}$ *so that*

$$AJ^p_{X\times S}(\mathcal{Z}+\mathcal{W})_m = \cdots = AJ^p_{X\times S}(\mathcal{Z}+\mathcal{W})_{2p-1} = 0\ .$$

Proof: The first non-trivial case is when $\operatorname{tr}\deg k = 1$ and $p = 2 = \dim X$. We then have

$$[\mathcal{Z}]_2 \in H^2(X)\otimes H^2(S)\ .$$

Denote by S_H a general hyperplane section $S\cap H$, assumed to be defined over $\mathbb{Q}$. Then by the Lefschetz theorem

$$H^0(S_H)\to H^2(S)$$

is onto, and there exists a Hodge class on $X\times S_H$ that maps to $[\mathcal{Z}]_2$.[5] By the HC (which in this case is known), there exists $\mathcal{W}\in Z^1(X\times S_H)$ so that

$$[i(\mathcal{W})] = m[\mathcal{Z}]_2$$

for some positive integer m. Moreover, we may assume that $\mathcal{W}$ is defined over $\mathbb{Q}$, and then we may use $-i(\mathcal{W})$ to remove the ambiguity $[\mathcal{Z}]_2$. □

An argument similar to those given above, together with an argument we have sketched behind the idea to keep things defined over $\mathbb{Q}$, may be used to establish (**??**)–(3.8).

A sketch of the proof of (3.10) goes as follows: The cycle $\mathcal{Z}\subset X\times S$ induces a map

$$\mathcal{Z}_* : CH_0(S)\to CH^p(X)$$

which preserves the filtrations and therefore induces transfer maps

$$Gr^m CH_0(S)\to Gr^m CH^p(X)$$

$$\ker\left\{Gr^{m-1}CH_0(S)\to Gr^{m-1}CH^p(X)\right\}\to Gr^mCH^p(X)/\text{image } Gr^mCH_0(S)$$

$$\ker\left\{Gr^{m-2}CH_0(S)\to Gr^{m-2}CH^p(X)\right\}\to Gr^mCH^p(X)/\text{image of previous map}$$

$$\vdots$$

[5]We general t, we use the surjectivity of $H^{t-1}(S_H)\to H^{t+1}(S)$.

which eventually surjects onto $Gr^m CH^p(X)$.[6] Let S_Λ be a linear section of S defined over $\mathbb{Q}$ and with $\dim S_\Lambda = m$. It is a consequence of (**??**) and the Lefschetz theorem that the natural map

$$F^k CH_0(S_\Lambda) \to F^k CH_0(S)$$

is surjective for $k \leqq m$. Thus every 0-cycle in $F^k CH_0(S)$ is rationally equivalent to a sum of 0-cycles supported on S_Λ, and therefore is defined over a field of tr deg $\leqq m$. Since X, S and $\mathcal{Z}$ are defined over $\mathbb{Q}$, it follows that $\mathcal{Z}_*(W)$ is defined over k for any 0-cycle W on S that is defined over k. This establishes the first assertion in (3.10), and the second follows by closer analysis of the argument just given.

To conclude this section we want to relate our invariants to the arithmetic cycle class of $\eta(Z)$ of $Z \in Z^p(X(k))$. Recall that from the work of Bloch-Gersten-Quillen we have (modulo torsion)

$$CH^p(X(\bar{k})) \cong H^p\left(\mathcal{K}_p^M\left(\mathcal{O}_{X(\bar{k})}\right)\right) \tag{4.4}$$

where $\mathcal{K}_p^M\left(\mathcal{O}_{X(\bar{k})}\right)$ is the sheaf associated to the p^{th} Milnor K-groups $K_p^M\left(\mathcal{O}_{X(\bar{k}),x}\right)$, $x \in X(\bar{k})$. There is a map

$$\Lambda^p d\log : \mathcal{K}_{\mathcal{P}}^M\left(\mathcal{O}_{X(\bar{k}),x}\right) \to \Omega^p_{X(\bar{k})/\mathbb{Q},x}$$

which induces

$$H^p\left(\mathcal{K}_{\mathcal{P}}^M\left(\mathcal{O}_{X(\bar{k})}\right)\right) \to H^p\left(\Omega^p_{X(\bar{k})/\mathbb{Q}}\right) . \tag{4.5}$$

Comparing (**??**) and (**??**) and noting that for any given Z we may work over a finite extension, still denoted by k, we have the *arithmetic cycle class*

$$\eta(Z) \in H^p\left(\Omega^p_{X(k)/\mathbb{Q}}\right) \subset H^p\left(\Omega^p_{X/\mathbb{Q}}\right) .$$

Since X is defined over $\mathbb{Q}$, the sequence

$$0 \to \Omega^1_{k/\mathbb{Q}} \otimes \mathcal{O}_{X(k)} \to \Omega^1_{X(k)/\mathbb{Q}} \to \Omega^1_{X(k)/k} \to 0$$

splits canonically. It follows that

$$H^p\left(\Omega^p_{X/\mathbb{Q}}\right) \cong \oplus H^p\left(\Omega^{p-m}_{X/\mathbb{C}}\right) \otimes \Omega^m_{\mathbb{C}/\mathbb{Q}} .$$

We write correspondingly

$$\eta(Z) = \eta_0(Z) + \eta_1(Z) + \cdots + \eta_p(Z)$$

where

$$\eta_m(Z) \in H^p\left(\Omega^{p-m}_{X/\mathbb{C}}\right) \otimes \Omega^m_{\mathbb{C}/\mathbb{Q}} .$$

On the other hand we have

$$[\mathcal{Z}] \equiv [\mathcal{Z}]_0 + \cdots + [\mathcal{Z}]_p \text{ mod ambiguities}$$

where

$$[\mathcal{Z}]_m \in H^{2p-m}(X) \otimes H^m(S) .$$

[6] It is a consequence of (**??**) that we need only go down two steps.

There are natural maps

$$\begin{array}{ll} \text{(i)} & H^{2p-m}(X)\otimes H^m(S)\to H^p\left(\Omega^{p-m}_{X/\mathbb{C}}\right)\otimes H^0\left(\Omega^m_{S/\mathbb{C}}\right) \\ \text{(ii)} & H^0\left(\Omega^m_{S/\mathbb{C}}\right)\to\Omega^m_{S/\mathbb{C},s_0} \qquad (s_0 = \text{ generic point of } S) \\ \text{(iii)} & \Omega^m_{S/\mathbb{C},s_0}\to\Omega^m_{k/\mathbb{Q}} \end{array}$$

and we have:

(4.6) *Under the composition of the maps (i)–(iii) above we have*

$$[\mathcal{Z}]_m \to \eta(Z)_m \ .$$

Moreover, the map

$$[\mathcal{Z}] \text{ mod amgibuities } \to \eta(Z)$$

is injective; i.e., $\eta(Z)=0$ *imples that* $[\mathcal{Z}]\in$ *ambiguities.*

Proof: Only the last statement needs proof. Since the map

$$H^0\left(\Omega^m_{S/\mathbb{C}}\right)\to\Omega^m_{S/\mathbb{C},s_0}$$

is injective, if $\eta(Z)_m=0$ then

$$[\mathcal{Z}]_m\in Hg^p(X\times S)\cap\left(H^{2p-m}(X)\otimes H^m(S)\right)$$

has no component in $H^{0,p-m}(X)\otimes H^{m,0}(S)$. Hence it gives a map

$$H_{2p-m}(X,\mathbb{Q})\to H^m(S,\mathbb{Q})$$

whose image lies in $H^{m-1,1}(S)\otimes\cdots\otimes H^{1,m-1}(S)$, and is therefore a Hodge structure of weight $m-1$. It then follows from the Lefschetz theorems and GHC that $[\mathcal{Z}]_m\in$ ambiguities. □

For X defined over $\mathbb{Q}$, we have defined a filtration $F^m\mathbb{C}H^p(X)$ on the Chow groups. For a general X we have remarked that the definition of $F^mCH^p(X)$ given above may be extended by considering the spread

$$\begin{array}{ccc} \mathcal{Z} & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ S & = & S \end{array} \tag{4.7}$$

of pairs $Z\subset X$ where Z is a codimension-p subvariety of X and both are defined over a field k with $\mathbb{Q}(S)\cong k$, and using the degeneration of the Leray spectral sequence associated to (4.7) when we use the intersection homology along the fibres of $\mathcal{X}\to S$ (cf. [R-M, S]). There is also a filtration on $H^p\left(\Omega^p_{X/\mathbb{Q}}\right)$ induced from the filtration

$$F^m\Omega^p_{X/\mathbb{Q}} = \text{image of } \left\{\Omega^m_{\mathbb{C}/\mathbb{Q}}\otimes\Omega^{p-m}_{X/\mathbb{Q}}\to\Omega^p_{X/\mathbb{Q}}\right\}$$

(as proved in [E-P], the spectral sequence associated to this filtration also degenerates at E_2). It may be verified that the arithmetic cycle class preserves these filtrations; i.e.,

$$\eta : F^mCH^p(X)\to F^mH^p\left(\Omega^p_{X/\mathbb{Q}}\right). \tag{4.8}$$

When X is defined over $\mathbb{Q}$, this is clear from (4.6).

What is *not* the case is that the filtration on Chow groups is induced from the filtration on the image of the arithmetic cycle class mapping. This is clear since the arithmetic cycle class $\eta(Z)$ does not "see" the $AJ_{X\times S}(\mathcal{Z})_m$ part of $[\mathcal{Z}]_{\mathcal{D}}$.

5. Remarks and examples

As illustrated in the preceeding section, the quantities $[\mathcal{Z}]_0, \dots, [\mathcal{Z}]_p$ and $AJ^p_{X\times S}(\mathcal{Z})_0, \dots, AJ^p_{X\times S}(\mathcal{Z})_{p-1}$ may be expressed as *period integrals*, both in the intuitive sense and in the precise sense given in [K-Z]. The conditions that

$$[\mathcal{Z}]_0 = \cdots = [\mathcal{Z}]_m = 0\ , \quad AJ^p_{X\times S}(\mathcal{Z})_0 = \cdots = AJ^p_{X\times S}(\mathcal{Z})_{m-1} = 0 \tag{5.1}$$

are therefore constructive and — at least in principle — computable in the same sense that the classical Abel's theorem and its converse gives "computable" conditions that a divisor on an algebraic curve be the divisor of a rational function. The test we are proposing that a cycle $Z \in Z^p(X)$ satisfy

$$Z \equiv_{\mathrm{rat}} 0$$

depends for $p \geqq 2$ on choosing a field of definition k for Z and checking the conditions (**??**) for $0 \leqq m \leqq p$. If $k \subset k'$ is a subfield, then as noted above there are maps

$$[\mathcal{Z}]_m \to [\mathcal{Z}']_m, \quad AJ^p_{X\times S}(\mathcal{Z})_m \to AJ^p_{X\times S}(\mathcal{Z}')_m$$

which, modulo torsion, are injective.

This mechanism is quite clear for the relative variety $X = (\mathbb{P}^2, T)$ discussed below and in section 9(iv) of [G-G1]. In this case the maps (with the evident notation)

$$\mathcal{H}_m(k) \to \mathcal{H}_m(k')$$

reduce to the standard maps

$$K_2(k) \to K_2(k')\ .$$

For $(\mathbb{P}^2, T)$ there is a "final object"

$$K_2(\mathbb{C}) = \lim_{k} K_2(k)\ .$$

In general, this final object is missing and is an important gap in our construction. Also, for $(\mathbb{P}^2, T)$ the maps (**??**) are surjective (analogue of Jacobi inversion). In general they cannot be surjective, and other conjectures of Beilinson (cf. [R]) provide possible qualitative descriptions of the images.

We now give some simple applications and examples of our construction. For the first example we observe that for any field k as above (i.e., k is of characteristic zero and finitely generated over $\mathbb{Q}$) the Hodge number

$$h^{q,0}(k)$$

may be defined to be $\dim H^0(\Omega^q_{S/\mathbb{C}})$ for any smooth variety S defined over $\mathbb{Q}$ and with $\mathbb{Q}(S) \cong k$. Then, as always assuming the GHC and (1.3) and working modulo torsion, we have

(5.2) *Let X be a regular algebraic surface defined over $\mathbb{Q}$ and $Z \in Z^2(X(k))$ a 0-cycle where*

$$\begin{cases} \deg Z = 0 \\ h^{2,0}(k) = 0\ . \end{cases}$$

Then $Z \equiv_{\mathrm{rat}} 0$. We may replace the assumption that X be regular with the two assumptions

$$\begin{cases} \mathrm{Alb}_X(Z) = 0 \\ h^{1,0}(k) = 0\ . \end{cases}$$

We now give some simple examples that illustrate our invariants together with a more or less familiar general principal that follows from them. Let X be a smooth curve defined over $\mathbb{Q}$ and $p_1, \dots, p_N$ points chosen generically. If k is the field of definition of $p_1, \dots, p_N$, then we may take

$$S = X^N$$

and the spread $\mathcal{Z} \subset X \times X^N$ of $Z = \sum_i n_i p_i$ is given by

$$\mathcal{Z} = \sum_i n_1 \Delta_{1,i+1} \subset X \times X^N$$

where Δ_{ij} is the diagonal $x_i = x_j$ in X^{N+1}. It is clear that

$$[\mathcal{Z}]_0 = \left(\sum_i n_i \right) \cdot \left(\text{generator of } H^2(X) \otimes H^0(X^N) \right) .$$

Next, using Poincaré duality $H^1(X) \cong H^1(X)^*$ there is a canonical element

$$\delta \in H^1(X) \otimes H^1(X)^*$$

corresponding to the identity, and we denote by δ_i the element δ in the i^{th} slot and zero elsewhere. Then

$$[\mathcal{Z}]_1 = \sum_i n_i \delta_i .$$

This is never zero, corresponding to the fact that

If $(p_1, \dots, p_N) \in X^N$ is $\mathbb{Q}$-Zariski dense in X^N, then for any $n_i \neq 0$

$$\sum_i n_i p_i \not\equiv_{\text{rat}} 0 .$$

If the p_i are not chosen generically and we let

$$W \subset X^N$$

be the $\mathbb{Q}$-Zariski closure of $(p_1, \dots, p_N)$ (assumed smooth), then we may take

$$\begin{aligned} S &= W \\ \mathcal{Z} &= \left(\sum_i n_i \Delta_{1,i+1} \right) \cdot (X \times W) \end{aligned}$$

and

$$[\mathcal{Z}]_1 = \text{image } \left\{ \sum_i n_i \delta_i \to H^2(X) \otimes H^1(W) \right\} .$$

As a simple example, if X is a plane cubic defined over $\mathbb{Q}$ and $L \subset \mathbb{P}^2$ is a general line, then for

$$Z = X \cdot L = p_1 + p_2 + p_3$$

we have

$$W \cong X \times X \subset X \times X \times X$$

by

$$(p, q) \to (p, q, \text{ third point of } \overline{pq} \cdot X) .$$

Now the mapping

$$\overset{3}{\underset{i=1}{\oplus}} H^1(X) \to H^1(W) \cong H^1(X) \oplus H^1(X)$$

is given by

$$(\alpha, \beta, \gamma) \to (\alpha - \gamma, \beta - \gamma)$$

and then

$$\delta_1 + \delta_2 + \delta_3 \to (0, 0)$$

giving

$$[\mathcal{Z}]_1 = 0$$

which geometrically reflects the fact that as we vary L the cycle $L \cdot X$ remains constant in $CH^1(X)$. The general principle is:

> *We cannot obtain a rational equivalence on points on X unless there are algebraic relations on the coordinates of the points (this is clear), and this condition is reflected cohomologically.*

As another example we let X be a regular surface defined over $\mathbb{Q}$ and

$$\begin{cases} (p_1, \ldots, p_N) \in X^N \\ W = \mathbb{Q}\text{-Zariski closure of } (p_1, \ldots, p_N) \\ \delta \in H^2(X) \otimes H^2(X) \end{cases}$$

as above. Then as before

$$[\mathcal{Z}]_2 = \text{image } \left\{ \sum_i n_i \delta_i \to H^2(X) \otimes H^2(W) \right\} ,$$

and

$$[\mathcal{Z}]_2 = 0 \Leftrightarrow \left\{ \begin{array}{c} \textit{The position of } W \subset X^N \\ \textit{imposes cohomological} \\ \textit{relations on the } \Delta_{1,i+1} \end{array} \right\} .$$

It is this principle, and its extension to the case where X is defined over a general field (see below), that has been used in a number of circumstances to show that

$$Z \not\equiv_{\text{rat}} 0 .$$

As another illustration, in section 2 above we mentioned related work of M. Saito and others; in [R-M.S] one may find a general theory that has the following consequence:

(5.3) *Let Y_1, Y_2 be smooth algebraic curves of genus $\geqq 1$, and on the surface $X = Y_1 \times Y_2$ consider a 0-cycle*

$$Z = [p_1 - q_1] \times [p_2 - q_2]$$

where $p_i, q_i \in Y_i$. Assume that Y_i, q_i are defined over k and that p_1, p_2 are algebraically independent over $\overline{k}$. Then

$$0 \neq [Z] \in F^2CH^2(X) .$$

The fact that

$$\begin{cases} \psi_0(Z) = 0 \\ \psi_1(X) = 0 \end{cases} \qquad (\psi_1 = \text{Alb}_X)$$

is true for any p_i, q_i. When $k = \mathbb{Q}$ we may see that our invariant

$$\varphi_3(Z) \neq 0$$

as follows: The 0-cycle Z has spread

$$\begin{array}{c} \mathcal{Z} \subset X \times S_1 \times S_2 \\ \downarrow \\ S_1 \times S_2 \end{array}$$

where the S_i are curves defined over $\mathbb{Q}$ and with $\mathbb{Q}(S_i) = k(p_i)$ being the field of definition of p_i. Choose $\omega_i \in H^0(\Omega^1_{Y_i/\mathbb{C}})$ with $\omega_i(p_i) \neq 0$ and set $\omega = \omega_1 \wedge \omega_2 \in H^0(\Omega^2_{X/\mathbb{C}})$. Then as noted above a part of $[\mathcal{Z}]_2$ is given by

$$\mathrm{Tr}_{\mathcal{Z}}\omega \in H^0\left(\Omega^2_{S_1\times S_2/\mathbb{C}}\right) ,$$

and by our construction this is non-zero.

A related 0-cycle was given by Faber-Pandharipande and studied in [G-G2]: Let Y be a smooth of genus $g \geqq 1$ and on $X = Y \times Y$ set

$$Z_K = K_1 \times K_2 - (2g-2)\Delta_K$$

where K_i is a canonical divisor on the i^{th} factor and Δ_K is a canonical divisor on the diagonal. Then

$$\begin{cases} \deg Z_K = 0 \\ \mathrm{Alb}_X(Z_K) = 0 \end{cases}$$

so that

$$[Z_K] \in F^2CH^2(X) .$$

Faber-Pandharipande showed that $[Z_K] = 0$ if $g = 2,3$ (the case $g = 2$ is trivial since $[Z_K] = 0$ whenever Y is hyperelliptic), and in [G-G2] it is proved that

$$\begin{array}{c} [Z_K] \neq 0 \textit{ if } g \geqq 4 \\ \textit{and } Y \textit{ is general.} \end{array} \tag{5.4}$$

Remark that the condition Y be general is necessary, since (**??**) implies that $[Z_K] = 0$ if Y is defined over a number field. The proof of (**??**) consists in considering the spread

$$\begin{array}{c} \mathcal{Z}_K \subset \mathcal{X} \\ \downarrow \\ S \end{array} \tag{5.5}$$

of the pair (Z_K, X) where S is (a finite covering of) the moduli space of curves genus g. Associated to (**??**) is an infinitesimal invariant

$$\delta\nu_{\mathcal{Z}_K}$$

constructed from the analogue of $[\mathcal{Z}]_2$ but where now the surface X varies also. A geometrically motivated but technically intricate calculation using Shiffer variations shows that $\delta\nu_{\mathcal{Z}_K} \not\equiv 0$.

To construct the analogue of the invariants defined above when X is not defined over $\mathbb{Q}$, one must take a spread analogous to (**??**) and use properties of the VHS associated to the intersection homology along the fibres of $\mathcal{X} \to S$.

We have defined Hodge theoretic invariants of a cycle $Z \in Z^p(X)$ when X is smooth and projective. It is reasonable to assume that this construction can be extended to more general situations; e.g., when $X = (\mathbb{P}^2, T)$ is the relative variety discussed in section 9(iv) of [G-G1]. Anticipating this we will suggest what these invariants will be in this case. For this, we let $\mathbb{P}^2 - T \cong \mathbb{C}^* \times \mathbb{C}^*$ have coordinates (x, y) and write $Z \in Z^2(\mathbb{P}^2, T)$ as

$$Z = \sum_i n_i(x_i, y_i) \; .$$

The spread of Z will be written as

$$Z(s) = \sum_i n_i \left(x_i(s), y_i(s)\right) \; , \qquad s \in S \; .$$

Here, Z is defined over k with $\mathbb{Q}(S) \cong k$, and we let $D \subset S$ be the divisor of points $s \in S$ where $Z(s) \in T$ and set $S^0 = S - D$. Our descriptions are as follows:

$$[\mathcal{Z}]_0 = \sum_i n_i \; .$$

Assuming $[\mathcal{Z}]_0 = 0$ then

$[\mathcal{Z}]_1$ *is the induced map*
$H_1(S^0, \mathbb{Z}) \to H_1\left(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z}\right)$,

Here, we are thinking of $\mathbb{C}^* \times \mathbb{C}^*$ as the Albanese variety of $(\mathbb{P}^2, T)$ and the above map is induced from the Albanese map

$$s \to \left(\int_{\gamma_s} \frac{dx}{x}, \int_{\gamma_s} \frac{dy}{y} \right) \in \mathbb{C}/\mathbb{Z}(1) \oplus \mathbb{C}/\mathbb{Z}(1)$$

where $\partial\gamma_s = Z_s$ and $\mathbb{C}/\mathbb{Z}(1) \cong \mathbb{C}^*$. Explicitly, this map is

$$s \to \left(\prod_i x_i(s)^{n_i}, \prod_i y_i(s)^{n_i} \right) \; .$$

If $[\mathcal{Z}]_1 = 0$ then this map is constant (assuming S is connected) and

$$AJ_{X\times S}(\mathcal{Z})_0 = \left(\prod_i x_i(s_0)^{n_i}, \prod_i y_i(s_0)^{n_i} \right) \; .$$

The interesting invariants are $[\mathcal{Z}]_2$ and $AJ_{X\times S}(\mathcal{Z})_1$. The former is the trace map

$$H^0\left(\Omega^2_{\mathbb{P}^2/\mathbb{C}}(\log T)\right) \to H^0\left(\Omega^2_{S/\mathbb{C}}(\log D)\right)$$

(recall that $\omega = (dx/x) \wedge (dy/y)$ generates $H^0\left(\Omega^2_{\mathbb{P}^2/\mathbb{C}}(\log T)\right)$). For the later, as discussed in (loc. cit.) this is given by the regulator

$$\lambda \to \int_{\mathcal{Z}(\lambda)} \log x \frac{dy}{y} - \log y(s_0) \frac{dx}{x}$$

where λ is a closed loop in S^0 and $\mathcal{Z}(\lambda)$ is the corresponding closed curve in $\mathbb{P}^2 - T$ traced out by $Z(s)$, $s \in \lambda$. If $\lambda = \partial\Lambda$ is a boundary, then since $[\mathcal{Z}]_2 = 0$ the integral above is

$$\int_{\mathcal{Z}(\Lambda)} \omega = \int_{\Lambda} \operatorname{Tr} \omega = 0 \; .$$

In closing we would like to discuss the conjecture (cf. [B], [Re] and the references cited therein) that for a flat vector bundle

(5.6) $$E \to X$$

we have

(5.7) $$c_p(E) \in F^p CH^p(X)\ , \qquad p \geqq 2\ .$$

Assuming that X is defined over $\mathbb{Q}$, it is reasonable to hope that the spread

$$\mathcal{E} \to X \times S$$

of (??) has a connection $\theta_{\mathcal{E}}$ whose curvature $\Theta_{\mathcal{E}}$ vanishes on each $X \times \{s\}$. If $P_p(A_1, \dots A_p)$ is the invariant polynomial giving the p^{th} Chern class, then since in de Rham cohomology

$$c_p(\mathcal{E}) = P_p\left(\Theta_{\mathcal{E}}, \dots, \Theta_{\mathcal{E}}\right)$$

it follows that

(5.8) $$[c_p(\mathcal{E})]_0 = \cdots = [c_p(\mathcal{E})]_{p-1} = 0 \qquad \text{mod torsion.}$$

To establish (??) it must be shown that, modulo torsion,

(5.9) $$AJ_{X\times S}(c_p(\mathcal{E}))_0 = \cdots = AJ_{X\times S}\left(c_p(\mathcal{E})\right)_{p-2} = 0\ , \qquad p \geqq 2\ .$$

Using the interpretation given in section 4 above, the first step in (??) follows from the important work of Reznikov [Re].

However, for $p \geqq 3$ the remainder of (??) does not seem to follow from [Re]. In fact, in general $\mathcal{E} \to X \times S$ is not flat and so the secondary Chern classes may not be defined. Using our proposed definition of $F^p CH^p(X)$, the conjecture (??) has to do with the behaviour of $c_p(\mathcal{E})$ and $AJ_{X\times S}(\mathcal{E})$ on the Künneth components of $H^{2p}(X \times S)$ and $H^{2p-1}(X \times S)$. The first of these are Hodge classes, and so assuming the HC the issue is roughly the following: Let $Y \subset X \times S$ be an irreducible subvariety of dimension p and where, setting $Y_s = X \times \{s\}$ for generic $s \in S$,

$$\dim Y_s = p - m\ , \qquad m = 0, \dots, p-1\ .$$

Now $\Theta_{\mathcal{E}}\big|_Y =: \Theta_{\mathcal{E},Y}$ is not flat, but we do have that

(5.10) $$\Theta^{m+1}_{\mathcal{E},Y} = \underbrace{\Theta_{\mathcal{E},Y} \wedge \cdots \wedge \Theta_{\mathcal{E},Y}}_{m+1} = 0\ .$$

It follows that, for $\mathcal{E}_Y = \mathcal{E}\big|_Y$ we have modulo torsion

$$c_q(\mathcal{E}_Y) = 0 \quad \text{for} \quad q \geqq m+1\ .$$

Passing to an integer multiple if necessary,

$$AJ_Y(c_q(\mathcal{E}_Y)) = J^q(Y)$$

is then defined. For $m = 0$, $\mathcal{E}_Y$ is flat and we are in the case considered by Reznikov. However, for $m \geqq 1$ the bundle $\mathcal{E}_Y$ is not flat but is what we may call *m-flat*.

In general, let W be a smooth algebraic variety and $F \to W$ on m-flat holomorphic vector bundle with curvature Θ_F satisfying

$$\Theta_F^{m+1} = 0\ .$$

Then, modulo torsion

(5.11) $$c_q(F) = 0 \qquad q \geqq m+1$$

and in this range

$$AJ_W(c_q(F)) \in J^q(W)$$

is defined. Moreover, for $q \geqq m+2$, $AJ_W(c_q(F))$ is rigid in the sense that it is constant on connected components in the moduli space of m-flat bundles. This follows from the result proved in [Gri] that for a family $F_t \to X$ of holomorphic bundles with $F_0 = F$

$$\frac{d}{dt}\left(AJ_W(c_q(F_t))\right)_{t=0} = P_q(\eta, \underbrace{\Theta_F, \ldots, \Theta_F}_{q-1})$$

where $\eta \in H^1_{\bar\partial}(\mathrm{Hom}(F,F))$ gives the Kodaira-Spencer class representing $dF_t/dt\big|_{t=0}$. Thus we may ask if, in addition to (**??**), we have modulo torsion

$$AJ_W(c_q(F)) = 0 \qquad q \geqq m+2 \tag{5.12}$$

for $F \to W$ on m-flat bundle. It is our feeling that this general result would imply (**??**).

Finally we would like to mention the relation between our proposed filtration on $CH^p(X)_{\mathbb{Q}}$ and those suggested by H. Saito-Jannsen and Murre. Regarding the latter, assuming the GHC it can be shown that these filtrations agree, thus providing a Hodge-theoretic interpretation for Murre's filtration. Regarding the former, there is an heuristic argument that

$$\left\{\begin{array}{c} m^{\text{th}}\text{-step of} \\ \text{S-J filtration}\end{array}\right\} \subseteq \left\{\begin{array}{c} m^{\text{th}}\text{-step of the} \\ \text{filtration defined above}\end{array}\right\}.$$

This heuristic argument depends on the GHC and global properties of VHS/intersection homology analogous to what is required to extend our construction to varieties defined over arbitrary fields.

Appendix: Reformulation of the construction

The following is extracted from the referee's report.

It was known for quite a while that (as yet conjectural) formalism of mixed motives provides a canonical filtration on Chow groups $CH^i(X)$ of any (smooth proper) algebraic variety X over an arbitrary field k. In this paper the authors define, by means of Hodge theory, a certain filtration on $CH^i(X)$ in the situation when $k = \mathbb{C}$. Their filtration coincides with the "motivic" once we assume the two extra conjectures relating the motivic and Hodge pictures, namely, (i): the Hodge conjecture, and (ii): the assertion that the composition of maps

$$CH^i(Y_{\mathbb{Q}}) \hookrightarrow CH^i(Y) \to H^{2i}_{\mathcal{D}}(Y_{\mathbb{C}}, \mathbb{Q}(i)) \tag{1}$$

is injective for every smooth proper variety $Y_{\mathbb{Q}}$ over $\mathbb{Q}$ (the right arrow is the class map for Deligne cohomology).

I believe that the construction in this paper can be summarized as follows. Notation: $\mathcal{H}$ is the category of mixed $\mathbb{Q}$-Hodge structures, $D\mathcal{H}$ its derived category. For an algebraic variety Z over $\mathbb{C}$ we denote by $C(Z) := R\Gamma(Z,\mathbb{Q}) \in D\mathcal{H}$ the complex of $\mathbb{Q}$-cochains of Z with its mixed Hodge structure $H^i(Z) := H^i(Z,\mathbb{Q}) \in \mathcal{H}$. The Deligne cohomology for arbitrary Z are defined as $H^a_{\mathcal{D}}(Z,\mathbb{Q}(b)) := \mathrm{Hom}_{D\mathcal{H}}(\mathbb{Q}(0), C(Z)(b)[a])$ (if Z is proper and smooth this is the usual Deligne cohomology); the class map from Chow group to Deligne cohomology is always well-defined.

I believe that the only Hodge-to-motivic assumption relevant to the story is the following conjecture (iii) (a corollary of (i), (ii) above) which says that the arrow (1) is injective for any smooth $Y_{\mathbb{Q}}$ regardless if it is proper or not.

So let X be our smooth proper variety over $\mathbb{C}$. It is defined over a subring of $\mathbb{C}$ finitely generated over $\mathbb{Q}$. So, considering X as a smooth affine scheme (of infinite type) over $\mathbb{Q}$, one can write $X = \varinjlim X_{S_\alpha}$ where S_α is a smooth affine scheme over $\mathbb{Q}$ and X_{S_α} is a smooth proper S_α-scheme. Here $S_\alpha = \text{Spec } R_\alpha$ where $R_\alpha \subset \mathbb{C}$, $\cup R_\alpha = \mathbb{C}$. The presentation $X = \varinjlim X_{S_\alpha}$ is essentially unique.

Set $H^i(X_S) := \varinjlim H^i((X_{S_\alpha})_{\mathbb{C}}) \in \varinjlim \mathcal{H}$; similarly, $C(X_S) := \varinjlim C((X_{S_\alpha})_{\mathbb{C}} \in D\varinjlim \mathcal{H}$ and $H^a_C(X_S, \mathbb{Q}(b)) := \varinjlim H^a_D((X_{S_\alpha})_{\mathbb{C}}, Q(b)) = \text{Hom}(\mathbb{Q}(0), C(X_S)(b)[a])$. Since $CH^i(X) = \varinjlim CH^i(X_{S_\alpha})$ the class maps for X_{S_α} yield the one

$$CH^i(X) \to H^{2i}_D(X_S, \mathbb{Q}(i)) \ . \tag{2}$$

One can compute the Deligne cohomology by means of Morihiko Saito's theory of Hodge complexes. Namely, one has $H^a_D((X_{S_\alpha})_{\mathbb{C}}, \mathbb{Q}(b)) = \text{Hom}(\mathbb{Q}(0), R\pi_{\alpha *}\mathbb{Q}(b)[a])$, the morphisms are taken in the derived category of mixed Hodge complexes on $(S_\alpha)_{\mathbb{C}}$ and $\pi_\alpha : (X_{S_\alpha})_{\mathbb{C}} \to (S_\alpha)_{\mathbb{C}}$. The Leray spectral sequence for π_α provides a canonical filtration on $H^{2i}_D(X_S, \mathbb{Q}(i))$ shifted by $2i$. The filtration F on $CH^i(X)$ considered in the note is the pull-back of this filtration by map (2).

If X is defined over $\mathbb{Q}$, $X = (X_{\mathbb{Q}})_{\mathbb{C}}$ (which is actually the case considered in the main body of the note) then there is no need to invoke Saito's theory: the old Deligne's mixed Hodge theory is sufficient. Indeed, we can assume that $X_{S_\alpha} = X_{\mathbb{Q}} \times S_\alpha$, so by Künneth $C(X_{S_\alpha}) = C(X) \otimes C(S_\alpha)$, and the filtration on the Deligne cohomology comes from the filtration $(\tau_{\leq} C(X_{\mathbb{Q}})) \otimes C(S_\alpha)$ on $C(X_{S_\alpha})$ where τ is the canonical filtration. Notice that a usual computation shows that every successive quotient $Gr^a H^2_D(X_S, \mathbb{Q}(i))$ can be represented canonically as an extention of $\text{Hom}_{\mathcal{H}}(\mathbb{Q}(0), H^{2i-a}(X) \otimes H^a(S)(i)) = \text{Hom}_{\mathcal{H}}(H_a(S), H^{2i-a}(X))$ by $\text{Ext}^1_{\mathcal{H}}(\mathbb{Q}(0), H^{2i-a}(X) \otimes H^{a-1}(S)(i)) = \text{Ext}^1_{\mathcal{H}}(H_{a-I}(S), H^{2i-a}(X)(i))$ where $H_a(S)$ is the projective limit of the homology groups of $(S_\alpha)_{\mathbb{C}}$.

Conjecture (iii) asserts that (2) is injective, and also that F coincides with the "motivic" filtration.

References

[S] S. Saito, Motives, algebraic cycles and Hodge theory, in *The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998)*, 235–253, *CRM Proc. Lecture Notes* **24**, A.M.S., Providence, RI, 2000.

[J] U. Jannsen, Equivalence relations on algebraic cycles, *The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998)*, 225–260, *NATO Sci. Ser. C Math. Phys. Sci.* **548**, Kluwer Acad. Publ., Dordrecht, 2000.

[M] J. P. Murre, On a conjectural filtration on the Chow groups at an algebraic variety, I., *Indag. Math.* **4** (1993), 177–188.

[G-G1] M. Green and P. Griffiths, On the tangent space to the space of algebraic cycles on a smooth algebraic variety, to appear.

[M.S] M. Saito, Modules de Hodge polamisibles, *Publ. Res.* ??? *Inst. Math. Sci.* **24** (1988), 849–995 and Mixed ???? *Res. Inst. Math. Sci.* **26** (1990), ???

[R] D. Ramakrishnan, Regulators, algebraic cycles and values of L-functions, *Contemp. Math.* **83** (1989), 183–310.

[S] C. Schoen, Zero cycles modulo rational equivalence for some varieties defined over fields of transcendence degree one, *Proc. Symposia in Pure Math.* **46** (1987), 463–473.

[G-G2] M. Green and P. Griffiths, An interesting 0-cycle, to appear.

[R-M.S] A. Rosenschon and M. Saito, Cycle map for strictly decomposable cycles, *RIMS*-1323, June2001.

[L] J. Lewis, A filtration on the Chow groups of complex projective variety, *Comp. Math.* **128** (2001), 299–322.

[E-P] H. Esnault and K. Paranjape, Remarks on absolute de Rham and absolute Hodge cycles, *C. R. Acad. Sci. Paris Sér I Math.* **319** (1) (194), 67–72.

[K-Z] M. Kontsevich and D. Zagier, Periods, *Mathematics Unlimited—2001 and Beyond*, 771–808, Springer, Berlin, 2001.

[G] M. Green, Algebraic cycles and Hodge theory, Lecture notes (Banff).

[B] S. Bloch, Applications of the dilogarithm function in algebraic k-theory and algebraic geometry, *Int. Symp. on Alg. Geometry, Kyoto, 1977*, 103–114.

[Gri] P. Griffiths, Some resluts on algebraic cycles on algebraic manifolds, *1969 Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, 93–191, Oxford Univ. Press, London.

[A] M. Asakura, Motives and algebraic de Rham cohomology, CRM *Proceedings and Lecture Notes*, Vol. 24 (2000), 193–154.

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