
Given two parabolic subgroups $P$ and $Q$ such that $P \subset Q$ we have defined $\tau_P^Q$ (resp. $\tau_P^{-Q}$) to be the characteristic function of the set of $H \in \mathfrak{g}_P^Q$ such that $\alpha(H) > 0$ for all $\alpha \in \Delta_P^Q$ (resp. $\pi(H) > 0$ for all $\pi \in \Delta_P^{-Q}$). By abuse of notation we also consider them as functions on $\mathfrak{g}_P$ depending only on the projection on $\mathfrak{g}_P^Q$.

When $P$ and $Q$ are $\varepsilon$-invariant we define $\varepsilon \tau_P^Q$ (resp. $\varepsilon \tau_P^{-Q}$) to be the restriction to $(\mathfrak{g}_P^Q)_\varepsilon$ the subset of $\varepsilon$-invariant vectors. They will also be considered as functions on $\mathfrak{g}_P^\varepsilon$ and even on $\mathfrak{g}_0^\varepsilon$. We introduce a new functions on $(\mathfrak{g}_P^Q)_\varepsilon \times (\mathfrak{g}_P^Q)_\varepsilon$:

$$\varepsilon \Gamma_P^Q(H, X) = \sum_{P \subset R \subset Q \\varepsilon(R) = R} (-1)^{\varepsilon R - \varepsilon Q} \varepsilon \Gamma_R^Q(H) \varepsilon \Gamma_R^Q(H-X).$$

The key observation for all that follows is the

LEMMA 13.1.1.

(i) Assume that $X$ remains in a compact subset $\omega$ then

$$H \rightarrow \varepsilon \Gamma_P^Q(H, X)$$

is supported in a compact subset of $(\mathfrak{g}_P^Q)_\varepsilon$ independent of $X \in \omega$. 


(ii) If X is regular then

\[ H \longrightarrow \varepsilon P^Q(H, X) \]

is the characteristic function of the set of \( H \in (\mathcal{P}^Q)_{\varepsilon} \) such that

\[ a(H) > 0 \text{ for all } \alpha \in \mathcal{A}^Q_P \]
\[ \varpi(H) \leq \omega(X) \text{ for all } \varpi \in \mathcal{A}^Q_S \].

(iii) \( \varepsilon P^Q(H, 0) = \delta_P^Q \) (the Kronecker symbol).

Given H we define \( S = S_H \) to be the \( \varepsilon \)-invariant parabolic subgroup \( S \) between \( P \) and \( Q \) such that

\[ \mathcal{A}^S_P = \{ \alpha \in \mathcal{A}^Q_P | a(H) > 0 \} \].

We have

\[ \varepsilon P^Q(H, X) = \sum_{P \subseteq R \subseteq S \subseteq (R) = R} (-1)^{a_R - a_S} \varepsilon R^Q(H-X). \]

This is non-zero only if \( \varpi(H-X) > 0 \) for all \( \varpi \in \mathcal{A}^Q_S \) and \( \varpi(H-X) \leq 0 \) for all \( \varpi \in \mathcal{A}^Q_P - \mathcal{A}^Q_S \). Choose \( X_1 \in (\mathcal{P}^Q)_{\varepsilon} \) such that

\[ a(X_1) \leq \inf_{X \in \omega \cup \{0\}} a(X) \]

for all \( \alpha \in \mathcal{A}^Q_P \). Since \( a(H) > 0 \geq a(X_1) \) for \( \alpha \in \mathcal{A}^S_P \) and \( \varpi(H) > \varpi(X) \geq \varpi(X_1) \) for \( \varpi \in \mathcal{A}^Q_S \) we have \( \varpi(H) > \varpi(X_1) \) for all \( \varpi \in \mathcal{A}^Q_P \). In the same way,
replacing \( \inf \) by \( \max \) and changing the sense of inequalities we define \( X_2 \); then for all \( \varpi \epsilon \Delta_P^Q \) we have

\[
\varpi(X_1) < \varpi(H) \leq \varpi(X_2)
\]

whenever \( \epsilon^Q_P(H, X) \neq 0 \) and \( X \epsilon \omega \). Assertion (i) follows.

Consider now a fixed \( X \) such that \( a(X) \geq 0 \) for all \( a \epsilon \Delta_P^Q \), then we may take \( X_1 = 0 \) and \( X_2 = X \); this implies \( S_H = Q \) if \( \epsilon P^Q(H, X) \neq 0 \) and assertion (ii) follows.

If \( X = 0 \) we may take \( X_1 = X_2 = 0 \) and this implies \( S_H = Q \) and \( S_H = P \) if \( \epsilon P^Q(H, 0) \neq 0 \). This yields assertion (iii). \( \square \)

Remark: Assertion (iii) above has already been proved, with other notations, in Lecture 2; see 13.1.2. below.

We now introduce matrices of function on \( \pi_0^e \) whose entries are indexed by pairs of \( e \)-invariant parabolic subgroups: let \( e^r = (e^r_P, Q) \) be such that

\[
e^r_P, Q = 0 \quad \text{if} \quad P \not\subset Q
\]

\[
e^r_P, Q = (-1)^{a_P^e} e^r_P, \quad \text{if} \quad P \subset Q
\]

considered as functions on \( \pi_0^e \). In the same way we define \( \hat{e}^r \). Assertion (iii) in the above lemma yields the

COROLLARY 13.1.2. \( e^r \hat{e}^r = 1 \). \( \square \)
We introduce a matrix \( \Gamma = (\Gamma_{P,Q}) \) whose entries are such that

\[
\Gamma_{P,Q} = 0 \quad \text{if} \quad P \notin Q \\
\Gamma_{P,Q} = (-1)^{a_P - a_Q} \Gamma_{Q,P} \quad \text{if} \quad P \subset Q.
\]

Using the definition of \( \gamma_Q \) we see that

\[
\gamma_{(H, X)} = \gamma_{(H)} \gamma_{(H-X)}.
\]

**Lemma 13.1.3.**

\[
\gamma_Q(H-X) = \sum_{P \subset R \subset Q, \varepsilon(R) = R} (-1)^{a_R - a_Q} \gamma_{(H)} \gamma_{(H-X)}.
\]

Using Corollary 13.1.2 we see that

\[
\gamma(H-X) = \gamma(H)^{-1} \gamma(H, X) = \gamma(H) \gamma_{(H, X)} \quad \square
\]

Since \( H \rightarrow \gamma_Q(H, X) \) is compactly supported on \((a_Q^0)^\varepsilon\)

the integral

\[
\gamma_Q(\lambda, X) = \int \gamma_Q(H, X)e^{\lambda(H)} d\text{H}
\]

is convergent for all \( \lambda \in \mathcal{A}_0^* \otimes \mathcal{C} \) and defines an analytic function. We want to compute \( \gamma_Q \). We define \( \Delta_Q \) to be the set of restrictions to
of $\varepsilon$-orbits of elements in $\Delta^Q_P$. Given $\alpha \in \Delta^Q_P$ the coroot $\check{\alpha}$ lies in $(\alpha^Q_P)^\varepsilon$. We define

$$\varepsilon^Q_P = |\det(\check{\alpha}, \check{\beta})|^{\frac{1}{2}} \quad \alpha, \beta \in \Delta^Q_P$$

and

$$\varepsilon^\theta_P(\lambda) = (\varepsilon^Q_P)^{-1} \prod_{\alpha \in \Delta^Q_P} \lambda(\check{\alpha})$$

Now assume that $\text{Re}(\lambda(\check{\alpha})) < 0$ for all $\alpha \in \Delta^Q_P$, then

$$\int_{(\alpha^Q_P)^\varepsilon} \varepsilon^\tau_P(H)e^{\lambda(H)}dH = \varepsilon^\theta_P(\lambda)^{-1}$$

Replacing roots by weights we define $\varepsilon^\delta_P$, $\varepsilon^\varepsilon_P$, and $\varepsilon^\theta_P$ is the Laplace transform of $\varepsilon^\tau_P$. This yields the following expression for $\varepsilon^\gamma_P$:

**Lemma 13.1.4.**

$$\varepsilon^\gamma_P(\lambda, X) = \sum_{P \in R \subset Q \quad \varepsilon(R) = R} a^\varepsilon_R a^\varepsilon_R \lambda(X^Q_R) \varepsilon^\theta_R(\lambda)^{-1} \varepsilon^\theta_R(\lambda)^{-1}$$

where $X^Q_R$ is the projection of $X$ on $(\alpha^Q_R)^\varepsilon$.

The left-hand side is analytic, the right-hand side is meromorphic and hence they are equal everywhere and the singularities of the right-hand side cancel. $\Box$
Letting $\gamma^Q_P(X) = \gamma^Q_P(0, X)$ we have the

**Lemma 13.1.5.** The function

$$X \longrightarrow \gamma^Q_P(X)$$

is a homogeneous polynomial of degree $k = a^P - a^Q$ given by

$$\frac{1}{k!} \sum_{P \subset R \subset Q} (-1)^{a^P - a^Q} \lambda^k (\epsilon^Q_R \lambda) k \hat{\theta}^P(\lambda)^{-1} \hat{\theta}^Q_R(\lambda)^{-1}$$

well defined if $\lambda$ is not a singular value of $\hat{\theta}(\lambda)^{-1}$ or $\hat{\theta}(\lambda)^{-1}$, and independent of $\lambda$.

It is clear that $X \longrightarrow \gamma^Q_P(X)$ is analytic and homogeneous of degree $k = a^P - a^Q$ and it is easy to compute the limit

$$\gamma^Q_P(0, X) = \lim_{t \to 0} \gamma^Q_P(t\lambda, X)$$

when $\lambda$ is not a singular value for $\hat{\theta}(\lambda)^{-1}$ or $\hat{\theta}(\lambda)^{-1}$. □

13.2. The trace formula as a polynomial.

The left-hand side of the trace formula for the group $G$ and the function $\phi$ is a sum over $\sigma \in \theta$ of terms $\int^{G, T}_G (\phi)$ which are the integral over $G \setminus G^1$ of $\int^{G, T}_G (\phi, x)$ which in turn are the sums over $\epsilon$-invariant parabolic subgroups $P \subset G$ (standard) of

$$(-1)^{a^P - a^Q} \sum_{\delta \in P \setminus G} \hat{\gamma}_P(\delta x - T) K^{\epsilon, \phi}_P, \sigma(\delta x, \delta x)$$
where

\[ K_{P, \sigma}^\epsilon, \phi(x, y) = \sum_{\gamma \in M_P \cap \sigma \cap N_P} \int \phi(x^{-1} \gamma n e(y)) \, dn. \]

It was proved in Lecture 4 that the integral over \( G \setminus G' \) is convergent provided \( T \) is suitably regular uniformly if \( \phi \) varies in some compact set of functions.

We want to compute \( J^{G, T+X} \) in terms of \( J^{Q, T} \) where \( Q \) runs over \( \epsilon \)-invariant parabolic subgroups. Using 13.1.3 we see that

\[
J^{T+X}_{\sigma}(\phi, x) = \sum_{P \subset Q} (-1)^{a_P - a_Q} \sum_{\epsilon(P) = P, \epsilon(Q) = Q} \int_{Q/H(\xi x), X} \int_{P/H(\delta \xi x)-T} K_{P, \sigma}^\epsilon, \phi(\delta \xi x, \delta \xi x). 
\]

But if \( x = nmk \) with \( n \in N_Q, m \in M_Q \) and \( k \in K \) we have (if \( P \subset Q \))

\[ K_{P, \sigma}^\epsilon, \phi(x, x) = K_{P, \sigma \cap Q}^\epsilon, \phi^k(m, m) \]

where

\[ \phi^k_Q(m) = \delta_Q(m)^{1/2} \int \phi(m^{-1} n m e(k)) \, dn. \]

Using the fact that the left-hand side of the trace formula is convergent
for \((Q, \phi_Q^k)\) uniformly for \(k \in K\) provided \(T\) is suitably regular we get when \(T\) and \(X\) are suitably regular

\[
\varepsilon^{G, T+X}_\sigma(\phi) = \sum_{\varepsilon(Q) = Q} \varepsilon^{G}_Q(\phi_Q) \varepsilon^{J, T}_\sigma \cap Q(\phi_Q)
\]

where

\[
\phi_Q = \int_{\phi_Q^k} \frac{dk}{K}
\]

The right-hand side is a polynomial in \(X\) and this allows one to define \(\varepsilon^{G, T}_\sigma(\phi)\) for all \(T\) as a polynomial in \(T\) of degree \(a_R^\varepsilon - a_G^\varepsilon\) where \(R\) is any \(\varepsilon\)-invariant parabolic subgroup whose rank is minimal for the property \(K_{R, \sigma}^{\varepsilon, \phi} \neq 0\).

A cuspidal datum \(\chi\) is a conjugacy class of pairs \((\pi, M_P)\) where \(\pi\) is a cuspidal automorphic representation for \(M_P\) the Levi subgroup of a standard parabolic subgroup. If one considers the partial spectral decomposition indexed by cuspidal data one is led to introduce partial kernels \(K_{P, \chi}(x, y)\) and one can show, using a refinement of the results in Lectures 7 and 8, that provided \(T\) is sufficiently regular

\[
\varepsilon^{G, T}_\chi(\phi, x) = \sum_{\varepsilon(P) = P} (-1)^{a_P^\varepsilon - a_G^\varepsilon} \sum_{\delta \in P \setminus G} \varepsilon^{x_P(H(\delta x) - T)K_{P, \chi}^\varepsilon(x, \delta x)}
\]

is integrable over \(G \setminus G^1_{\varepsilon}\); we shall denote by \(\varepsilon^{G, T}_\chi\) its integral. As above we get
\[ \varepsilon^J_{\chi, T+X}(\phi) = \sum_{\varepsilon(Q)=Q} \varepsilon^G_{Q(x)} \varepsilon^J_{\chi}(\phi) \]

provided \( T \) and \( X \) are suitably regular. The right-hand side is a polynomial in \( X \) of degree \( a^\varepsilon_R - a^\varepsilon_G \) where \( R \) is any \( \varepsilon \)-invariant parabolic subgroup whose rank is minimal for the property \( K_{R, \chi} \neq 0 \).

13.3. Changing the minimal parabolic.

Let \( \Omega^G, \varepsilon \) be the subgroup of \( \varepsilon \)-invariant elements in the Weyl group; let \( w \in G \) be an element which represents \( s \in \Omega^G, \varepsilon \). Simple changes of variable yield

\[ \varepsilon^J_T(\phi) = \int_{G \setminus G^1_{\varepsilon}} \sum_{\varepsilon(P)=P} (-1)^{a^\varepsilon_P} \sum_{\delta \in w^{-1}(P) \setminus G} \varepsilon^\tau_P(H(w\delta x)-T)K_{w^{-1}(P)}(\delta x, \delta x) \]

where \( w^{-1}(P) = w^{-1}Pw \) and where \( K_{w^{-1}(P)} \) is defined in an obvious way.

It is natural to define \( \varepsilon^\hat{\tau}_{w^{-1}(P)} \) such that

\[ \varepsilon^\hat{\tau}_{w^{-1}(P)}(H) = \varepsilon^\tau_{w^{-1}(P)}(w^{-1}(H)) \]

If \( y = nma_k \) is a Langlands-Iwasawa decomposition corresponding to \( Q = w^{-1}(P_0) \) we define \( H_Q \) such that \( H_Q(y) = H(a) \) and hence

\[ w^{-1}H(wy) = H_Q(y) + w^{-1}H(w) \]

and

\[ \varepsilon^\hat{\tau}_P(H(wy)-T) = \varepsilon^\hat{\tau}_{w^{-1}(P)}(H_Q(y)-T_Q) \]
where $T_Q = w^{-1}(T-H(w))$. With these notations we get

$$e^{J^T(\phi)} = \int \sum_{G \backslash G^i} \sum_{\varepsilon(R) = R} \sum_{\delta \in R \backslash G} R \supset Q \varepsilon^R \left[H_Q(\delta x)^{-T}Q \right]K^R_x(\delta x, \delta x)$$

which can be written

$$e^{J^T(\phi)} = e^{J^T_Q(\phi)}$$

where $e^{J^T_Q}$ is the trace formula computed using the minimal $e$-invariant parabolic subgroup $Q$ in place of $P_0$.

### 13.4. Action of conjugacy.

We now want to compare $J^T(\phi)$ with $J^T(\phi^y)$ where

$$\phi^y(x) = \phi(yx \varepsilon(y)^{-1})$$

We have

$$J^T(\phi^y) = \int \sum_{G \backslash G^i} \sum_{\varepsilon(P) = P} \sum_{\delta \in P \backslash G} P \supset P_0 \varepsilon^P \left[H(\delta xy)^{-T} \right]K^P_x(\delta x, \delta x)$$

but

$$H(\delta xy) = H(\delta x) + H(k(\delta x)y)$$
where \( k(\delta x) \) is the \( K \)-component of an Iwasawa decomposition of \( (\delta x) \).

Using 13.1.3 we are led to introduce

\[
\varepsilon^Q_T(x, y) = \int_{\varepsilon Q \setminus \alpha^Q_T} \varepsilon^Q_T(H, -H(k(x)y))dH
\]

and

\[
\phi_Q, y(m) = \delta_Q(m) \frac{1}{4} \int_{\mathbb{K}} \int_{\mathbb{N}_Q} \phi(k^{-1}mn\varepsilon(k)) \varepsilon^G_{Q, y}dkdn
\]

with these notations we obtain as in 13.2

\[
\varepsilon^G_T(\phi^y) = \sum_{\varepsilon(Q) = Q} \varepsilon^Q_T(\phi_Q, y).
\]

13.5. On some regularity property.

In 13.1 we introduced

\[
\varepsilon^Q_T(\lambda, X) = \int_{(\mathbb{K}^Q)^\varepsilon} \varepsilon^Q_T(H, X)e^{\lambda(H)}dH.
\]

We shall now study this function when \( \lambda \) is imaginary. Consider \( D \) a differential operator with constant coefficients on \( i(\alpha^Q_T)^{\varepsilon\varepsilon} \) then if \( \lambda \in i(\alpha^Q_T)^{\varepsilon\varepsilon} \) we have

\[
|D^\varepsilon^Q_T(\lambda, X)| \leq \int_{(\mathbb{K}^Q)^\varepsilon} |P_D(H)\varepsilon^Q_T(H, X)|dH
\]

where \( P_D \) is the polynomial associated to \( D \). Using that

\[
\Gamma(tH, tX) = \Gamma(H, X)
\]
for \( t \in \mathbb{R}_+^x \) and Lemma 13.1.1(i) it is not difficult to see that

**Lemma 13.5.1.**

\[
|D_{\epsilon} \gamma_P^Q(\lambda, X)| < c(1 + \|X\|)^N
\]

for some \( N \) independent of \( \lambda \) when \( \lambda \) is imaginary. \( \square \)

In other words, \( X \rightarrow \gamma(\lambda, X) \) is a "slowly increasing" function.

Now consider \( \varphi \) a Schwartz-Bruhat function on \( i(\alpha_P^Q)^* \epsilon \), let \( \hat{\varphi} \)
be its Fourier transform so that

\[
\varphi(\lambda) = \int_{i(\alpha_P^Q)^* \epsilon} \hat{\varphi}(H)e^{\lambda(H)}dH.
\]

We define

\[
\epsilon \gamma_P^Q(\lambda, \varphi) = \int_{i(\alpha_P^Q)^* \epsilon} \hat{\varphi}(X) \epsilon \gamma_P^Q(\lambda, X)dx.
\]

This makes sense also when \( \hat{\varphi} \) is a "rapidly decreasing" distribution.

Lemma 13.5.1 above shows that on \( i(\alpha_P^Q)^* \epsilon \) the function

\[
\lambda \rightarrow \epsilon \gamma_P^Q(\lambda, \varphi)
\]

is smooth and by 13.1.4 we obtain the following expression

\[
\epsilon \gamma_P^Q(\lambda, \varphi) = \sum_{P \in R \subseteq Q \subseteq \mathcal{L}} (-1)^{a_R^{\epsilon} - a_R^{\epsilon}} \varphi_{\epsilon \gamma_P^Q}(\epsilon \lambda_R^Q)
\]

\[
\epsilon \delta_P^R(\lambda)^{-1} \epsilon \gamma_P^Q(\lambda)^{-1}
\]
which is valid at least when $\lambda$ is imaginary and not a singular value of $\hat{\theta}(\lambda)^{-1}$ or $\theta(\lambda)^{-1}$ and where $\lambda^Q_R$ is the projection of $\lambda$ on $(\sigma^Q_R)^e \otimes \mathbb{C}$.

The left-hand side is smooth and hence the singularities of the right-hand side cancel when $\varphi$ is any Schwartz-Bruhat function. This implies that more generally we have the

**Lemma 13.5.2.** Given any smooth function $\varphi$

$$
\sum_{P \in \mathcal{R} \subset \mathcal{Q}} (-1)^{\epsilon^P - \epsilon^Q} \varphi(\lambda^Q_R) \theta^R(\lambda)^{-1} \theta^Q(\lambda)^{-1}
$$

extends to a smooth function of $\lambda \in i(\sigma^Q_P)^e$. □