Lecture 5

$\mathcal{O}$-EXPANSION AND WEIGHTED ORBITAL INTEGRALS

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5.1. The second form of the $\mathcal{O}$-expansion.

Let P be an ε-invariant parabolic and $\mathfrak{o}$ an ε-semisimple-conjugacy class. For $\gamma \in \mathfrak{o} \cap M$ let $\gamma' = \gamma\varepsilon$ and $N(\gamma'_s)$ the centralizer in N of the semisimple part γ'_s of γ'. We introduce

$$N(\phi, x, \gamma') = \int_{\mathbf{N}(\gamma'_s)} \omega(x)\phi(x^{-1}n^{-1}\gamma\varepsilon(x))dn$$

and

$$j_{P,\mathfrak{o}}(x) = \sum_{\gamma \in M \cap \mathfrak{o}} \sum_{\eta \in N(\gamma'_s)\backslash N} N(\phi, \eta x, \gamma') .$$

Using Lemma 3.1.1 we see that the series are in fact finite sums since ϕ is compactly supported. We now define a truncated term by

$$j^T_{\mathfrak{o}}(x) = \sum_{\varepsilon(P)=P} \sum_{\delta \in P\backslash G} (-1)^{a^\varepsilon_P} \hat{\tau}_P(H(\delta x) - T) j_{P,\mathfrak{o}}(\delta x) .$$

Here also the series are finite sums; this is a consequence of Lemma 2.1. The aim of this section is to prove the

THEOREM 5.1.1. (i) For a sufficiently regular T

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{Ⓖ^1} |j^T_{\mathfrak{o}}(x)|dx$$

is finite.

(ii) For any $\sigma \in \mathcal{O}$

$$\int_{Ⓖ^1} j_{\sigma}^{T}(x)\,dx = \int_{Ⓖ^1} k_{\sigma}^{T}(x)\,dx \quad .$$

The proof of statement (i) is, with minor modifications, the same as the proof of Theorem 3.1.2 and will not be repeated (see Lectures 3 and 4).

To prove the statement (ii) we need the

LEMMA 5.1.2.

$$K_{P,\sigma}(x, x) = \int_{Ⓝ} j_{P,\sigma}(nx)\,dn \quad .$$

Recall that

$$K_{P,\sigma}(x, x) = \sum_{\gamma \in M \cap \sigma} \omega(x) \int_{\mathbb{N}} \phi(x^{-1} n \gamma \varepsilon(x))\,dx \quad .$$

The continuous analogue of Lemma 3.1.1 shows that

$$\int_{\mathbb{N}} \omega(x)\phi(x^{-1}n^{-1}\gamma\varepsilon(x))\,dn = \int_{\mathbb{N}(\gamma_s')\backslash \mathbb{N}} N(\phi, nx, \gamma')\,dn \quad .$$

The lemma is now an immediate consequence of the definition of $j_{P,\sigma}$. □

COROLLARY 5.1.3. <u>Given</u> $P_1 \subset P$ <u>we have</u>

$$\int_{Ⓝ_1} j_{P,\sigma}(nx)\,dn = \int_{Ⓝ_1} K_{P,\sigma}(nx, nx)\,dn \quad .$$

We need only to remark that $P \supset N_1 \supset N$. □

In Lecture 3 we introduced a function $H_1^2(x)_{\sigma}^T$ such that

$$\int_{Ⓖ^1} k_{\sigma}^T(x)\,dx = \sum_{P_1 \subset P_2} \int_{P_1\backslash \mathbf{G}^1} H_1^2(x)_{\sigma}^T dx \quad .$$

If we substitute $j_{P,\sigma}(x)$ for $K_{P,\sigma}(x, x)$ in the definition of $H_1^2(x)_{\sigma}^T$ we obtain a function $J_1^2(x)_{\sigma}^T$. Then Corollary 5.1.3 tells us that

$$\int_{Ⓝ_1} H_1^2(nx)_{\sigma}^T dn = \int_{Ⓝ_1} J_1^2(nx)_{\sigma}^T dn$$

and the assertion (ii) in the above theorem follows from the fact that integration over $P_1\backslash \mathbf{G}^1$ can be seen as an integration over $Ⓝ_1$ followed by an integration over $P_1\mathbf{N}_1\backslash \mathbf{G}^1$. □

Another variant of the θ-expansion will be of interest. Let P be an ε-invariant parabolic subgroup, the group E of connected components of G' acts on Δ_P and to each orbit $\bar{\alpha}$ we may attach an averaged weight $\varpi_{\bar{\alpha}}$:

$$\varpi_{\bar{\alpha}} = \frac{1}{\ell} \sum_{r=0}^{\ell-1} \varepsilon^r \varpi_{\alpha}$$

where α is any element in $\bar{\alpha}$. We define ${}_{\varepsilon}\hat{\tau}_P$ as the characteristic function of the $X \in \mathfrak{a}_0$ such that $\varpi_{\bar{\alpha}}(X) > 0$ for any $\alpha \in \Delta_P$. If we substitute ${}_{\varepsilon}\hat{\tau}_P$ for $\hat{\tau}_P$ in the definition of k_{σ}^T and j_{σ}^T we obtain new functions which we shall denote by ${}_{\varepsilon}k_{\sigma}^T$ and ${}_{\varepsilon}j_{\sigma}^T$: their definition makes sense since the analogue of Lemma 2.1 is available. We may reproduce the proofs in Lectures 2, 3, 4 with minor changes; we simply have to replace from time to time weights by averaged weights and σ_1^2 by ${}_\varepsilon\sigma_1^2$ the characteristic functions of the H such that $\alpha(H) > 0$ if $\alpha \in \Delta_1^2$, $\alpha(H) \leq 0$ if $\alpha \in \Delta_1 - \Delta_1^2$ and $\varpi_{\bar{\alpha}}(H) > 0$ if $\alpha \in \Delta_Q$ where Q is the maximal ε-invariant parabolic subgroup contained in P_2 if $Q \supset P_1$, and ${}_\varepsilon\sigma_1^2 = 0$ if there is no ε-invariant P between P_1 and P_2. More details will be given in Lecture 9.

5.2. Conjugacy classes and parabolic subgroups.

Let P be a (not necessarily standard) parabolic subgroup and let $P' = N_{G'}(P)$ be its normalizer in G'. We shall say that P' is a parabolic subgroup in G' if its projection on E, the group of connected components of G', is surjective.

LEMMA 5.2.1. Assume P' is a parabolic subgroup in G' whose neutral connected component P is standard, then $\varepsilon \in P'$.

By assumption there is an element $\varepsilon_1 \in P'$ which projects on ε_0 the given generator of E. We have $P_0 \subset P'$, let $P_1 = \varepsilon_1(P_0)$; this is a minimal parabolic subgroup and hence there exist $\delta_1 \in P$ such that $\delta_1 P_1 \delta_1^{-1} = P_0$. Then $\delta_1\varepsilon_1$ leaves P_0 invariant; so does ε and hence $\delta = \delta_1\varepsilon_1\varepsilon^{-1}$ normalizes P_0 and is an element of G so that $\delta \in P_0$ and $\varepsilon = \delta^{-1}\delta_1\varepsilon_1 \in P'$. □

Such parabolic subgroups in G' will be called standard; P' is standard if and only if P is standard and ε-invariant; moreover $P' \supset P'_0$. Let M be the Levi component of P containing M_0, then M

and ε generate a subgroup M' in P' which will be called "the" Levi component of P'. Let A be the split component of the center of M, then A^{ε} is the split component of the center of M'. The weights of A^{ε} in G are the orbits under E of the weights of A; since E preserves positivity of weights, the centralizers of A and A^{ε} in G (which are connected) are equal to M. The centralizer of A^{ε} in G' is M'.

Consider $\gamma_1 \in G$ such that $\gamma'_1 = \gamma_1 \varepsilon$ is semisimple and P'_1 a standard parabolic subgroup of G' such that $\gamma'_1 \in M'_1$ its Levi component and such that moreover no strictly smaller standard parabolic subgroup contains an M'_1-conjugate of γ'_1 in its Levi component.

Let $A_1 \varepsilon$ be the split component of the center of M'_1.

LEMMA 5.2.2. <u>The torus</u> A_1^{ε} <u>is a maximal split torus in</u> $G'(\gamma'_1)$ <u>the centralizer of</u> γ'_1 <u>in</u> G'.

Let B be a maximal split torus in $G'(\gamma'_1)$. Since γ'_1 is semisimple $G'(\gamma'_1)$ is reductive and up to conjugacy in $G'(\gamma'_1)$ we may assume $A_1^{\varepsilon} \subset B$. Let M_2 (resp. M'_2) be the centralizer of B in G (resp. G'), we have $M'_2 \subset M'_1$. Up to conjugacy inside M_1 we may assume that M_2 is the Levi component of a standard parabolic subgroup $P_2 \subset P_1$ of G. Since γ'_1 commutes with B we have $\gamma'_1 \in M'_2$ and γ'_1 normalizes N_2 the unipotent radical of P_2 (γ'_1 fixes the weights of B) and hence $\gamma'_1 \in P'_2$. This implies that P'_2 projects surjectively on E. The minimality property of P'_1 implies $P'_1 = P'_2$; moreover $\varepsilon \in M'_1 = M'_2$ so that $B = B^{\varepsilon} = A_1^{\varepsilon}$. $\square$

COROLLARY 5.2.3. *Up to conjugacy* M'_1 *is well defined by* $c(\gamma'_1)$ *the* G-*conjugacy class of* γ'_1.

Given $\gamma'_1 \in M'_1 \subset P'_1$ and $\gamma'_2 \in M'_2 \subset P'_2$ minimal as above we know that A^{ε}_1 and A^{ε}_2 are maximal split tori in $G'(\gamma'_1)$ and $G'(\gamma'_2)$. If γ'_1 and γ'_2 are conjugate then A^{ε}_1 and A^{ε}_2 are also conjugate and the same is true for the M'_i. □

COROLLARY 5.2.4. *Given* P' *a standard parabolic subgroup of* G' *with Levi component* M' *and* $\gamma' \in M' \cap C(\gamma'_1)$ *there exists a standard parabolic* P'_2 *of* G' *associated with* P'_1 *such that* $P'_2 \subset P'$ and $m\gamma' m^{-1} \in M'_2$ *for some* $m \in M'$. □

Given P'_1 and P'_2 as above, let $\mathfrak{a}_i$ be the Lie algebra of $A_i(\mathbb{R})^o$. Let us denote as usual by $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ the set of restrictions to $\mathfrak{a}_1$ of elements $s \in \Omega$, the Weyl group of G, such that $s(\mathfrak{a}_1) = \mathfrak{a}_2$. Given $\sigma \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ there exist a unique element $s \in \Omega$ such that s induces σ and such that moreover $s^{-1}\alpha > 0$ for all $\alpha \in \Delta^2_0$; it is the element with minimal length in the class σ. This provides us with an injective map from $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ into Ω. We shall identify $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ with its image.

Let us denote by $\Omega(\mathfrak{a}^{\varepsilon}_1, \mathfrak{a}^{\varepsilon}_2)$ the set of restrictions to $\mathfrak{a}^{\varepsilon}_1$ of elements $s \in \Omega$ such that $s(\mathfrak{a}^{\varepsilon}_1) = \mathfrak{a}^{\varepsilon}_2$. Since M_i is the centralizer of A^{ε}_i (in G) such an s defines an element in $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ and hence $\Omega(\mathfrak{a}^{\varepsilon}_1, \mathfrak{a}^{\varepsilon}_2)$ may be regarded as a subset of $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ and be identified with a subset of Ω.

LEMMA 5.2.5. $\Omega(\mathfrak{a}_1^{\varepsilon}, \mathfrak{a}_2^{\varepsilon})$ <u>is the set of</u> $s \in \Omega$ <u>such that</u>

(i) $s(\mathfrak{a}_1) = \mathfrak{a}_2$

(ii) $s^{-1}(\alpha) > 0 \qquad \forall \alpha \in \Delta_0^2$

(iii) $\varepsilon s = s\varepsilon$.

The first two conditions define $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$; if an element satisfies the three conditions it clearly defines an element in $\Omega(\mathfrak{a}_1^{\varepsilon}, \mathfrak{a}_2^{\varepsilon})$. Conversely if $s(\mathfrak{a}_1^{\varepsilon}) = \mathfrak{a}_2^{\varepsilon}$ and $s^{-1}(\alpha) > 0$ for all $\alpha \in \Delta_0^2$ the same is true for $s_1 = \varepsilon s \varepsilon^{-1}$ since P_2 is ε-invariant. Moreover s_1 and s have equal restrictions to $\mathfrak{a}_1^{\varepsilon}$ and hence equal restrictions to $\mathfrak{a}_1$. This implies $s = s_1$. $\square$

Given P' a standard parabolic subgroup of G', let us denote by $\tilde{\Omega}(\mathfrak{a}_1^{\varepsilon}, P')$ the set of elements $s \in \Omega$ such that there exists a parabolic subgroup $P_2' \subset P'$ standard in G' with $s(\mathfrak{a}_1^{\varepsilon}) = \mathfrak{a}_2^{\varepsilon}$. The Weyl group of M, denoted by Ω^M, acts on the left on $\tilde{\Omega}(\mathfrak{a}_1^{\varepsilon}, P')$ and each class $\sigma \in \Omega^M \backslash \tilde{\Omega}(\mathfrak{a}_1^{\varepsilon}, P')$ contains a unique element s such that $s^{-1}\alpha > 0$ for any $\alpha \in \Delta_0^P$. As usual s is the element of minimal length in σ. Thanks to Lemma 5.2.5 we see that such an s commutes with ε. We shall identify $\Omega^M \backslash \tilde{\Omega}(\mathfrak{a}_1^{\varepsilon}, P')$ with the set $\Omega(\mathfrak{a}_1^{\varepsilon}, P')$ of those s in Ω.

We can now describe rather explicitly the set $M' \cap c(\gamma_1')$. Given $\gamma' \in M' \cap c(\gamma_1')$ there exists $s \in \Omega(\mathfrak{a}_1^{\varepsilon}, P')$ and $m \in M$ such that

$$\gamma' = m^{-1} w_s \gamma_1' w_s^{-1} m$$

where $w_s \in G$ represents s. But s is not always uniquely defined by γ'; it defines only a double coset in Ω:

$$\Omega^M . s . \Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$$

where $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ is the subgroup of the $\sigma \in \Omega(\mathfrak{a}_1^\varepsilon, \mathfrak{a}_1^\varepsilon)$ such that

$$w_\sigma \gamma_1' w_\sigma^{-1} = m_1 \gamma_1' m_1^{-1}$$

for some $m_1 \in M_1$. The element $m \in M$ is defined by γ' and $w_s \gamma_1' w_s^{-1}$ up to an element in $M(w_s \gamma_1' w_s^{-1})$ its centralizer in M.

5.3. Tame semisimple conjugacy classes.

The aim of this section is to give a simple expression for $j_{\mathcal{O}}^T(x)$ when $\mathcal{O}$ contains only semisimple elements. Such classes will be called tame semisimple. Given such a class $\mathcal{O}$ and $\gamma \in \mathcal{O}$, then $\gamma' = \gamma\varepsilon$ is semisimple and for any parabolic subgroup P' of G' containing γ' we have $N(\gamma') = N(\gamma_1') = \{1\}$.

An element γ' defines a tame semisimple class if and only if its centralizer $G(\gamma')$, in G, contains no unipotent element. In particular, regular semisimple elements give rise to tame semisimple classes.

Let γ_1', P_1', M_1' be as in Lemma 5.2.2 with γ' conjugate to γ_1' (in G) and assume that $G(\gamma_1')$ contains no unipotent elements. Recall that A_1^ε is a maximal split torus in $G(\gamma_1')$, since $G(\gamma_1')$ contains no unipotent element the neutral component $G(\gamma_1')^o$ lies in the centralizer of A_1^ε that is M_1'. Hence $M_1(\gamma_1')$ is of finite index say $d(\gamma_1')$ in $G(\gamma_1')$.

More generally given P' standard in G' with Levi component M' such that $\gamma' \in M'$ let us denote by $d(M, \gamma')$ the index of $M(\gamma')$ in $G(\gamma')$.

Let $s \in \tilde{\Omega}(\mathfrak{a}_1^\varepsilon, P')$ be such that $\gamma' = m^{-1} w_s \gamma_1' w_s^{-1} m$ where w_s represents s and $m \in M$.

LEMMA 5.3.1. <u>The cardinality of the set</u>

$$\Omega^M \backslash \Omega^M . s . \Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$$

<u>is</u> $d(M, \gamma)$.

Consider first the case where $\gamma' = \gamma_1'$, $s = 1$ and $P' = P_1'$, then all we have to prove is that the order of $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ is $d(\gamma_1')$ and this follows from the

LEMMA 5.3.2. <u>There is a natural map from</u> $G(\gamma_1')$ <u>onto</u> $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ <u>with kernel</u> $M_1(\gamma_1')$.

An element $g \in G(\gamma_1')$ normalizes A_1^ε the center of $G(\gamma_1')^o$ and hence it normalizes M_1. Then g defines an element s_g of $\Omega(\mathfrak{a}_1^\varepsilon, \mathfrak{a}_1^\varepsilon)$ and since g commutes with γ_1' it lies in $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$. By the very definition of $\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ this map is surjective and its kernel is $M_1 \cap G(\gamma_1') = M_1(\gamma_1')$. □

We can now return to the general case. We need only to prove it when $\gamma' = \gamma_1'$, $w_s = 1$, $P_1' \subset P'$, in which case it amounts to saying that the index of $M_1(\gamma_1')$ in $M(\gamma_1')$ is the cardinality of $\Omega^M \cap \Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$ which is clear. □

Given $\mathfrak{o}$ a tame semisimple class we have

$$j_{P,\mathfrak{o}}(x) = \sum_{\gamma \in M \cap \mathfrak{o}} \sum_{\eta \in N} \omega(x)\phi(x^{-1}\eta^{-1}\gamma\varepsilon(\eta x))$$

since $N(\gamma_s') = N(\gamma') = \{1\}$. Now since in such a case $\mathfrak{o}$ is the twisted conjugacy class of some γ_1 with $\gamma_1' = \gamma_1\varepsilon$ semisimple in M_1', minimal as above, we may use the description of $c(\gamma_1') \cap M$ obtained at the end of 5.2 to see that $j_{P,\mathfrak{o}}(x)$ is the sum over

$$s \in \Omega(\mathfrak{a}_1^\varepsilon, P', \gamma_1') = \Omega^M \backslash \tilde{\Omega}(\mathfrak{a}_1^\varepsilon, P')/\Omega(\mathfrak{a}_1^\varepsilon, \gamma_1')$$

and over $\xi \in M(w_s\gamma_1'w_s^{-1}) \backslash P$, where w_s represents s of

$$\omega(x)\phi(x^{-1}\xi^{-1}w_s\gamma_1\varepsilon(w_s^{-1}\xi x)) \quad .$$

We may replace the sum over $\Omega(\mathfrak{a}_1^\varepsilon, P', \gamma_1')$ by a sum over $\Omega(\mathfrak{a}_1^\varepsilon, P')$ but we must divide each term by the integer $d(M, w_s\gamma_1'w_s^{-1})$ as follows from the Lemma 5.3.1. We may also replace the sum over $M(w_s\gamma_1'w_s^{-1}) \backslash P$ by a sum over $w_sM_1(\gamma_1')w_s^{-1} \backslash P$ but we must divide each term by the index of $w_sM_1(\gamma_1')w_s^{-1}$ in $M(w_s\gamma_1'w_s^{-1})$ which equals

$$d(\gamma_1')/d(M, w_s\gamma_1'w_s^{-1}) \quad .$$

We finally obtain $j_{P,\sigma}(x)$ as the sum over $s \in \Omega(\mathfrak{a}_1^\varepsilon, P')$ of the sum over $\xi \in w_s M_1(\gamma_1') w_s^{-1} \backslash P$ of

$$d(\gamma_1')^{-1}\omega(x)\phi(x^{-1}\xi^{-1}w_s\gamma_1\varepsilon(w_s^{-1}\xi x)) \quad .$$

This yields immediately the following expression for $j_\sigma^T(x)$:

$$j_\sigma^T(x) = \sum_{\delta \in M_1(\gamma_1') \backslash G} d(\gamma_1')^{-1}\omega(x)\phi(x^{-1}\delta^{-1}\gamma_1\varepsilon(\delta x))e_1(\delta x, T)$$

where

$$e_1(x, T) = \sum_{\varepsilon(P)=P} \sum_{s \in \Omega(\mathfrak{a}_1^\varepsilon, P')} (-1)^{a_P^\varepsilon} \hat{\tau}_P(H(w_s x) - T)$$

depends only on the parabolic subgroup P_1'. We may get rid of the factor $d(\gamma_1')^{-1}$ if we replace $M_1(\gamma_1')$ by $G(\gamma_1')$; we obtain the

LEMMA 5.3.3. Given σ a tame semisimple class we have

$$j_\sigma^T(x) = \sum_{\delta \in G(\gamma_1') \backslash G} \omega(x)\phi(x^{-1}\delta^{-1}\gamma_1\varepsilon(\delta x))e_1(\delta x, T) \quad .$$

Replacing $\hat{\tau}_P$ by ${}_\varepsilon\hat{\tau}_P$ we define e_1^ε the analogue of e_1 and we have the

LEMMA 5.3.4. Given σ a tame semisimple class we have

$${}_\varepsilon j_\sigma^T(x) = \sum_{\delta \in G(\gamma_1') \backslash G} \omega(x)\phi(x^{-1}\delta^{-1}\gamma_1\varepsilon(\delta x))e_1^\varepsilon(\delta x, T) \quad .$$

The reason for introducing the e_1^ε is so that the weighted orbital integrals have a usable form, that is, can be treated along the lines suggested by Y. Flicker in "Base change for GL(3)" and used again in his preprints on GL(3) and SU(3).

Given $s \in \Omega(\mathfrak{a}_1^\varepsilon, \mathfrak{a}_2^\varepsilon)$ we define Δ_0^s to be the set of $\alpha \in \Delta_0$ such that $s^{-1}\alpha > 0$. The Lemma 5.2.5 tells us that this is the set of simple roots attached to a standard parabolic subgroup P'_s of G' containing P'_2. Given s as above we introduce a function on $\mathfrak{a}_0$:

$$
{}_\varepsilon B_1^s(x) = \sum_{P'_2 \subset P' \subset P'_s} (-1)^{a_P^\varepsilon} \, {}_\varepsilon\hat{\tau}_P(sX) \quad .
$$

This is the product of $(-1)^{a_{P_s}^\varepsilon}$ and of the characteristic functions of the $X \in \mathfrak{a}_0$ such that $\varpi_{\bar\alpha}(sX) > 0$ for any $\alpha \in \Delta_0 - \Delta_0^s$ and $\varpi_{\bar\alpha}(sX) \leq 0$ for any $\alpha \in \Delta_0^s - \Delta_0^2$.

Given $s \in \Omega$ we introduce

$$
H_s(x, T) = s^{-1}(T - H(w_s x)) \quad .
$$

With these notations we have

$$
e_1^\varepsilon(x, T) = \sum_{s \in \Omega(\mathfrak{a}_1^\varepsilon)} {}_\varepsilon B_1^s(-H_s(x, T))
$$

where $\Omega(\mathfrak{a}_1^\varepsilon)$ is the (disjoint) union over the P'_2 of $\Omega(\mathfrak{a}_1^\varepsilon, \mathfrak{a}_2^\varepsilon)$. Let $c_1^\varepsilon(x, T)$ be the set of $H \in \mathfrak{a}_0$ whose projection on $\mathfrak{z}^\varepsilon \backslash \mathfrak{a}_1^\varepsilon$ lies in the convex hull of the projections on $\mathfrak{z}^\varepsilon \backslash \mathfrak{a}_1^\varepsilon$ of the set of

$H_s(x, T)$ with $s \in \Omega(\mathfrak{a}_1^\varepsilon)$.

LEMMA 5.3.5. Assume T is sufficiently regular then

$$H \longrightarrow \sum_{s \in \Omega(\mathfrak{a}_1^\varepsilon)} {}_\varepsilon B_1^s(H - H_s(x, T))$$

is the characteristic function of $c_1^\varepsilon(x, T)$.

This lemma is essentially Lemma 3.2 in Arthur's paper [Inventiones Math. 32, 1976]. More details will be given in Lecture 9 below. □

We shall denote by $v_1^\varepsilon(x, T)$ the volume of the projection on $\mathfrak{z}^\varepsilon \backslash \mathfrak{a}_1^\varepsilon$ of $c_1^\varepsilon(x, T)$. We obtain the

PROPOSITION 5.3.6. Given σ a tame semisimple class we have

$$\int_{\textcircled{G}^1} {}_\varepsilon j_\sigma^T(x)\,dx = \int_{G(\gamma_1')\mathbf{G}(\gamma_1')^\circ \backslash \mathbf{G}} v(\gamma_1')\omega(x)\phi(x^{-1}\gamma_1\varepsilon(x))v_1^\varepsilon(x, t)\,dx$$

where $v(\gamma_1')$ is the volume of $A_1^\varepsilon(\mathbf{R})^\circ G(\gamma_1')^\circ \backslash \mathbf{G}(\gamma_1')^\circ$.