Lecture 6

PROPERTIES OF THE TRUNCATION OPERATOR

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The most important property of $\Lambda^T$ is that it converts smooth slowly increasing functions into rapidly decreasing functions but we begin by studying its formal properties.

Recall that $\Lambda^T$ is defined for $T$ suitably regular in $\mathfrak{a}_0^+$ and that it is defined first of all for continuous or, better, bounded measurable $\varphi$ by

$$\Lambda^T \varphi(g) = \sum \frac{(-1)^{a_P}}{\delta \epsilon P \setminus G} \int_{N \setminus N} \varphi(n \delta g) \widehat{\chi}_P(H(\delta g) - T) \quad ,$$

where

$$a_P = \dim \mathfrak{a}_P / \mathfrak{a}_G \ .$$

By Lemma 2.1 the sums appearing on the right are finite.

PROPOSITION 6.1. The operator $\Lambda^T$ is an idempotent, so that

$$\Lambda^T (\Lambda^T \varphi) = \Lambda^T \varphi \quad .$$

This proposition is of course an immediate consequence of the following lemma.

LEMMA 6.2. If $\varphi$ is bounded measurable then

$$\int_{N_1 \setminus N_1} \Lambda^T \varphi(n_1 g) dn_1 = 0$$
unless \( \varpi(H(g) - T) \leq 0 \) for every \( \varpi \in \hat{\Delta}_1 \).

We first consider

\[
(1) \quad \int_{N_1 \setminus N_1} \sum_{\delta \in P \setminus G} \int_{N \setminus N} \varphi(n \hat{\delta} n_1 g) \hat{T}_P(H(\hat{\delta} n_1 g) - T) \ dn \ dn_1.
\]

Let \( \Omega(\alpha_0, P) \) be the set of \( s \) in \( \Omega(\alpha_0, \alpha_1) \) such that \( s^{-1} \alpha > 0 \) for all \( \alpha \in \hat{\Delta}_0^P \). The Bruhat decomposition assures us that \( P \setminus G \) is a disjoint union

\[
\bigcup_{s \in \Omega(\alpha_0, P)} P w_s N_0,
\]

\( w_s \) being a representative of \( s \).

Thus the expression (1) is equal to the sum over \( \Omega(\alpha_0, P) \) of

\[
(2) \quad \int_{N_1 \setminus N_1} \sum_{\nu \in w_s N_0 w_s \cap N_0 \setminus N_0} \int_{N \setminus N} \varphi(nw_s \nu n_1 g) \hat{T}_P(H(w_s \nu n_1 g) - T) \ dn \ dn_1.
\]

The outer integral and the sum can be fused to obtain an integral over

\[
w_s^{-1} N_0 w_s N_0 \setminus N_0 \setminus N_1,
\]

which we then decompose as an iterated integral, so that (2) becomes a triple integral

\[
\int_{w_s^{-1} N_0 w_s \cap N_0 \setminus N_0 \setminus N_1} \int_{w_s^{-1} N_0 w_s \cap N_0 \setminus N_0 \setminus N_1} \int_{w_s^{-1} N_0 w_s \cap N_0 \setminus N_0 \setminus N_1} \varphi(nw_s \nu n_1 g) \hat{T}_P(H(w_s \nu n_1 g) - T) \ dn \ dn_1 \ dn_1.
\]
The domain of integration in the outer integral depends on the choice of \( N_1 \) and on \( s \) but not on \( P \). Since it is the alternation over \( P \) that will force the vanishing we ignore the final integration and concentrate on the inner double integral. A little reflection convinces one that

\[
w_s^{-1}N_0w_s \cap N_0 \backslash w_s^{-1}N_0w_s \cap N_1 = w_s^{-1}N_0w_s \cap N_1 \backslash w_s^{-1}N_0w_s \cap N_1.
\]

Since \( s \in \Omega(\alpha_0, P) \) the intersection \( w_s^{-1}N_0w_s \cap M \) is \( N_0 \cap M \).

Thus \( w_s^{-1}N_0w_s \cap M \) is a parabolic subgroup of \( M \) with unipotent radical \( w_s^{-1}N_1w_s \cap M \). If we pass the variable in \( w_s^{-1}N_0w_s \cap N_1 \) through \( w_s \) we obtain a variable in \( N_0 \cap w_sN_1w_s^{-1} = (N \cap w_sN_1w_s^{-1})(M \cap w_sN_1w_s^{-1}) \).

Thus the second integration in the double integral can be taken over the product

\[
(N \cap w_sN_1w_s^{-1} \backslash N \cap w_sN_1w_s^{-1}) \times (M \cap w_sN_1w_s^{-1} \backslash M \cap w_sN_1w_s^{-1}).
\]

The volume of the first factor is 1 and since the first integration is taken over \( N \backslash N \) the integral does not depend on the first variable in the product.

Thus the double integral becomes finally

\[
\int_M \cap w_sN_1w_s^{-1} \backslash M \cap w_sN_1w_s^{-1} \int_{N \backslash N}.
\]

However

\[
(M \cap w_sN_1w_s^{-1}).N.
\]
is the unipotent radical of a parabolic subgroup $P_s$ of $G$. So the double integral becomes a single integral over $N_s \setminus N_s$, which we now write out explicitly.

\[ (3) \quad \int_{N_s \setminus N_s} \varphi(n w_{s} n_{1} g) \hat{\tau}(H(w_{s} n_{1} g) - T) \, dn, \]

the $n_{1}$ being the variable for the outer integration, which does not concern us at the moment.

The group $P_s$ is contained in $P$. The group $N_{1}$ is fixed but $s$ varies over $\Omega(\mathfrak{a}_{0}, P)$ and we are to sum over $P$ and $\Omega(\mathfrak{a}_{0}, P)$. What we do is fix $s$ and a $P^{0} \supseteq P_{0}$ and sum over all $P$ with $s \in \Omega(\mathfrak{a}_{0}, P)$ and $P_s = P^0$.

The set $\{a \in \Delta_{0} | s^{-1}a > 0\}$ is the disjoint union of two subsets, the first $S^1$ consisting of those $a$ in it for which $s^{-1}a$ is orthogonal to $\mathfrak{a}_{1}$ and the second $S_{1}$ of those for which it is not. It is clear that $\Delta_{0}^{P} \subset \Delta_{0}^{P}$ and that

\[ \Delta_{0}^{P} = \Delta_{0}^{P} \cap S^1, \]

for $a \in S_{1}$ if and only if $s^{-1}a$ is a root in $N_{1}$. Thus the freedom of $P$ is that the intersection of $\Delta_{0}^{P}$ with $S_{1}$ can be chosen at will.

The dependence of (3) on $P$ is through the function $\hat{\tau}(H(w_{s} n_{1} g) - T)$. The sum

\[ \sum (-1)^{a} \hat{\tau}(P(H(w_{s} n_{1} g) - T)) \]
over the allowed $P$ is therefore $0$ unless

$$\overline{w}(H(w_s n_1 g) - T) > 0$$

for $\alpha \notin \Delta_0^O \cup S_1$ and

$$\overline{w}(H(w_s n_1 g) - T) \leq 0$$

for $\alpha \in S_1$.

To complete the proof of the lemma we have to show that these inequalities imply that

$$\overline{w}(H(g) - T) \leq 0$$

for $\overline{w} \in \hat{\Delta}_1$. We have

$$s^{-1}(H(w_s n_1 g) - T) = H(g) - T + s^{-1}H(w_s v) + T - s^{-1}T$$

with $v \in N_0(A)$.

We write, identifying $\mathfrak{M}_0$ and its dual,

$$H(w_s n_1 g) - T = \sum_{\alpha \in \Delta_0} t_\alpha \alpha$$

with $t_\alpha > 0$ for $\alpha \notin \Delta_0^O \cup S_1$ and $t_\alpha \leq 0$ for $\alpha \in S_1$. Then

$$\overline{w}(s^{-1}(H(w_s n_1 g) - T)) = \sum t_\alpha \overline{w}(s^{-1}\alpha)$$

$$= \sum_{\alpha \notin S_1} t_\alpha \overline{w}(s^{-1}\alpha)$$
for \( s^{-1}a \) is orthogonal to \( m_1 \) if \( a \in S^1 \). If \( a \notin S^1 \cup S_1 \) then \( t_a > 0 \) and \( \varpi(s^{-1}a) \leq 0 \) and if \( a \in S_1 \) then \( t_a \leq 0 \) and \( \varpi(s^{-1}a) > 0 \). Thus this expression is less than or equal to zero.

To complete the proof of the lemma we need only show that for sufficiently regular \( T \)

\[
\varpi(s^{-1}H(w_s v)) + \varpi(T - s^{-1}T) \geq 0 .
\]

There is certainly no harm in replacing \( G \) by a Levi factor of the smallest standard parabolic containing \( s \), which to simplify the notation we suppose is \( G \) itself. Then given any constant \( C \) we can take \( T \) sufficiently regular and suppose that

\[
\varpi(T - s^{-1}T) > C .
\]

It therefore remains to show that there exists a constant \( C \) such that

\[
(4) \quad \varpi(s^{-1}H(w_s v)) \geq -C 
\]

for all \( v \in N_0(A) \). This is a statement which is easily seen to be independent of the choice of \( K \). Indeed it is enough to prove it over a field which splits \( G \). So we can suppose \( G \) is split and semi-simple.

Then one has the usual optimal choice of \( K \) and for this one proves by induction on the length of \( s \) the following lemma.

**Lemma 6.3.** If \( v \) lies in \( N_0 \) then
\[ s^{-1}H(w_s v) = \sum_{\alpha > 0} c_{\alpha} \]

with \( c_{\alpha} > 0 \).

This gives the relation (4) with \( C = 0 \). To prove the lemma one begins with \( SL(2) \), taking \( w_s \in K \). So, for the non-trivial \( s \),

\[ w_s v w_s^{-1} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}. \]

Moreover \( H(w_s v w_s^{-1}) \) is the sum of its local contributions and these are

(i) \( v \) real

\[ -\frac{1}{2} \ln(1 + |x_v|^2)(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

(ii) \( v \) complex

\[ -\ln(1 + |x_v|^2)(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

(iii) \( v \) non-archimedean

\[ -\ln \max\{|1|, |x_v|\}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Thus for \( SL(2) \) and hence in general the lemma is proved for an \( s \) of length one.

For a Chevalley group and an optimal choice of \( K \) we may take \( w_s \in K \). If \( s = s_1 s_2 \) with \( s_1 \) a reflection associated to the root \( \beta \)
and $1 + \text{length } s_2 = \text{length } s$ then

$$s_2^{-1}H(w_2 v) = s_2^{-1}s_1^{-1}H(w_1 w_2 v) = s_2^{-1}s_1^{-1}H(w_1 v') + s_2^{-1}s_1^{-1}(s_1H(w_1 v)) .$$

The induction assumption allows us to write this as

$$s_2^{-1}d_\beta + \sum_{\alpha > 0} c_\alpha s_2^{-\alpha}$$

with $d_\beta \geq 0$, $c_\alpha \geq 0$. Since

$$\{\alpha > 0 | s_\alpha < 0\} = \{\alpha > 0 | s_2 \alpha < 0\} \cup \{s_2^{-1} \beta\}$$

the lemma follows.

**PROPOSITION 6.4.** Suppose that $\varphi_1$ and $\varphi_2$ are continuous functions on $G \setminus \mathbf{G}$ and that on $G \setminus \mathbf{G}^1$ we have an inequality

$$|\varphi_1(g)| \leq c|g|^N$$

for some $N$ and that on any Siegel domain in $G^1$ we have an inequality

$$|\varphi_2(g)| \leq c_N|g|^{-N}$$

for all $N$. Then

$$\int_{G \setminus \mathbf{G}^1} \Lambda^T \varphi_1(g) \varphi_2(g) dg = \int_{G \setminus \mathbf{G}^1} \varphi_1(g) \Lambda^T \varphi_2(g) dg .$$

This clearly reduces to showing that
\[ \int_{G \setminus G_1} \left\{ \sum_{\delta \in P \setminus G} \int_{N \setminus N} \varphi_1(n \delta g) \mathcal{d}n \mathcal{h}_p(H(\delta g) - T) \right\} \varphi_2(g) \, dg \]

is equal to

\[ \int_{G \setminus G_1} \varphi_1(g) \left\{ \sum_{\delta \in P \setminus G} \int_{N \setminus N} \varphi_2(n \delta g) \mathcal{d}n \mathcal{h}_p(H(\delta g) - T) \right\} \, dg . \]

It follows readily from Lemma 7.8 of the next lecture that the second integral is absolutely convergent when \( \varphi_1 \) and \( \varphi_2 \) are replaced by their absolute values. Thus a formal proof of the equality assures us of both the equality and the convergence of the first integral.

The formal proof is of course easy, the second expression reducing to

\[ \int_{P \setminus G} \varphi_1(g) \mathcal{h}_p(H(g) - T) \left\{ \int_{N \setminus N} \varphi_2(n g) \, dn \right\} \, dg \]

which equals

\[ \int_{N \setminus G} \mathcal{h}_p(H(g) - T) \left\{ \int_{N \setminus N} \varphi_1(n g) \, dn \right\} \left\{ \int_{N \setminus N} \varphi_2(n g) \, dn \right\} \, dg , \]

an expression symmetric in \( \varphi_1 \) and \( \varphi_2 \).

**COROLLARY 6.5.** \( \Lambda^T \) extends to an orthogonal projection on the Hilbert space \( L \).

We will not need any of these assertions in the next two lectures. What we will need is the fact that \( \Lambda^T \) transforms smooth slowly increasing functions into rapidly decreasing functions. For now we
content ourselves with a relatively simple statement.

To any element $Y$ of the universal enveloping algebra of the Lie algebra of $G$ we can associate a left-invariant differential operator $R(Y)$ on $G$.

**LEMMA 6.6.** Suppose $T$ is sufficiently regular. Let $G$ be a Siegel domain on $G^1$. For any pair of positive numbers $N$ and $N'$ and any open compact subgroup $K_0$ of $G(A_f)$ we can find a finite subset \{\(Y_1, \ldots, Y_r\)\} in the universal enveloping algebra such that

\[
|\lambda^T \varphi(g)| |g|^{-N'} \leq \sum_i \sup_{h \in G^1_g} |R(Y_i) \varphi(h)| |h|^{-N}
\]

for $g \in G$ provided $\varphi$ is invariant on the right under $K_0$ and sufficiently smooth that all the operators $R(Y_i)$ can be applied to it.

This is proved by an argument similar to that used for the proof of the $\sigma$-expansion. Its structure is more transparent, many of the incidental difficulties met with the $\sigma$-expansion no longer arising. However the alternating sum is used in a slightly different way and it is best to dispose of the necessary technical lemma immediately.

For this purpose we fix $P_1 \subset P_2$ and consider a continuous function $\psi$ on $N_1 \setminus N$. If $P_1 \subset P \subset P_2$ then

\[
\varpi_p : n_1 \rightarrow \int_{N \setminus N_1} \psi(nn_1) dn_1
\]

is also a function on $N_1 \setminus N$ because $N$ is a normal subgroup of $N_1$. We want to consider
\[ \prod \psi = \sum_{\mathcal{P}} (-1)^{a_P} \prod_{\mathcal{P}} \psi. \]

Let \( \Delta_0^2 - \Delta_0^1 = \{\alpha_1, \ldots, \alpha_s\} \) and let \( \sum_{\alpha} \) be the set of positive roots \( \alpha \) of the form

\[ \alpha = \sum_{\beta \in \Delta_0^2} b_\beta \beta \]

with \( b_\beta \neq 0 \) for \( \beta = \alpha_i \) or \( \beta \in \Delta_0^2 \). There is a parabolic \( \mathcal{P} \) between \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) such that the Lie algebra of \( N^i \) is spanned by the root vectors attached to the roots \( \alpha \) in \( \sum_{\alpha} \). For any \( P \) between \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) there is a unique subset \( \sum_{\mathcal{P}} \) of \( \{\alpha_1, \ldots, \alpha_r\} \) such that \( \Delta_0^2 \) is the disjoint union of \( \sum_{\mathcal{P}} \) and \( \Delta_0^1 \). Moreover

\[ N = \prod_{i \in \sum_{\mathcal{P}}} N^i. \]

It follows easily, all the groups \( N \) being normal in \( N_1 \), that

\[ \prod = \prod_{i=1}^{r} \left( \prod_{\mathcal{P}_2}^{r} - \prod_{i}^{r} \right), \]

where for simplicity of notation we have set \( \prod_{\mathcal{P}}^{i} = \prod_{i}^{r} \).

Let \( \sum_{\alpha} \) be the set of positive roots which when written as in (5) have \( b_{\alpha_i} \neq 0 \). Let an integer \( r \geq 0 \) be given. For the purposes of the next lemma we define a left-invariant differential operator of type \( r \) to be a product

\[ \prod_{i=1}^{r} \prod_{j=1}^{r} X_{ij}. \]
the order being immaterial and \( X_{ij} \) being a root vector of type \( \alpha \) with 
\[ \alpha \in \sum_{i}. \]

**LEMMA 6.7.** For any integer \( r > 0 \) and any open compact subgroup \( U \) of \( N_1^f = N_1(A^f) \), there is a constant \( c = c(r, U) \) and a finite collection \( Y_1, \ldots, Y_m \) of differential operators of type \( r \), the collection depending on \( r \) alone and not on \( U \), such that

\[ \|\mathcal{T}_r \psi\|_\infty \leq c \sum_{i} \|R(Y_i) \psi\|_\infty \]

for any function \( \psi \) on \( N_1 \setminus N_1 / U \) which has continuous derivations up to order \( rs \).

The norms in the inequality are of course \( L_\infty \)-norms. A little reflection shows that we can make a number of simplifications. First of all replacing \( \psi \) by \( \mathcal{T}_r \psi \) we can work in the group \( M_2 \) rather than in \( G \). In other words we may suppose that \( G = P_2 \). Then the formula (6) reduces to the case that \( P_1 \) is a maximal proper parabolic of \( G \) over \( Q \).

We choose a composition series of groups over \( Q \)

\[ N_1 = V_1 \supseteq V_{r-1} \supseteq \cdots \supseteq V_0 = \{1\} \]

with \( V_{i+1} / V_i \) isomorphic to the additive group. Since

\[ (1 - \mathcal{T}_r) \psi(n_1) = \sum_{i=0}^{r-1} \int_{V_i \setminus V_1} \psi(vn_1) dv - \int_{V_{i+1} \setminus V_{i+1}} \psi(vn_1) dv \]
it is enough to prove the following lemma.

**Lemma 6.8.** Let \( r \geq 0 \) be an integer and let \( U \) be an open subgroup of \( \mathbf{A} \). There is a constant \( c = c(r, U) \) such that for any function \( \psi \) on \( \mathbb{Q} \setminus \mathbf{A}/U \) which is continuously differentiable of order \( r \)

\[
\sup_{\mathbf{Q} \setminus \mathbf{A}} |\psi(x) - \int \psi(Y) dy| \leq c \| \frac{\partial^r \psi}{\partial x^r} \|_{\infty}.
\]

To be a function on \( \mathbb{Q} \setminus \mathbf{A}/U \) is to be a function on a quotient \( L \setminus \mathbf{R} \) where \( L = L(U) \) is a lattice in \( \mathbf{R} \). The inequality thus follows readily from

\[
\sum_{n \neq 0} |a_n| \leq \left( \sum_{n \neq 0} |n^r a_n|^2 \right)^{1/2} \left( \sum_{n \neq 0} \frac{1}{n^{2r}} \right)^{1/2},
\]

at least for \( r > 0 \), but the case \( r = 0 \) is quite trivial.

We shall apply Lemma 6.7 to a function

\[
n \rightarrow \psi(na)
\]

where \( \psi \) is a function on \( \mathbb{G} \setminus \mathbb{G} \) and \( a \in A_0(\mathbf{A}) \). If we want to regard the \( Y_i \) as left-invariant differential operators on \( \mathbb{G} \) we must write the inequality of Lemma 6.7 as

\[
\sup_{\mathbb{N}^\infty} |\prod_{i \neq 1} \psi(n_i a)| \leq c \sum_{i \neq 1} \sup_{i} |R(\text{ad}^{-1}(Y_i)) \psi(n_i a)|.
\]

This will be to our advantage.

We now take up the proof of Lemma 6.8. The first step is to
replace
\[ \hat{\tau}_P(H(x) - T) \int_{N \setminus N} \varphi(n) \, dn \]

by
\[ \sum_{P_1 \subset P} \sum_{P_2 \setminus P_1} F^1_P(x, T) \sigma^2_1(H(x) - T) \int_{N \setminus N} \varphi(n) \, dn \]

the sum being over the pairs \( P_1, P_2 \). There is then a sum over \( P \setminus G \)
and an alternating sum over \( P \). The final result is a sum over pairs \( P_1 \subset P_2 \)
of
\[ \sum_{P_1 \setminus G} \sum_{\{P \setminus P_1 \subset P \subset P_2\}} (-1)^{a_P} F^1_P(\delta g, T) \sigma^2_1(H(\delta g) - T) \int_{N \setminus N} \psi(n\delta g) \, dn \]

However Corollary 4.1.2 allows us to replace \( F^1_P \) by \( F^1_2 \). The upshot is that we are forced to estimate
\[ \sum_{P_1 \setminus G} F^1_2(\delta g, T) \sigma^1_1(H(\delta g) - T) \sum_{P} (-1)^{a_P} \int_{N \setminus N} \varphi(n\delta g) \, dn \]

Lemma 2.1 shows that
\[ \sum_{P_1 \setminus G} F^1_2(\delta g, T) \sigma^2_1(H(\delta g) - T) \leq c \| g \|^M \]

for some \( M, T \) being held constant. Thus the problem is to estimate
\[ \sum_{P} (-1)^{a_P} \int_{N \setminus N} \varphi(n\delta g) \, dn \]
It is now best to be more precise about Siegel domains. In contrast to the previous definition the elements \( g \) of \( \mathcal{G}_P(T_0) \) will now be required to have all of the following properties:

(i) If \( g = pk \) and \( a = a(g) \) is the projection of \( p \) on \( A_0 \) then \( g \in a\Omega \) where \( \Omega \) is a fixed compact set.

(ii) \( a(H(g) - T_0) > 0 \) for all \( a \in \Delta_0^P \).

(iii) There are constants \( c_1 \) and \( c_2 \) so that

\[
|2n|a| \leq c_1(1+|H(g)|) \leq c_2(1+|2n|a|) \ .
\]

The final condition is easily seen to force the component of \( a \) in \( A_0(A^f) \) to lie in a compact set. This modification entails a modification in \( \mathcal{G}_P^1(T_0, T) \) but the set

\[
P_1 \mathcal{G}_P^1(T_0, T)
\]

and thus the function \( F_1^P \) is not changed; provided of course \( c_1, c_2 \) and \( \Omega \), which affect the size of \( \mathcal{G}_P(T_0) \), are all chosen large enough.

This definition has the advantage that for a given \( \mathcal{G}_G(T_0) \), for example that of Lemma 6.6, there are positive constants \( c_1 \) and \( \varepsilon \) such that

\[
|\delta g|^{-1} \leq c |g|^{-\varepsilon}
\]

for all \( \delta \in G \) and all \( g \in \mathcal{G}_G(T_0) \). (I know no reference for this fact. It can be deduced from Prop. II.1.5 of A. Borel, Ensembles fondamentaux.)
pour les groupes arithmétiques et formes automorphes. Cours à l'IHP (1964).)

Thus all we need do is show that if \( g \in G^1(T_0, T) \) and 
\[ \sigma^2_1(H(g) - T) \neq 0 \] 
then, for a suitable choice of \( g \) modulo \( P_1 \),

\[
|g|^M N' \sum_P (-1)^{a_P} \int_{N \setminus N} \varphi(ng) \leq \sum_i \sup_{h \in G^1g} |R(Y_i) \varphi(h)| |h|^{-N}.
\]

The suitable choice of \( g \) will be an element in \( G^1(T_0, T) \).

Then conditions (i) and (iii) yield

\[
|g|^M N' \leq c e^{M'} \|H\|
\]

with \( H = H(g) = H(a) \). Thus denoting the right side of (8) by \( A \) we need only show that

\[
ce^{M'} \|H\| \sum_P (-1)^{a_P} \int_{N \setminus N} \varphi(ng) \leq A.
\]

Since we can readily deal with right translations by elements from a compact set in \( G^1 \) we may suppose that \( g = a(g) = a \). As in Lemma 4.2 we may write \( H = H_1 + H_2 \) with \( H_1 \in \mathcal{H}_1^2 \) and \( H_2 \in \mathcal{H}_2 \) and deduce from the fact that \( \sigma^2_1(H-T) \neq 0 \) that

\[
\|H_2\| \leq c(1 + \|H_1\|)
\]

the constant \( c \) depending of course on \( T \), but that is of no consequence.

Thus it will be enough to prove (9) with \( H \) replaced by \( H_1 \) but with a larger
$M'$. This is an easy consequence of (7), for the inequality (ii) applied with $P_2$ replacing $P$ assures us that the coefficients of $\text{ada}^{-1}(Y)$ with respect to a fixed basis of the universal enveloping algebra are bounded by $e^{-M''\|H_1\|}$, where $M'' \to \infty$ with $r$. 
Appendix

Truncation has been seen to have two essential properties. It is an idempotent and it converts slowly increasing smooth functions to rapidly decreasing functions. It may be worthwhile to see how this comes to pass in a simple case.

A function \( f \) on the upper half-plane which is invariant under \( \text{SL}(2, \mathbb{Z}) \) may also be considered as a function on \( \text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R}) \) if we set

\[
\phi(g) = f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

In particular

\[
\phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = f(a^{2}i+x).
\]

The function \( f \) is determined by its values on a fundamental domain

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fundamental_domain.png}
\end{array}
\]

Truncation is achieved by leaving \( f \) untouched below a certain line \( y = y_{0} \) in the fundamental domain and by removing the constant term.
of its Fourier expansion above the line. So it is clearly idempotent.

The inequality

\[ \sum_{n \neq 0} |a_n| \leq \sqrt{\sum_{n \neq 0} n^{2r} |a_n|^{2r}} \sqrt{\sum_{n \neq 0} \frac{1}{n^{2r}}} \]

shows that for \( r \geq 1 \) and \( y > y_0 \)

(1) \[ |Af(u+iy)| \leq c \int_{-\frac{1}{2}}^{\frac{1}{2}} |\frac{dr}{dx} f(x+iy)|^2 dx \]

However if \( X \) is the element

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]

in the Lie algebra then

\[ \frac{dr}{dx} f(x+iy) = \frac{dr}{dx} \phi(\begin{pmatrix}1 & x \\ 0 & 1\end{pmatrix} \begin{pmatrix}a & 0 \\ 0 & a^{-1}\end{pmatrix}) \]

with \( y = a^2 \) and right side of this equality is

\[ a^{-2r} R(X)^r \phi(\begin{pmatrix}1 & x \\ 0 & 1\end{pmatrix} \begin{pmatrix}a & 0 \\ 0 & a^{-1}\end{pmatrix}) \]

Thus bounds on \( R(X)^r \phi \) of the form

\[ |R(X)^r \phi(\begin{pmatrix}1 & x \\ 0 & 1\end{pmatrix} \begin{pmatrix}a & 0 \\ 0 & a^{-1}\end{pmatrix})| \leq c(r)a^{2s} \]

where \( s \) is a constant independent of \( r \) - and this is the kind of bound that will be available to us - yield
\[ | \frac{d^r}{dx^r} f(x+iy) | \leq c(r) y^{s-r}. \]

The inequality (1) then implies that

\[ |Af(u+iy)| \leq cc(r) y^{s-r} \]

for any \( r \).