Lecture 7

PREPARATION FOR THE COARSE χ-EXPANSION

I: STATEMENT OF LEMMAS

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Recall that the right side of the basic identity is

$$\sum_{P_1 \subset P_2} \sum_{P_1 \backslash G} \sigma_1^2(H(\delta g)) \sum_{P_1 \subset P \subset P_2} (-1)^{a_P^{\varepsilon}} \Lambda^{T,P_1} K_P(\delta g, \delta g) \quad ,$$

where

$$a_P^{\varepsilon} = \dim \mathfrak{a}_P^{\varepsilon}/\mathfrak{a}_G^{\varepsilon} \quad .$$

Since the χ-expansion can not be introduced without recalling facts from the theory of Eisenstein series, we begin by proving the absolute convergence of

$$\int_{G \backslash G'} \sum_{P_1 \backslash G} \sigma_1^2(H(\delta g) - T)\left(\sum_{P_1 \subset P \subset P_2} (-1)^{a_P^{\varepsilon}} \Lambda^{T,P_1} K_P(\delta g, \delta g)\right) dg \quad .$$

This will not be a simple matter and will provide us with techniques and lemmas necessary for the proof of the absolute convergence of the χ-expansion.

We prove in fact the stronger assertion that

$$\int_{P_1 \backslash \mathbf{G}^1} \sigma_1^2(H(g) - T) \left|\sum_{P_1 \subset P \subset P_2} (-1)^{a_P^{\varepsilon}} \Lambda^{T,P_1} K_P(g, g)\right| dg < \infty \quad .$$

Notice – Clozel uses an even stronger assertion which is indeed proved later on.

Recall that

$$\mathbf{G}^1 = \{g \in \mathbf{G} \mid |\chi(g)| = 1 \ \forall \chi \in X^*(G)\} \quad .$$

We begin the proof with a sequence of preliminary reductions. Since

$$\mathbf{G}^1 = (\mathbf{P}_1 \cap \mathbf{G}^1)\cdot K \quad ,$$

the integral can be replaced by an integral over $P_1 \backslash (\mathbf{P}_1 \cap \mathbf{G}^1)^{\times K}$, the measure on $\mathbf{P}_1 \cap \mathbf{G}^1$ being the left-invariant Haar measure. Let

$$\mathbf{P}_1^1 = \{p \in \mathbf{P}_1 \mid |\chi(p)| = 1 \ \forall \chi \in X^*(P_1)\}$$

and let $A_1^G(\mathbf{R})^o = \exp \mathfrak{a}_1^G$ be the connected component of $A_1^G(\mathbf{R})$. Then

$$\mathbf{P}_1 \cap \mathbf{G}^1 = \mathbf{P}_1^1 \cdot A_1^G(\mathbf{R})^o$$

and the left-invariant Haar measure on $\mathbf{P}_1 \cap \mathbf{G}^1$ is given by

$$d(pa) = \rho_{P_1}^{-2}(a)\, dp\, da \quad ,$$

with

$$\rho_{P_1}^{-2}(a) = \left| \mathrm{ad}(a)|_{\mathfrak{p}_1} \right| \quad ,$$

$\mathfrak{p}_1$ being the Lie algebra of P_1. To simplify the notation in this lecture I shall abbreviate $A_1^G(\mathbf{R})^o$ to A_1^G.

Let Ω be a compact subset of $\mathbf{G}$ satisfying $K\Omega K = \Omega$ and set

$$\Phi(g) = \sup_{k,k' \in \Omega} \Big| \sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_P(gk,\ gk') \Big| \quad .$$

It is clearly enough to show that for any real number r

$$\int_{P_1\backslash \mathbf{P}_1^1\times A_1^G} \sigma_1^2(H(p)+H(a)-T)(1+\|H(p)\|+\|H(a)\|)^r\phi(pa)(\rho_{P_1}^{-2}(a)dpda < \infty \quad .$$

Let $\mathfrak{S} = \mathfrak{S}^{P_1}(T_0, \omega) \cap \mathbf{P}_1^1$ be a Siegel domain in $\mathbf{P}_1^1$. It is certainly enough to show that for any arbitrary $\mathfrak{S}$ the integral

$$(1) \quad \int_{\mathfrak{S}\times A_1^G} \sigma_1^2(H(p)+H(a)-T)(1+\|H(p)\|+\|H(a)\|)^r\phi(pa)\rho_{P_1}^{-2}(a)dpda$$

is finite. We may suppose that on $\mathfrak{S}$

$$c'|\ell n|p\| - c_2' \le \|H(p)\| \le c_1|\ell n|p\| + c_2 \quad .$$

The proof has two aspects. One first shows that the integrand is zero on a large subset of the domain of integration, and then uses the estimates of the previous lecture on the set on which it does not vanish.

We begin by finding a convenient expression for

$$(2) \qquad \sum_{P_1\subset P\subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_P(h, g) \quad .$$

Recall that

$$K_P(h, g) = \omega(g)\sum_{Z_0\backslash M} \int_{\mathbf{N}} \phi(h^{-1}\gamma n\varepsilon(g))dn \quad .$$

It follows immediately from the definition of Λ^{T,P_1} that

$$\Lambda^{T,P_1}\varphi = \Lambda^{T,P_1}\psi$$

if

$$\psi(g) = \int_{N_1 \backslash \mathbb{N}_1} \phi(n_1 g)\, dn_1 \quad .$$

Thus when studying the expression (3) we may replace $K_P(h, g)$ by

$$\int_{N_1 \backslash \mathbb{N}_1} K_P(n_1 h, g)$$

and this equals

$$\omega(g) \sum_{Z_0 N \backslash P} \int_{\mathbb{N}_1 / N_1} \int_{\mathbb{N}} \phi(h^{-1} n n_1 \gamma \varepsilon(g))\, dn\, dn_1 \quad ,$$

because $\mathbb{N}$ is a normal subgroup of $\mathbb{N}_1$. This expression is in turn equal to

$$\omega(g) \sum_{\gamma \in P_1 \backslash P} \sum_{\delta \in Z_0 N_1 \backslash P_1} \int_{\mathbb{N}_1} \phi(h^{-1} n_1 \delta \gamma \varepsilon(g))\, dn_1 = \omega(g) \sum_{\gamma \in P_1 \backslash P} K_{P_1}(h, \gamma\varepsilon(g)) \quad ,$$

where K_{P_1} now denotes the kernel for the case that ε is the trivial automorphism.

The expression (3) becomes

$$\omega(g) \sum_{\gamma \in P_1 \backslash P_2} \left(\sum_{\substack{P_1 \subset P \subset P_2 \\ \gamma \in P}} (-1)^{a_P^{\varepsilon}} \right) \Lambda^{T, P_1} K_{P_1}(h, \gamma\varepsilon(g)) \quad .$$

If there is no ε-invariant parabolic subgroup between P_1 and P_2 the sum is empty, equals 0, and the convergence of (1) is trivial. Otherwise let Q be the largest such subgroup. For a given γ let P_γ

be the smallest such subgroup containing γ. Then

$$\sum_{\substack{P_1 \subset P \subset P_2 \\ \gamma \in P}} (-1)^{a_P^\varepsilon} = \sum_{P_\gamma \subset P \subset Q} (-1)^{a_P^\varepsilon} ,$$

and this is clearly 0 unless $P_\gamma = Q$, when it is 1.

Let $F_\varepsilon(P_1, P_2)$ be the set of all $\gamma \in Q$ (taken modulo P_1) for which $P_\gamma = Q$. Then ($\cancel{3}$ 2) is equal to

$$(3) \qquad (-1)^{a_Q^\varepsilon} \omega(g) \sum_{\gamma \in F_\varepsilon(P_1,P_2)} \Lambda^{T,P_1} K_{P_1}(h, \gamma\varepsilon(g)) \quad .$$

To prove the convergence we need a number of lemmas. These we next state, explaining how the convergence follows from them, postponing the proof of the lemmas until the next lecture. It will be convenient to introduce a notational convention. We denote by C a compact set and by c a constant both depending only on Ω and the support of ϕ, and by $c(\phi)$ a semi-norm on $C_c^\infty(\mathbf{G})$. All three are allowed to vary from line to line. The first lemmas are concerned with the support of the integrand in (2).

LEMMA 7.1. <u>There is a compact set</u> C <u>in</u> $\mathfrak{a}_Q$ <u>such that</u>

$$\Lambda^{T,P_1} K_{P_1}(gk, \gamma\varepsilon(gk')) \neq 0$$

<u>for some</u> $\gamma \in Q = Q(\mathbf{Q})$ <u>and some</u> $k, k' \in \Omega$ <u>implies that</u> $H(g) \in \mathfrak{a}_0^Q + \mathfrak{a}_Q^\varepsilon + C$.

Until now Ω has needed only to contain K. Now we suppose that in addition it contains $\exp C \cdot K$, with the C of the lemma. Then, at the cost of taking a slightly different T, we can replace the integral in ($\overset{1}{\cancel{2}}$) by an integral over $\mathfrak{G} \times A_1^Q A_Q^\varepsilon$. Indeed T is fixed. Thus there is a $C \in \mathfrak{a}_1^Q + \mathfrak{a}_Q^\varepsilon$ such that

$$\sigma_1^2(H(p) + H(a) - T) = \sigma_1^2(H(a) - T) \neq 0$$

implies that $H(a) = H + X$ with $X \in C$ and $\sigma_1^2(H) \neq 0$. This allows us, again at the cost of enlarging Ω, to take $T = 0$.

LEMMA 7.2. <u>If</u> $p, p' \in \mathbb{P}^1$, $k, k' \in \Omega$, $\gamma \in Q$, $a, a' \in A_1^Q$, $b \in A_Q^\varepsilon$ <u>then</u>

$$\Lambda^{T,P_1} K_{P_1}(pabk, \gamma\varepsilon(p'abk')) = \Lambda^{T,P_1} K_{P_1}(pak, \gamma\varepsilon(p'ak'))\rho_{P_1}^2(\overset{b}{\cancel{a}}) \quad .$$

LEMMA 7.3. <u>If</u> $H = H_1^Q + H_Q^\varepsilon$ <u>with</u> $H_1^Q \in \mathfrak{a}_1^Q$, $H_Q^\varepsilon \in \mathfrak{a}_Q^\varepsilon$ <u>and</u> $\sigma_1^2(H) \neq 0$ <u>then</u> $\alpha(H_1^Q) > 0$ <u>for all</u> $\alpha \in \Delta_1^Q$ <u>and</u>

$$\|H_Q^\varepsilon\| \leq c(1 + \|H_1^Q\|) \quad .$$

These two lemmas and the previous reductions allow us to majorize (2) by

$$(4) \qquad c \int_{\mathfrak{G} \times A_1^Q} \tau_1^Q(H(a))(1+\|H(p)\|+\|H(a)\|)^r \Phi(pa)\rho_{P_1}^{-2}(a)\,dp\,da \quad ,$$

it being however understood that the integration over A_Q^ε, or to be more precise over

$$\{b \in A_Q^{\varepsilon} \mid \|H(b)\| \leq c(1+\|H(a)\|) \quad ,$$

has forced us to increase the exponent r. The function τ_1^Q is the characteristic function of

$$\{H \in \mathfrak{a}_1^Q \mid \alpha(H) > 0 \, \forall \alpha \in \Delta_1^Q\} .$$

To estimate $\Phi(pa)$ in the integral $(\overset{4}{\cancel{5}})$ we observe that it is dominated by

$$\sum_{\gamma \in F_{\varepsilon}(P_1, P_1)} |\Lambda^{T,P_1} K_{P_1}(pak, \gamma\varepsilon(pak'))|$$

for some $k, k' \in \Omega$. At this point there are three more steps left for the proof. The first is to show that if this expression does not vanish then $\|H(a)\|$ is controlled by $\|H(p)\|$ and that is the purpose of the next lemma. Then we have to show that the truncation provides us with functions rapidly decreasing at infinity in $\mathfrak{G}$, and finally that the summation over γ, although it tempers the rate of the decrease, does not destroy it.

LEMMA 7.4. Suppose γ lies in $F_{\varepsilon}(P_1, P_2)$, $nm \in \mathfrak{G}$, $a \in A_1^Q$, and that $\tau_1^Q(H(a)) \neq 0$. Suppose in addition that for some $k, k' \in \Omega$ and some $m' \in \mathbf{M}_1^1 = \mathbf{M}_1 \cap \mathbf{P}_1^1$ we have

$$\Lambda^{T,P_1} K_{P_1}(m'ak', \gamma\varepsilon(nmak)) \neq 0 \quad .$$

Then for some other m'

$$K_{P_1}(m'ak',\ \gamma\varepsilon(nmak)) \neq 0$$

and

$$\|H(a)\| \le c(1+\|H(m)\|) \quad .$$

On A_1^Q we have $|a| \le ce^{N_1\|H(a)\|}$. It therefore follows from this and the following lemma together with Lemma 6.6 of the previous lecture that for certain M_1 and N but for an arbitrary M and thus for an arbitrary M'

$$|\Lambda^{T,P_1} K_{P_1}(pak,\ \gamma\varepsilon(pak')| = c|p|^{-M}|a|^N|\gamma\varepsilon(p)|^{M_1} \le c'|p|^{-M'}|\gamma\varepsilon(p)|^{M_1}$$

provided $\tau_1^Q(H(a)) \neq 0$.

If Y is an element of the universal enveloping algebra of the Lie algebra of G then we can associate to Y a left-invariant differential operator $R(Y)$ in $\mathbf{G}$ and a right-invariant differential operator $L(Y)$.

LEMMA 7.5. *Let* Y *lie in the universal enveloping algebra of* M_1 *and let* $R(Y)K_{P_1}(h,\ g)$ *be the result of applying* $R(Y)$ *to* K_{P_1} *regarded as a function of the first variable, the second argument being held fixed. Then there are constants* $c = c(\phi)$ *and* N *such that for* $k,\ k' \in \Omega$

$$|R(Y)K_{P_1}(hk,\ gk')| \le c(|h||g|)^N \quad .$$

Proceeding with the proof of convergence of (5) we estimate

$$\text{(5)} \qquad \sum_{\gamma \in F_\varepsilon(P_1,P_2)}{}' \ |\gamma\varepsilon(pa)|^{M_1} \quad ,$$

where $p \in \mathbf{G}$ and the prime indicates that we sum only over those $\gamma \in F_{\varepsilon}(P_1, P_2)$ for which

$$\Lambda^{T,P_1} K_{P_1}(pak,\ \gamma\varepsilon(pak')$$

is non-zero for some k, k' in Ω and the given pa.

The next lemma limits the γ which appear in the sum (6).

LEMMA 7.6. <u>There exists a point</u> T_0 <u>in</u> $\mathfrak{a}_0$ <u>depending only on the support of</u> ϕ <u>and on the compact set</u> Ω <u>such that</u>

$$\hat{\tau}_1(H(g) - H(h) - T_0) = 1$$

<u>whenever</u>

$$K_{P_1}(mhk,\ gk') \neq 0$$

<u>for some</u> $m \in \mathbf{M}_1^1$ <u>and some</u> $k, k' \in \Omega$. <u>Here</u> h <u>and</u> g <u>lie in</u> $\mathbf{G}$.

The following lemma and Lemma 7.6 taken together allow us to estimate the sum (6).

LEMMA 7.7. <u>Suppose that</u> $T \in \mathfrak{a}_0$ <u>and</u> $\mathbf{M}_1 \geq 0$. <u>Then we can find constants</u> c <u>and</u> M_1^1 <u>and a set</u> $[P_1 \backslash G]$ <u>of representatives for</u> $P_1 \backslash G$ <u>such that for any</u> $h, g \in \mathbf{G}$

$$\sum_{\delta \in [P_1 \backslash G]} |\delta g|^{M_1} \hat{\tau}_{P_1}(H(\delta g) - H(h) - T_0) \leq c|h|^{M_1^1} |g|^{\mathbf{M}_1^1} .$$

The upshot of these considerations is that the domain of integration in (5) can be taken to be

$$\{(m,\ a)\,|\,m \in \mathfrak{G}\ ,\ a \in A_1^Q,\ \|H(a)\| \leq c(1+\|H(m)\|)$$

and that the integrand is dominated on this domain by a constant times

$$(1+\|H(m)\|)^r \rho_{P_1}^{-2}(a)\,|m|^{-M'}\,|m|^{2M_1'}|a|^{M_1'}\quad ,$$

where M_1' is some perhaps large but well determined number and M' is arbitrary. Thus this expression is dominated by a constant times

$$(1+\|H(m)\|)^r\,|m|^{-M''}\quad ,$$

where M'' can be taken arbitrarily large. We can integrate over the variable a. Since the integration is over a ball of radius $c(1+\|H(m)\|$ we are left with

$$c\int_{\mathfrak{G}} (1+\|H(m)\|^{r'}\,|m|^{-M''}dm$$

and this integral is finite.