PREPARATION FOR THE COARSE $\chi$-EXPANSION

II: PROOF OF THE LEMMAS

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We begin with an easy one, Lemma 7.2. Since the truncation operator applied to $K_{P_1}(pabk, \gamma \varepsilon(p'abk'))$ is applied to it as a function of $p$, it is enough to show that

$$K_{P_1}(pabk, \gamma \varepsilon(p'abk')) = \phi_{P_1}^{-2}(b)K_{P_1}(pak, \gamma \varepsilon(p'ak')).$$

We recall that

$$K_{P_1}(pabk, \gamma \varepsilon(p'abk')) = \sum_{\delta \in N_1 \setminus P_1} \int_{N_1} \phi(k^{-1}b^{-1}a^{-1}p^{-1}n_1^{-1} \gamma \varepsilon(pa) \varepsilon(b) \varepsilon(k)) dn_1,$$

and notice that $\varepsilon(b) = b$. Since $P_1 \subset Q$,

$$b^{-1}a^{-1}p^{-1} = a^{-1}p^{-1}b^{-1}n_2,$$

with $n_2 \in N_1$, and a change of variables allows us to absorb $n_2$ in $n_1$. Since $Q$ is $\varepsilon$-invariant, $\varepsilon(pa)$ still lies in $Q(A)$ and

$$\gamma \varepsilon(pa)b = \gamma_{n_3}b \gamma \varepsilon(pa),$$

with $n_3 \in N_Q$. Since $N_Q \subset N_1$, we can also absorb $n_3$ in $n_1$, and the desired equality follows from the observation that
\[ \left| \frac{\mathrm{d} n_1}{\mathrm{d}(b^{-1} n_1 b)} \right| = \rho_p^2 (b) . \]

We next turn to Lemma 7.7. Recall that we established during the proof of Lemma 2.1 that if

\[ \hat{\tau}_p (H(\delta g) - H(h) - T_0) \neq 0 \]

then we could find a representative \( \delta' \) for \( \delta \) such that

\[ |\delta' g| \leq c \left( |g| e^{\|H(h)+T_0\|} \right)^N . \]

Thus Lemma 7.7 follows from Lemma 2.1 provided we note that

\[ \|H(h)\| \leq c(1 + |ln| h|) . \]

Working our way backwards we next prove Lemma 7.6. Since \( K \cap K = \Omega \) we may suppose that \( g \) and \( h \) lie in \( P_1 \). Then

\[ K_{P_1}(mhk, gk') \neq 0 \]

implies that for some \( n \in N_1 \) and some \( m' = \gamma m \in M_1 \)

\[ g^{-1} nm'h \in \cap (\text{supp } \phi)^{-1} \Omega^{-1} = C . \]

Thus if we choose \( \rho \) and \( v_\alpha \) as in the second lecture, \( \alpha \in \Delta_0 - \Delta_0 \)

we have

\[ \|\rho (g^{-1} nm'h) v_\alpha\| \leq c . \]
On the other hand we have chosen $g$ and $h$ to lie in $P_1$ and $nm'$ lies in $P_1$. Thus

$$
\|\phi(g^{-1}nm'h)\| = e^{\frac{d}{\sigma_a}(H(h) - H(g))} \leq c
$$

The conclusion is that

$$\pi_a(H(h) - H(g)) \leq c$$

for all $\pi_a \in \hat{P_1}$. This clearly implies the statement of the lemma.

To prove Lemma 7.5 we observe that $R(Y)K_{P_1}$ is obtained by replacing $\phi$ by $\psi = L(Y)\phi$ and then building the kernel attached to $\psi$. Thus it is enough to prove the lemma for $Y = 1$.

We have, for $\gamma \in P_1$

$$
\int_{N_1} \phi(k^{-1}h^{-1}n_{1,\gamma}g) = \rho_1^2(g) \int_{N_1} \phi(k^{-1}h^{-1}g_{1,\gamma}k)\mathrm{d}n_1
$$

Moreover, since $K\Omega K = \Omega$ we may suppose that $h$ and $g$ lie in $P_1$, and indeed in $M_1$ because, for example,

$$|m| \leq c|g|^N$$

if $g = mn$, $n \in N_1$, $m \in M_1$. Then the integral can be taken over a fixed compact set in $N_1$ which does not depend on $h$ and $g$. So we need only estimate
\[ \sum_{M_1} \chi(h^{-1}\gamma g) \]

where \( \chi \) is the characteristic function of a compact set \( C \) in \( M_1 \).
This is the number of \( \gamma \in M_1 \) for which \( \gamma \in hCg^{-1} \) and is easily estimated by Lemma 2.1.

Lemma 7.3 is geometrical. Since \( \Delta_1^0 \subseteq \Delta_1^2 \) we have

\[ \alpha(H_1^Q) = \alpha(H) > 0 \]

for \( \alpha \in \Delta_1^0 \). Moreover

\[ \pi(H_Q^e) > 0, \quad \pi \in \hat{\Delta}_P^2. \]

Since \( \hat{\Delta}_Q \) is the \( E \)-orbit of \( \hat{\Delta}_P^2 \), \( E \) being \( \{1, \varepsilon, \ldots, \varepsilon^{-1}\} \), and since \( H_Q^e \) is \( \varepsilon \)-invariant we obtain the same inequality for \( \pi \in \hat{\Delta}_Q \). On the other hand

\[ \alpha(H_Q^e) < -\alpha(H_1^Q) \]

for \( \alpha \in \Delta_1^2 \). Thus

\[ \alpha(H_Q^e) \leq C\|H_1^Q\| \]

for \( \alpha \in \Delta_Q \), and the lemma follows.

This leaves us with Lemmas 7.1 and 7.4. The first is the easier, and we begin with it.

The assumption of the lemma implies that for some \( p \) in \( \mathbb{P}_1^1 \)
\[ \phi(k^{-1}g^{-1}p_{\gamma}\varepsilon(gk')) \neq 0 \, . \]

Thus
\[ g^{-1}p_{\gamma}\varepsilon(g) \in C \, , \]

\( C \) being a compact set depending only on \( \Omega \) and the support of \( \phi \).

Taking \( \alpha \in \Delta_0 - \Delta_0^Q \) we evaluate \( \|\rho(g^{-1}p_{\gamma}\varepsilon(g))v_{\alpha}\| \)

\[
-\text{d}_{\alpha} \varepsilon (H(g) - \varepsilon H(g))_{\alpha} ||v_{\alpha}|| \]

and \( \|\rho(\varepsilon(g^{-1})y^{-1}p^{-1}g)v_{\alpha}\| \) as

\[
\text{d}_{\alpha} \varepsilon (H(g) - \varepsilon H(g))_{\alpha} ||v_{\alpha}|| \, . \]

Both expressions are bounded above. If \( \chi \in X^*(G) \) we can also bound

\[ |\chi(g^{-1}\varepsilon(g))| = |\chi(g^{-1}p_{\gamma}\varepsilon(g))| \]

above and below, concluding that \( \|H(g) - \varepsilon H(g)\| \) is bounded. The lemma follows.

We deal finally with Lemma 7.4. It is clear that the assumption implies that

\[ K_{\mathbf{1}}(m'ak, \gamma\varepsilon(nmak)) \neq 0 \]

for some \( m' \in \mathbf{M}_1 \). For brevity we write \( H = H(a) \).

We write \( \gamma = n'\omega n \) with \( n' \in P_1 \), \( n \in N_0 \), and with \( \omega \) in the normalizer of \( A_0 \). It represents an element \( s \) in the Weyl group of
$A_0$ or $\alpha_0$. We are free to modify $\omega$ on the left by an element of $M_1$, incorporating the modification in $\eta'$. So we can suppose that $s^{-1}a > 0$ if $a \in \Delta_0^1$. Since $H$ lies in $A^Q_1$ and $a(H) > 0 \forall a \in \Delta_1^Q$ we have $H \in \alpha_1^+ \subset \alpha_0^+$. Thus $\varepsilon H$ is also in $\alpha_0^+$ and if $\alpha \in \Delta_0^1$ then
\[
\alpha(H - s\varepsilon H) = -s^{-1}\alpha(\varepsilon H) < 0 .
\]

We shall now show in addition that, for $\alpha \in \Delta_0^1 - \Delta_0^1$,

\[
(2) \quad \overline{\omega}_\alpha(H - s\varepsilon H) \leq c(1 + \|H(m)\|).
\]

This will allow us to infer from (iv) of Lecture 2 that

\[
(3) \quad H - s\varepsilon H \in X - \Delta_0^1
\]

where $\|X\| \leq c(1 + \|H(m)\|)$.

To prove (2) we notice that (1) implies that for some $m' \in M_1$ and some $n_1 \in N_1$

\[
\varepsilon(a^{-1}m^{-1}n^{-1}) \eta^{-1} \omega^{-1} n_1 m' a \in C ,
\]

$\eta'$ having been absorbed in $n_1 m'$. Thus

\[
\|\rho(\varepsilon(a^{-1}m^{-1}n^{-1}) \eta^{-1} \omega^{-1} n_1 m' a) \varepsilon_a\| \leq c .
\]

We choose $\alpha \in \Delta_0^1 - \Delta_0^1$. Then

\[
\|\rho(n_1 m' a) \varepsilon_a\| = \|\xi_a \overline{\omega}_\alpha(a) \| \|\varepsilon_a\| .
\]
If $w = \omega^{-1}_{\alpha}$ then $w$ is a weight vector corresponding to the weight
$s^{-1}_{\alpha} : H \rightarrow \alpha(sH)$. Moreover

$$\rho(\varepsilon(n^{-1})_{\alpha}w = w + u$$

where $u$ is an adelic linear combination of weight vectors for weights of the form

$$s^{-1}_{\alpha} \sum_{\beta \in \Delta_0} c_{\beta} \beta$$

with $c_{\beta} > 0$, $\sum c_{\beta} \neq 0$. Consequently

$$\|\rho(\varepsilon(a^{-1}m^{-1})\varepsilon(n^{-1})_{\alpha}w\| \geq c\|\rho(\varepsilon(a^{-1}m^{-1}))w\|$$

$$= c e^{-d \frac{\pi}{\alpha_{\alpha}} (s \varepsilon H + s \varepsilon H(m))} \|w\| .$$

We deduce that

$$d \frac{\pi}{\alpha_{\alpha}} (H - s \varepsilon H + s \varepsilon H(m)) \geq e^{a} .$$

Taking logarithms we obtain (2).

We write the left side of (3) as

$$H - s \varepsilon H = H - \varepsilon H + s \varepsilon H .$$

Since $H \varepsilon R^+_0$, its transform $\varepsilon H$ also lies in $\alpha^{\alpha}_{0}$ and $\varepsilon H - s \varepsilon H \varepsilon R^+_0$.

We conclude that
$H - \varepsilon H = X - Y$

with $Y \in \mathcal{H}_0$. Applying $\varepsilon^k$, $0 \leq k \leq 2$ to this relation and summing over $k$ we obtain

$$0 = X' - \sum_{k=0}^{2-1} \varepsilon^k(Y)$$

with $\|X'\| \leq c(1 + \|H(m)\|)$. Since every $\varepsilon^k(Y)$ lies in $\mathcal{H}_0$ we infer that

$$\|Y\| \leq c(1 + \|H(m)\|).$$

This implies first that

$$\|H - \varepsilon H\| \leq c(1 + \|H(m)\|)$$

and thus that there is an $\varepsilon$-invariant $H_0$ with

$$\|H - H_0\| \leq c(1 + \|H(m)\|).$$

We are reduced to showing that

$$\|H_0\| \leq c(1 + \|H(m)\|)$$

knowing that

$$H_0 - sH_0 = X^{-} \mathcal{H}_0.$$
\[ \|X\| \leq c(1+\|H(m)\|), \] a relation which we deduce from (3). Since we may take

\[ H_0 = \frac{1}{2} \sum_{k=0}^{\kappa-1} \varepsilon^k H, \]

we may suppose that \( H_0 \in \sigma_0^\varepsilon \). Then

\[ H_0 - sH_0 \in \sigma_0^+ \]

and we conclude that

\[ \|H_0 - sH_0\| \leq c(1+\|H(m)\|). \]

At this point we introduce the assumption that \( \gamma \in F_\varepsilon(P_1, P_2) \).

We know that for any \( H \in \mathcal{N}_0 \),

\[ H - sH = \sum_{\alpha \in \Delta_0} c_{\alpha}(H, s)\alpha \]

where \( H \rightarrow c_{\alpha}(H, s) \) is a linear form on \( \mathcal{A}_0 \), non-negative on \( \mathcal{A}_0^+ \). In particular \( c_{\alpha}(\overline{w}_\beta, s) \geq 0 \). If \( c_{\alpha}(\overline{w}_\beta, s) = 0 \) for all \( \alpha \) then \( s\overline{w}_\beta = \overline{w}_\beta \).

Thus

\[ \sup_{\alpha} c_{\alpha}(H, s) \geq c_\beta(H) \]

provided \( s\overline{w}_\beta \neq \overline{w}_\beta \).

Applying this to \( H_0 \) we conclude that

\[ (4) \quad |\beta(H_0)| \leq c(1+\|H(m)\|) \]
unless \( s \bar{\omega}' = \bar{\omega}' \) for every \( \beta' \) in the \( E \)-orbit of \( \beta \), because \( \beta(H_0) \) is constant on such orbits.

To prove the lemma we need to establish (4) for all \( \beta \in \Delta_0 \). If the \( E \)-orbit of \( \beta \) does not meet \( \mathcal{P} \) then it cannot happen that \( s \bar{\omega}' = \bar{\omega}' \) for all \( \beta' \) in this \( \epsilon \)-orbit for then \( \gamma \in \mathcal{Q}' \) if

\[
\Delta_{Q'} = \Delta_0 - \{ \epsilon^k \beta \}
\]

and \( \mathcal{Q}' \supset \mathcal{P} \) and is \( \epsilon \)-invariant. On the other hand if for some \( \beta' \) in the \( E \)-orbit of \( \beta \) we have \( \beta' \in \Delta_0 \) then

\[
\beta'(H_0) = \beta(H_0)
\]

and

\[
\beta'(H) = 0
\]

Thus

\[
|\beta(H_0)| = |\beta'(H-H_0)| \leq c(1+\|H(m)\|)
\].