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Duality for Representations of a Reductive Group over a Finite Field, II

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This paper is a continuation of [6] whose notations and references we shall preserve.

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Let l be a prime number, $(l, q) = 1$, and let $\bar{\mathbb{Q}}_l$ be an algebraic closure of $\mathbb{Q}_l$. Let $\mathscr{R}(G)$ be the Grothendieck group of virtual G-modules over $\bar{\mathbb{Q}}_l$. We denote $D: \mathscr{R}(G) \to \mathscr{R}(G)$ the homomorphism such that

$$D_G(E) = \sum_{I \subset \bar{S}} (-1)^{|I|} E_{(I)}$$

for any finite-dimensional G-module E over $\bar{\mathbb{Q}}_l$.

Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$, defined over F_q and let T be the group of its F_q-rational points. For any one dimensional T-module θ (over $\bar{\mathbb{Q}}_l$) we have defined in [5, 1.20] a virtual representation $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$. Let $\sigma(\mathbf{G})$ be the F_q-rank of $\mathbf{G}$.

We shall prove the following:

THEOREM.

$$D_G(R^{\mathbf{G}}_{\mathbf{T}}(\theta)) = (-1)^{\sigma(\mathbf{G}) - \sigma(\mathbf{T})} R^{\mathbf{G}}_{\mathbf{T}}(\theta).$$

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Let us recall the definition of $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ or, more generally, that of "twisted induction" from Levi subgroups (cf. [5, 7]).

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Let **P** be a parabolic subgroup of **G**, with unipotent radical **U** and let **L** be a Levi subgroup of **P**. We assume that **L** is defined over F_q, but we make no assumption on **P**. Let L be the group of F_q-rational points of **L**. Let $F: \mathbf{G} \to \mathbf{G}$ be the Frobenius map corresponding to the F_q-rational structure of **G**. The variety

$$X_{\mathbf{L} \subset \mathbf{P}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}$$

has a $G \times L$ action

$$(g_0, l): g \to g_0 g l^{-1}.$$

Hence we have a $G \times L$-action $((g_0, l)^*)^{-1}$ of $G \times L$ on the cohomology groups $H_c^i(X_{\mathbf{L} \subset \mathbf{P}})$. (Here $H_c^i(\ )$ denotes étale cohomology with compact support with coefficients in $\bar{\mathbb{Q}}_l$.) Let π be a finite-dimensional L-module over $\bar{\mathbb{Q}}_l$. Then $(H_c^i(X_{\mathbf{L} \subset \mathbf{P}}) \otimes \pi)^L$, with L acting diagonally, is in a natural way a G-module. By definition,

$$R_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}}(\pi) = \sum_i (-1)^i (H_c^i(X_{\mathbf{L} \subset \mathbf{P}}) \otimes \pi)^L \in \mathscr{R}(G).$$

In particular, if **B** is a Borel subgroup containing **T**, with unipotent radical **V**, the virtual representation $R_{\mathbf{T} \subset \mathbf{B}}^{\mathbf{G}}(\theta)$ is defined. It is independent of **B** [5, 4.3] and is, by definition, $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$.

Let $\langle\ ,\ \rangle_G: \mathscr{R}(G) \times \mathscr{R}(G) \to \mathbb{Z}$ be the bilinear form defined by $\langle E, E' \rangle_G = \dim \operatorname{Hom}_G(E, E')$ for any two finite-dimensional G-modules E, E' over $\bar{\mathbb{Q}}_l$.

The following result is a generalization of the orthogonality formula [5, 6.8].

THEOREM.

$$\langle R_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}}(\pi), R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle_G = |L|^{-1} \sum_{\substack{n \in G \\ n^{-1}\mathbf{T}n \subset \mathbf{L}}} \langle \pi, R_{n^{-1}\mathbf{T}n}^{\mathbf{L}}({}^n\theta) \rangle_L, \tag{7.1}$$

where, for n in the sum, ${}^n\theta$ denotes the representation of $n^{-1}Tn$ defined by ${}^n\theta(n^{-1}tn) = \theta(t)$, $t \in T$.

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We shall now give the proof of Theorem 6 assuming Theorem 7. If $\mathbf{G} = \mathbf{T}$, the theorem is obvious. Assume now that $\mathbf{G} \neq \mathbf{T}$ and that the theorem is already proved for groups of dimension $< \dim \mathbf{G}$. We shall distinguish two cases.

Case 1. $\mathbf{T}$ is contained in a proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, defined over F_q. Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{P}$, containing $\mathbf{T}$ and defined over F_q. Let L, P denote the groups of F_q-rational points of $\mathbf{L}$, $\mathbf{P}$. According to [5, 8.2] we have $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \operatorname{Ind}(R_{\mathbf{T}}^{\mathbf{L}}(\theta))$, where $\operatorname{Ind}: \mathscr{R}(L) \to \mathscr{R}(G)$ is the homomorphism defined by lifting a representation of L to P and then inducing it from P to G. According to [1], we have $\operatorname{Ind} \circ D_L = D_G \circ \operatorname{Ind}$. Since the theorem is assumed to be true for $\mathbf{L}$, we have

$$\begin{aligned} D_G(R_{\mathbf{T}}^{\mathbf{G}}(\theta)) &= D_G \operatorname{Ind}(R_{\mathbf{T}}^{\mathbf{L}}(\theta)) = \operatorname{Ind}(D_L(R_{\mathbf{T}}^{\mathbf{L}}(\theta))) \\ &= \operatorname{Ind}((-1)^{\sigma(\mathbf{L}) - \sigma(\mathbf{T})} R_{\mathbf{T}}^{\mathbf{L}}(\theta)) = (-1)^{\sigma(\mathbf{G}) - \sigma(\mathbf{T})} R_{\mathbf{T}}^{\mathbf{G}}(\theta) \end{aligned}$$

since $\sigma(\mathbf{G}) = \sigma(\mathbf{L})$.

Case 2. For any proper parabolic subgroup $\mathbf{P}$ defined over F_q, we have $\mathbf{T} \not\subset \mathbf{P}$.

Given such $\mathbf{P}$, let $\mathbf{L}$ be a Levi subgroup of $\mathbf{P}$ defined over F_q and let P, L, Ind be defined as in Case 1. Our hypothesis shows that the set $\{n \in G \mid n^{-1}\mathbf{T}n \subset \mathbf{L}\}$ is empty. By Theorem 7, we then have $\langle R_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}}(\pi), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_G = 0$, for any L-module π of finite dimension. An argument entirely similar to that in the proof of [5, 8.2] shows that $R_{\mathbf{L} \subset \mathbf{P}}^{\mathbf{G}}(\pi) = \operatorname{Ind} \pi$. Thus we have $\langle \operatorname{Ind} \pi, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_G = 0$ for any L-module π of finite dimension or, equivalently, $(R_{\mathbf{T}}^{\mathbf{G}}(\theta))^U = 0$, where $M \to M^U$ is the homomorphism $\mathscr{R}(G) \to \mathscr{R}(L)$ such that, whenever M is a G-module, M^U is the space of vectors of M, invariant under U (=the unipotent radical of P). From the definition of D_G, we see then that $D(R_{\mathbf{T}}^{\mathbf{G}}(\theta)) = (-1)^{|\bar{S}|} R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. It remains to use the fact that, for our T, we have $\sigma(\mathbf{G}) - \sigma(\mathbf{T}) \equiv |\bar{S}| \pmod 2$.

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Let $\mathbf{X}$ be a scheme of finite type over an algebraic closure of F_p and let $f: \mathbf{X} \to \mathbf{X}$ be a morphism. We set

$$\mathscr{L}(f, \mathbf{X}) = \sum_i (-1)^i \operatorname{Tr}(f^*, H_c^i(\mathbf{X})).$$

In the proof of Theorem 7, we shall need the following:

LEMMA. *Assume that f is an automorphism (of finite order) of* $\mathbf{X}$, *and let* $\mathbf{T}$ *be a torus acting on* $\mathbf{X}$ *such that f commutes with each element of* $\mathbf{T}$. *Then* $\mathscr{L}(f, \mathbf{X}) = \mathscr{L}(f, \mathbf{X}^{\mathbf{T}})$, *where* $\mathbf{X}^{\mathbf{T}}$ *is the fixed point set of* $\mathbf{T}$ *on* $\mathbf{X}$.

Proof. Write $f = f_s \cdot f_u = f_u \cdot f_s$ where f_s (resp. f_u) is a power of f and has order prime to p (resp. power of p). According to [5, 3.2], we have $\mathscr{L}(f, \mathbf{X}) = \mathscr{L}(f_u, \mathbf{X}^{f_s})$, $\mathscr{L}(f, \mathbf{X}^{\mathbf{T}}) = \mathscr{L}(f_u, (\mathbf{X}^{\mathbf{T}})^{f_s}) = \mathscr{L}(f_u, (\mathbf{X}^{f_s})^{\mathbf{T}})$. Thus, we are reduced to the case where f has order a power of p. We can find an element $t \in \mathbf{T}$ such that $\mathbf{X}^t = \mathbf{X}^{\mathbf{T}}$. Assume that we know that

$$\mathscr{L}(f, \mathbf{X}) = \mathscr{L}(ft, \mathbf{X}). \tag{9.1}$$

Using again [5, 3.2], we have $\mathscr{L}(ft, \mathbf{X}) = \mathscr{L}(f, \mathbf{X}^t) = \mathscr{L}(f, \mathbf{X}^{\mathbf{T}})$; hence, by (9.1), we have $\mathscr{L}(f, \mathbf{X}) = \mathscr{L}(f, \mathbf{X}^{\mathbf{T}})$, as desired. It remains to prove (9.1). We choose F_{p^e}-rational structures on $\mathbf{T}$ and $\mathbf{X}$ with Frobenius maps $F\colon \mathbf{T} \to \mathbf{T}$, $F\colon \mathbf{X} \to \mathbf{X}$, such that $F(t'x) = F(t')\,F(x)$, for all $t' \in \mathbf{T}$, $x \in \mathbf{X}$, $F(t) = t$ and $F(fx) = fF(x)$ for all $x \in X$. Using the trace formula for Frobenius maps, we see that (9.1) would be a consequence of

$$|\mathbf{X}^{F^n f}| = |\mathbf{X}^{F^n f t}| \qquad \text{for all integers } n \geqslant 1. \tag{9.2}$$

Given $n \geqslant 1$, we can find $t_1 \in \mathbf{T}$ such that $t_1^{-1}F^n(t_1) = t$. Then the map $x \to t_1^{-1}x$ is a bijection $\mathbf{X}^{F^n f} \simeq \mathbf{X}^{F^n f t}$. This completes the proof of the lemma.

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We shall now prove Theorem 7. As in the proof of [7, Theorem 8] we see that the left-hand side of (7.1) is equal to

$$|T|^{-1}\,|L|^{-1} \sum_{\substack{t \in T \\ l \in L \\ j \geqslant 0}} (-1)^j \operatorname{Tr}(t, l), H_c^j(\bar{S}))\operatorname{Tr}(t, \theta)\operatorname{Tr}(l^{-1}, \pi), \tag{10.1}$$

where

$$\bar{\mathbf{S}} = \{(x, x', y) \in \mathbf{V} \times \mathbf{U} \times \mathbf{G} \mid xF(y) = yx'\},$$

and $T \times L$ acts on $\bar{\mathbf{S}}$ by

$$(t, l)\colon (x, x', y) \to (txt^{-1}, lx'l^{-1}, tyl^{-1}).$$

Let $\mathbf{C} = \{n \in \mathbf{G} \mid n^{-1}\mathbf{T}n \subset \mathbf{L}\}$ and let $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_m$ be the orbits of $\mathbf{L}$ acting on $\mathbf{C}$ by right translation. For each i, $1 \leqslant i \leqslant m$, let $\mathbf{G}_i = F^{-1}(\mathbf{B})\,\mathbf{C}_i F^{-1}(\mathbf{P}) = F^{-1}(\mathbf{V})\,\mathbf{C}_i F^{-1}(\mathbf{U})$. Then $\mathbf{G}_i$ $(1 \leqslant i \leqslant m)$ are the double cosets of $\mathbf{G}$ with respect to $F^{-1}(\mathbf{B})$ and $F^{-1}(\mathbf{P})$.

We set $\bar{\mathbf{S}}_i = \{(x, x', y) \in \bar{\mathbf{S}} \mid y \in \mathbf{G}_i\}$. The $\bar{\mathbf{S}}_i$ form a partition of $\mathbf{S}$, stable under the action of $T \times L$. Corresponding to this partition, we have a decomposition of expression (10.1) into the sum of m terms, the terms being obtained from (10.1) by replacing $\mathbf{S}$ by $\mathbf{S}_i$ $(1 \leqslant i \leqslant m)$. Let (10.1) ($i$) be the term corresponding to i. Let

$$\hat{\mathbf{S}}_i = \{(x, x', v, n, u) \in \mathbf{V} \times \mathbf{U} \times F^{-1}(\mathbf{V}) \times \mathbf{C}_i \times F^{-1}(\mathbf{U}) \mid xF(vnu) = vnux'\}.$$

The map $\hat{\mathbf{S}}_i \to \bar{S}_i$ defined by $(x, x', v, n, u) \to (x, x', vnu)$ is a locally trivial fibration all of whose fibres are affine spaces of fixed dimension. Moreover, this map is $T \times L$-equivariant, for the action of $T \times L$ on $\hat{\mathbf{S}}_i$ given by

$$(t, l)\colon (x, x', v, n, u) \to (txt^{-1}, lx'l^{-1}, tvt^{-1}, tnl^{-1}, lul^{-1}). \tag{10.2}$$

It follows that expression (10.1) (i) is equal to

$$|T|^{-1}\,|L|^{-1} \sum_{t \in T, l \in L, j \geqslant 0} (-1)^j \operatorname{Tr}((t, l), H_c^j(\hat{\mathbf{S}}_i)) \operatorname{Tr}(t, \theta) \operatorname{Tr}(l^{-1}, \pi). \tag{10.3}$$

We now make the change of variable $xF(v) \to x$, $x'F(u)^{-1} \to x'$. Then $\hat{\mathbf{S}}_i$ becomes

$$\{(x, x', v, n, u) \in \mathbf{V} \times \mathbf{U} \times F^{-1}(\mathbf{V}) \times \mathbf{C}_i \times F^{-1}(\mathbf{U}) \mid xF(n) = vnux'\}.$$

The action of $T \times L$ is given by the same formula (10.2) as in the old coordinates. Next, we note that the same formula (10.2) defines an action of the group

$$\mathbf{H}_i = \{(t, l) \in \mathbf{Z} \times \mathbf{Z}' \mid l^{-1}F(l) = F(n_i)^{-1}\, t^{-1}F(t)\, F(n_i)\}$$

on $\hat{\mathbf{S}}_i$, where $\mathbf{Z}$ (resp. $\mathbf{Z}'$) is the identity component of the centre of $n_i \mathbf{L} n_i^{-1}$ (resp. of $\mathbf{L}$) and n_i is any element of $\mathbf{C}_i$. (Clearly, $\mathbf{H}_i$ does not depend on the choice of n_i in $\mathbf{C}_i$.)

We shall now determine the fixed point set $(\hat{\mathbf{S}}_i)^{\mathbf{H}_i^0}$ of the torus $\mathbf{H}_i^0$ (=identity component of $\mathbf{H}_i$) on $\hat{\mathbf{S}}_i$.

By Lang's theorem, the homomorphism $\mathbf{H}_i \to \mathbf{Z}$, $((t, l) \to t)$ is onto, hence its restriction $\mathbf{H}_i^0 \to \mathbf{Z}$ must be also onto since $\mathbf{Z}$ is connected. Similarly, the homomorphism $\mathbf{H}_i^0 \to \mathbf{Z}'$, $((t, l) \to l)$ is onto. Therefore, if $(x, x', v, n, u) \in (\hat{\mathbf{S}}_i)^{\mathbf{H}_i^0}$, we must have

$$txt^{-1} = x, \qquad tvt^{-1} = v \qquad \text{for all} \quad t \in \mathbf{Z},$$
$$lx'l^{-1} = x', \qquad lul^{-1} = u \qquad \text{for all} \quad l \in \mathbf{Z}'.$$

Since $\mathbf{L}$ is the centralizer of $\mathbf{Z}'$ and $n_i \mathbf{L} n_i^{-1}$ is the centralizer of $\mathbf{Z}$, it follows that $x, v \in \mathbf{V} \cap n_i \mathbf{L} n_i^{-1}$ and $x', v' \in \mathbf{U} \cap \mathbf{L} = \{1\}$. We then have $xF(n) = vn$, hence $F(n) = x^{-1}vn \in n_i \mathbf{L} n_i^{-1}n$. Since $n \in n_i \mathbf{L}$, we have $F(n) \in n\mathbf{L}$, hence $F(\mathbf{C}_i) = \mathbf{C}_i$. In particular, $(\hat{\mathbf{S}}_i)^{\mathbf{H}_i^0}$ is empty, unless $F(\mathbf{C}_i) = \mathbf{C}_i$. Assume now that $F(\mathbf{C}_i) = (\mathbf{C}_i)$. Then $n_i \in \mathbf{C}_i$ may be chosen so that $n_i \in G$. The equation defining $\mathbf{H}_i$ may now be written: $F(ln_i^{-1}t^{-1}n_i) = ln_i^{-1}t^{-1}n_i$. It follows that

$$\mathbf{H}_i^0 = \{(t, l) \in \mathbf{Z} \times \mathbf{Z}' \mid l = n_i^{-1}tn_i\}$$

and

$$(\hat{\mathbf{S}}_i)^{\mathbf{H}_i^0} = \{(x, 1, v, n_i y^{-1}, 1) \mid x, v \in \mathbf{V} \cap n_i \mathbf{L} n_i^{-1}, y \in \mathbf{L},$$
$$y^{-1}F(y) = n_i^{-1} v^{-1} x n_i\}.$$

Since the action of $T \times L$ on $\hat{\mathbf{S}}_i$ comutes with the action of the torus $\mathbf{H}_i^0$, expression (10.3) is equal, by Lemma 9, to

$$|T|^{-1} |L|^{-1} \sum_{\substack{t \in T \\ l \in L \\ j \geqslant 0}} (-1)^j \operatorname{Tr}((t, l), H_c^j(\hat{\mathbf{S}}_i)^{\mathbf{H}_i^0}) \operatorname{Tr}(t, \theta) \operatorname{Tr}(l^{-1}, \pi).$$

In the case where $F(\mathbf{C}_i) = \mathbf{C}_i$, the map $(\hat{\mathbf{S}}_i)^{\mathbf{H}_i^0} \to \{y \in \mathbf{L} \mid y^{-1}F(y) \in n_i^{-1} \mathbf{V} n_i\}$ given by $(x, 1, v, n_i y^{-1}, 1) \to y$ is a locally trivial fibration all of whose fibres are affine spaces of the same dimension. The action of (t, l) on $(\hat{\mathbf{S}}_i)^{\mathbf{H}_i^0}$ corresponds to the action $y \to l y n_i^{-1} t n_i$. Hence expression (10.1) is equal to

$$|T|^{-1} |L|^{-1} \sum_{\substack{1 \leqslant i \leqslant m \\ F(\mathbf{C}_i) = \mathbf{C}_i}} \sum_{\substack{t \in T \\ l \in L \\ j \geqslant 0}} (-1)^j$$
$$\times \operatorname{Tr}((t, l), H_c^j\{y \in \mathbf{L} \mid y^{-1}F(y) \in n_i^{-1} \mathbf{V} n_i\}) \operatorname{Tr}(t, \theta) \operatorname{Tr}(l^{-1}, \pi)$$

which is clearly equal to the right hand side of (7.1). This completes the proof of Theorem 6.

References

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