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Algèbres de Lie/*Lie Algebras*

On the exceptional series, and its descendants

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Abstract Many of the striking similarities which occur for the adjoint representation of groups in the exceptional series (cf. [1–3]) also occur for certain representations of specific reductive subgroups. The tensor algebras on these representations are easier to describe (cf. [4,5,7]), and may offer clues to the original situation.

The subgroups which occur form a Magic Triangle, which extends Freudenthal's Magic Square of Lie algebras. We describe these groups from the perspective of dual pairs, and their representations from the action of the dual pair on an exceptional Lie algebra. ***To cite this article: P. Deligne, B.H. Gross, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 877–881.***

La série exceptionnelle, et sa descendance

Résumé Les articles [1–3] exhibent des ressemblances entre les propriétés des représentations adjointes des groupes de la série exceptionnelle. Nous obtenons des ressemblances analogues pour certaines représentations préférées de séries de sous-groupes. L'algèbre tensorielle de ces représentations est plus accessible (cf. [4,5,7]). Ceci pourrait aider à comprendre ce qui se passe.

Les sous-groupes en question forment un « triangle magique » qui prolonge le carré magique d'algèbres de Lie de Freudenthal. Nous décrivons ces sous-groupes en termes de paires duales, et leur représentations préférées en termes de leur action sur l'algèbre de Lie du groupe de la série exceptionnelle ambiant. ***Pour citer cet article : P. Deligne, B.H. Gross, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 877–881.***

We consider the series of algebraic groups

$$e \subset A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8, \tag{1}$$

where an indexed capital letter stands for the corresponding simple simply connected complex group. Each group is a subgroup of the next, the inclusion morphism being unique up to conjugacy. The inclusions of the trivial group e in A_1 in A_2 and of E_6 in E_7 in E_8 are given by inclusions of Dynkin diagram. The long roots of G_2 (resp. F_4) form a root system of type A_2 (resp. D_4), providing $A_2 \subset G_2$ and $D_4 \subset F_4$. Finally, G_2 (resp. F_4) is the subgroup of D_4 (resp. E_6) fixed by the group S_3 (resp. S_2) of automorphisms respecting a pinning (épinglage).

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Table 1. – The centralizer of H in K.

								A_1	$\leftarrow$	$E_7 \supset \mu_2$
							$\mathbf{G}_m$	A_2	$\leftarrow$	$E_6 \supset \mu_3$
						μ_3	A_1	G_2	$\leftarrow$	$F_4 \supset 1$
					μ_2^2	$\mathbf{G}_m^2$	A_1^3	D_4	$\leftarrow$	$D_4 \supset \mu_2^2$
				μ_2^2	A_1/μ_2	A_2	C_3	F_4	$\leftarrow$	$G_2 \supset 1$
			μ_3	$\mathbf{G}_m^2$	A_2	A_2^2	A_5	E_6	$\leftarrow$	$A_2 \supset \mu_3$
		$\mathbf{G}_m$	A_1	A_1^3	C_3	A_5	D_6	E_7	$\leftarrow$	$A_1 \supset \mu_2$
	A_1	A_2	G_2	D_4	F_4	E_6	E_7	E_8	$\leftarrow$	$e \supset 1$
	$\uparrow$	$\uparrow$		$\uparrow$		$\uparrow$	$\uparrow$			
	$/\mu_2$	$/\mu_3).2$		$/\mu_2^2).S_3$		$/\mu_3).2$	$/\mu_2$			
ν:	1/3	1/2	2/3	1	3/2	2	3	5		

The centralizers in E_8 of the subgroups (1) form (a conjugate of) the same series of subgroups of E_8, taken in the reverse order, giving rise to the dual pairs (e, E_8), (A_1, E_7), (A_2, E_6), (G_2, F_4) and (D_4, D_4) in E_8.

For each pair $H \subset K$ of distinct subgroups in (1), Table 1 gives the centralizer of H in K. Rows are indexed by H, columns by K. If H' and K' are the centralizers of H and K in E_8, the centralizer $H' \cap K$ of H in K is also the centralizer of K' in H'. This explains the symmetry of the table. If we go over to Lie algebras, the lower right 4×4 square is Freudenthal's "magic square" of Lie algebras.

For each pair $H \subset K$, H injects into the automorphism group Aut k of the Lie algebra k of K. Our *magic triangle* is the system of the centralizers G of H in Aut k. As H is connected, the centralizer of H in the adjoint group K^{ad} is the image of the centralizer of H in K: to obtain G from the corresponding entry of Table 1, one divides by the center of K and takes an extension with, as quotient group, the group of outer automorphisms of K. This modifying isogeny is indicated under each column. In all but one case, the required extension is a semi-direct product, with the quotient group acting as a group of automorphisms respecting a pinning. The exception is the nonsplit extension $(\mathbb{G}_m/\mu_3).2$ at $A_1 \subset A_2$.

As H injects into Aut(k), the center $Z(H)$ of H is a central subgroup of G. It is shown at the right of Table 1. In most cases, G and H form a dual pair in Aut(k), that is, they are each other's centralizer, hence have the same center. The exceptions are $E_6 \subset E_7$ and $F_4 \subset E_6$.

The last row of the magic triangle, corresponding to $H = e$, is the exceptional series of [2]: the groups of automorphisms of the Lie algebras $a_1, a_2, g_2, d_4, f_4, e_6, e_7, e_8$. Under each column, we also give the value of a parameter ν. For the exceptional series, it is $h^\vee/6$, for $h^\vee$ the dual Coxeter number: the inverse of the parameter μ used in [3].

For each pair $H \subset K$, giving rise to G, the representation of $G \times H$ on $k = \mathrm{Lie}(K)$ has the following structure:

$$k \simeq g \otimes 1 + 1 \otimes h + \sum V_\alpha \otimes W_\alpha,$$

with W_α running over the irreducible representations of H occurring in $k/(g+h)$. If $H = A_1, A_2, D_4, E_6$ or E_7, the W_α are the minuscule representations of H. They can be indexed by their central character, which is an arbitrary nontrivial character of the center, and are conjugate by the outer automorphisms of H. For $H = G_2$ or F_4, there is only one W: the representation whose nonzero weights are the short roots of H.

The V_α are always irreducible. In this way, for each row of the magic triangle, we obtain a system of irreducible representations of G. When H is simply laced, it is indexed by the nontrivial characters of

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Table 2. – The preferred representations V of G.

						2	(1)	$\overset{2}{\wedge}$
					1	3	(2)	$\overset{3}{\wedge}$, $V' \otimes V''$
				1	3	7	(1)	$\overset{3}{\wedge}$, S^2
			1	2.1	4	8	(3)	S^2, $V' \otimes V'' \otimes V'''$
		2.1	5	8	14	26	(1)	S^2, S^3
	1	3.1	6	9	15	27	(2)	S^3, $V' \otimes V''$
2.1	4	8	14	20	32	56	(1)	$\overset{2}{\wedge}$, S^4

$Z(H)$. When $H = G_2$ or F_4, it is reduced to a single representation. For $H = e$, it is empty. We call those representations the *preferred* representations.

The preferred representations of G are conjugate by outer automorphisms and, when there are two of them, are in duality. In particular, the preferred representations V of G are all of the same dimension, given in Table 2. When the restriction $V|G^0$ of V to the centralizer G^0 of H in K^{ad} is reducible, the dimension is given in the form $a.b$, a being the length of $V|G^0$ and b the dimension of its irreducible constituents. At $G_2 \subset D_4$ (resp. $F_4 \subset E_6$), we get the irreducible representation of S_3 of dimension 2 (resp. the sign character of S_2). In all other cases, there is a subgroup A of index a of $\mathrm{Out}(K)$ such that V is induced from a representation V_1 of $A \ltimes G^0$ with A acting trivially on the dominant weight subspace of V_1. As could be inferred from the knowledge of central characters, at $A_1 \subset F_4$ (resp. $G_2 \subset E_7$), the dominant weight of the representation of C_3 is $[0,0,1]$ (resp. $[0,1,0]$). For the rows 1 to 7, the group G is the subgroup of $\prod \mathrm{GL}(V_\alpha)$ fixing tensors whose types depend only on the row. The number of preferred representations, and those types, are given in Table 2 as well; an entry such as S^3 stands for $S^3(V_\alpha)$, all α.

One of us (B.G.) first encountered the Magic Triangle (for Lie algebras) in the Harvard Ph.D. thesis of K. Rumelhart. The approach to it here was suggested by the work of G. Savin on dual pairs.

In each row, the dimension of a preferred representation is a linear function of ν: for the row of centralizers to $E_7, \ldots, A_1$, it is

$$\begin{aligned} &E_7\colon\ \nu - 3, \quad E_6\colon\ \nu - 2, \quad F_4\colon\ 2\nu - 3, \quad D_4\colon\ 2\nu - 2, \\ &G_2\colon\ 6\nu - 4, \quad A_2\colon\ 6\nu - 3, \quad A_1\colon\ 12\nu - 4. \end{aligned} \tag{2}$$

If, in the definition of the magic triangle, one allows $H = K$, prefixing each row by a trivial group e and taking for its preferred representation(s) the zero representation(s), the dimension formulas (2) remain valid.

Other representations correspond to each other across the groups in a row. They give rise to uniform decomposition formulas, and their dimensions are the values at $\nu(G)$ of a rational function, which is the quotient of products of rational linear factors. The case of the last row, the exceptional series, is considered in [1–3]; the adjoint representations correspond to each other, and they have the dimension $2(5\nu - 1)(6\nu + 1)/(\nu + 1)$. This formula, due to Vogel, was the beginning of the story on the exceptional series. The case of the seventh row, corresponding to $H = A_1$, is strikingly similar. The vertices of the Dynkin diagram E_7 are labelled

$$\begin{matrix} 2 & 6 & 7 & 5 & 3 & 1 \\ & & 4 & & & \end{matrix}$$

and we write $\alpha(i)$, Ω_i, $s(i)$ for the simple roots, fundamental weights, and generators of the Weyl group so numbered. Let V_r be the ($\mathbb{Q}$-) linear span of the Ω_i $(i \leqslant r)$; it is the orthogonal complement of the $\alpha(i)$, $i > r$. For $r = 1, 3, 5, 6$ the trace on V_r of the E_7-root system is empty, resp. a root system A_1^3, A_5, D_6.

We take as its positive roots those which are positive for E_7. For $r = 1, 3$, the orthogonal projection to V_r of the E_7-root system is a root system A_1, resp. C_3. We take as positive roots the nonzero projections of positive roots. For G in row seven and of rank r, this construction allows us to view the Ω_i $(i \leqslant r)$ as dominant weights of the connected component G^0 of G. To an irreducible representation $V(\lambda)$ of E_7, whose dominant weight λ is linear combination of the Ω_i $(i \leqslant r)$, will correspond the following representation $V(G, \lambda)$ of G^0: take the irreducible representation $V(G^0, \lambda)$ of the "same" dominant weight λ, and the sum of its distinct conjugates by the component group G/G^0. This representation is the restriction to G^0 of an irreducible representation of G. We expect there is a best choice for this representation of G, giving rise to nicer decomposition formulas, but we have not pinned it down. Corresponding representations have the same central character χ, and are orthogonal (resp. symplectic) when χ is the trivial (resp. nontrivial) character of μ_2.

For each G, $V(\Omega_1)$ is the preferred representation. For $\operatorname{rank}(G) \geqslant 2$, $V(\Omega_2)$ is the adjoint representation, and $S^2V(\Omega_1) = V(2\Omega_1) + V(\Omega_2)$. For $\operatorname{rank}(G) \geqslant 3$, $\overset{2}{\wedge} V(\Omega_1) = V(\Omega_3) + 1$. For $\operatorname{rank}(G) \geqslant 3$ and $p \geqslant 1$, one has

$$V(\Omega_1) \otimes V(p\Omega_1) = \sum V(p\Omega_1 + \mu)$$

with μ running over the weights Ω_1, $-\Omega_1$, $\Omega_2 - \Omega_1$ and $\Omega_3 - \Omega_1$. Experiments using the program LiE [6] suggest that more generally

$$V(p\Omega_1) \otimes V(q\Omega_1) = \sum V(a\Omega_1 + b\Omega_2 + c\Omega_3),$$

the sum being over the triples (a, b, c) with a of the same parity as $p+q$ and $|p-q| \leqslant a \leqslant p+q-2(b+c)$, and that for $\operatorname{rank}(G) \geqslant 4$, a similar formula holds for $V(p\Omega_1) \otimes V(q\Omega_2)$.

We will exhibit rational functions $D(\lambda; X)$ depending on a dominant weight λ of E_7 which is a linear combination of the Ω_i $(i \leqslant 6)$, with the property that if the coefficient of Ω_i in λ vanishes whenever $i > \operatorname{rank}(G)$, then

$$\dim V(G, \lambda) = D\big(\lambda, \nu(G)\big)$$

for $\nu(G)$ the parameter of G, given in Table 1.

To describe $D(\lambda; X)$, we will use the language of multisets (sets with integer multiplicities, positive or negative). For $\mathcal{E}$ a set, a multiset E in $\mathcal{E}$ is simply a function $m_E \colon \mathcal{E} \to \mathbb{Z}$, its multiplicity function. We will only use multisets whose multiplicity function has finite support. For e in $\mathcal{E}$, the characteristic function of $\{e\}$ is noted $\delta[e]$. Multisets can be added, and $E = \sum m_E(e)\delta[e]$.

Let $(\ ,\ )$ be the Weyl group invariant symmetric bilinear form on the E_7 root system for which roots have square length 2. It induces a bilinear form $(\ ,\ )$ on V_r, and hence on the root systems A_1, A_1^3, C_3, A_5, D_6. For those of rank $\geqslant r$, define

$$\begin{aligned} W(G^0, \lambda) = {} & (\text{multiset of the } (\lambda+\rho, \alpha), \text{ for } \alpha \text{ a positive root}) \\ & - (\text{multiset of the } (\rho, \alpha), \text{ for } \alpha \text{ a positive root}). \end{aligned}$$

In this definition, ρ is the sum of fundamental weights of G^0 and the multiplicity $m(x)$ of a rational number x in $W(G^0, \lambda)$ is the number of positive roots such that $(\lambda + \rho, \alpha) = x$, minus the number of positive roots such that $(\rho, \alpha) = x$. Weyl's dimension formula says

$$\dim V\big(G^0, \lambda\big) = \prod x \quad \big(x \text{ in } W\big(G^0, \lambda\big)\big) := \prod x^{m(x)}.$$

It is not only the integers $\dim V(G, \lambda)$ which we will interpolate from group to group, but also the multisets $W(G^0, \lambda)$. Define $W'(G^0, \lambda)$ by its multiplicity function

$$m'(x) := m(x) - m(x+1). \tag{3}$$

One has $m(x)=\sum m'(x+n)$ (sum over $n \geqslant 0$). We will exhibit a multiset $\mathcal{W}'$ of affine linear forms in the coordinates p_i ($i \leqslant 6$) of $\lambda=\sum p_i\Omega_i$, and an indeterminate X, such that each $W'(G^0,\lambda)$ is deduced from $\mathcal{W}'$ by specialization at λ and $\nu(G)$. Further, for a given dominant weight $\lambda=\sum p_i\Omega_i$, if we specialize $\mathcal{W}'$ at λ to obtain a multiset $\mathcal{W}'_\lambda$ of linear forms in X, $\mathcal{W}'_\lambda$ is derived by (3) from a finite multiset $\mathcal{W}_\lambda$, and

$$D(\lambda,X)=\prod w \quad (w \text{ in } \mathcal{W}_\lambda).$$

As in [3], the discrepancy between $\dim V(G^0,\lambda)$ and $\dim V(G,\lambda)$ is accounted for by true values of factors $0/0$ (see example below).

Let $\mathcal{W}'_{\mathrm{mv}}$ be the sum of the following contributions ("mv" is for "moving": linear forms actually depending on λ). A positive root $\alpha=\sum a_i\alpha(i)$ of E_7 fixed by $s(7)$ contributes

$$\delta\Big[a_7X+n+\sum a_ip_i\Big]-\delta\Big[a_7X+n-1+\sum a_ip_i\Big],$$

with $5a_7+n=\sum_{i\leqslant 7}a_i$. A pair of positive roots exchanged by $s(7)$, $\alpha=\sum a_i\alpha(i)$ and $\alpha'=\alpha-\alpha(7)=\sum a'_i\alpha(i)$, contributes

$$\delta\Big[a_7X+n+\sum a_ip_i\Big]-\delta\Big[a'_7X+n'-1+\sum a'_ip_i\Big],$$

with $5a_7+n=\sum_{i\leqslant 7}a_i$ and $5a'_7+n'=\sum_{i\leqslant 7}a'_i$. One has $a'_7=a_7-1$, $a'_i=a_i$ for $i\leqslant 6$ and $n'-1=n+3$.

Finally one obtains $\mathcal{W}'$ by adding to $\mathcal{W}'_{\mathrm{mv}}$ a sum of $\delta[BX+n]$, such that $\mathcal{W}'$ specialized at $\lambda=0$ is the empty multiset of linear forms $BX+n$. It follows that

$$D(\lambda,X)=\prod\Big[(BX+n+1)\dots\Big(BX+n+\sum a_ip_i\Big)\Big],$$

the product being over $\mathcal{W}'_{\mathrm{mv}}$. Example: for $\lambda=p\Omega_1$,

$$\dim V(G,p\Omega_1)=\frac{(4X-3+1)\dots(4X-3+p).(3X-2+1)\dots(3X-2+p).(2X+p-1)}{(X-1+1)\dots(X-1+p).(1)\dots(p).(2X-1)}$$

evaluated at $X=\nu(G)$. For $p=1$, we obtain the dimension of the preferred representation:

$$\dim V=\frac{(4X-2)(3X-1)(2X)}{X.1.(2X-1)}$$

evaluated at $X=\nu(G)$. For $G=(\mathbb{G}_m/\mu_3).2$, one has $\nu:=\nu(G)=1/2$; the numbers $4\nu-2$ and $2\nu-1$, $3\nu-1$ and ν, as well as 2ν and 1 cancel to give $W(\mathbb{G}_m,\Omega_1)=\emptyset$, and $4X-2/2X-1$ ($=0/0$ at $X=1/2$) contributes the factor 2 in the dimension 2.1 of V.

We expect a similar story for row 6, if the vertices of E_6 are labelled

$$\begin{matrix}1 & 4 & 6 & 5 & 2\\ & & 3 & & \end{matrix}$$

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