Nonvanishing of L-functions on $\Re(s) = 1$.

by

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To: J. Shalika on his 60^{th} birthday.

Abstract

In [Ja-Sh], Jacquet and Shalika use the spectral theory of Eisenstein series to establish a new result concerning the nonvanishing of L-functions on $\Re(s) = 1$. Specifically they show that the standard L-function $L(s, \pi)$ of an automorphic cusp form π on GL_m is nonzero for $\Re(s) = 1$. We analyze this method, make it effective and also compare it with the more standard methods. This note is based on the letter [Sa1].

§1. Review of de la Vallée Poussin's method

A celebrated result of Hadamard and de la Vallée Poussin is the Prime Number Theorem. Their proof involved showing that the Riemann zeta function $\zeta(s)$ is not zero for $\Re(s) = 1$. In fact these two results turn out to be equivalent. de la Vallée Poussin (1899) extended this method to give a zero free region for $\zeta(s)$ of the form;

$$\zeta(s) \neq 0 \text{ for } \sigma \geq 1 - \frac{c}{\log(|t| + 2)} \,. \tag{1}$$

Here c is an absolute positive constant and $s = \sigma + it$. We will call a zero free region of the type (1), a standard zero free region.

Poussin's method is based on the construction of an auxillary L-function, $D(s)$ with positive coefficients. $D(s)$ should be analytic in $\Re(s) > 1$, have a pole at $s = 1$ of order say k and if $L(\sigma + it_0) = 0$,* $D(s)$ should vanish to order at least k at $s = \sigma$. † This is enough to ensure that $L(1 + it_0) \neq 0$ and if the order of vanishing at σ is bigger than k then one obtains an effective standard zero free region for $L(s)$. To arrange for $D(s)$ to have positive coefficients one often uses

*Here $L(s)$ stands for a generic L-function whose nonvanishing we seek to establish.

†Since the coefficients of $D(s)$ are positive, the Euler product for $D(s)$ converges absolutely for $\Re(s) > 1$.

a positive definite function on an appropriate group. For example $L(s,\pi\times\tilde{\pi})$, where π is any unitary isobaric representation of GL_m (see for example [Ho-Ra] for definitions and examples) has this property. See also [De] for such positive definite functions on other groups.

In the most basic case of $\zeta(s)$ and $t_0\in\mathbb{R}$ with $|t_0|\geq 2$, one can take

$$\begin{aligned} D(s) &= \zeta^3(s)\,\zeta^2(s+it_0)\,\zeta^2(s-it_0)\,\zeta(s+2it_0)\,\zeta(s-2it_0) \\ &= L(s,\Pi\times\widetilde{\Pi})\,. \end{aligned} \tag{2}$$

Here $\Pi = \mathbf{1}\boxplus\alpha^{-it_0}\boxplus\alpha^{it_0}$ is an isobaric representation of GL_3 and α the principal quasi-character of $\mathbb{A}^*_{\mathbb{Q}}$. $D(s)$ has a pole of order 3 at $s=1$ and if $\zeta(\sigma+it_0)=0$ then $D(s)$ will have a zero of order 4 at $s=\sigma$. Hence, by a standard function theoretic argument (see [Ho-Ra] for example) we have that $\zeta(\sigma+it_0)\neq 0$ for $\sigma\geq 1-\frac{c}{\log|t_0|}$. This establishes the standard zero free region (1) for $\zeta(s)$.

We note that (1) is not the best zero free region that is known for $\zeta(s)$. Vinogradov [Vi] and his school have developed sophisticated techniques which lead to zero free regions of the type; $\zeta(s)\neq 0$ for $\sigma\geq 1-c_\alpha/(\log(|t|+2))^\alpha$ for $\alpha>\frac{2}{3}$ and $c_\alpha>0$.

Another well-known example of nonvanishing is that of a Dirichlet L-function $L(s,\chi)$, with χ a quadratic character of conductor q. For $D(s)$ we can take $\zeta(s)L(s,\chi)$ (or if one prefers $(\zeta(s)L(s,\chi))^2 = L(s,(\mathbf{1}\boxplus\chi)\times\widetilde{(\mathbf{1}\boxplus\chi)})$. In this case the order of zero at $s=1$ is equal to the order of pole. Hence $L(1,\chi)\neq 0$ (see Landau's Lemma [Da , pp34]) but this does not yield a standard zero free region near $s=1$ for $L(s,\chi)$ - ie in terms of the conductor.[‡] In fact no such zero free region is known for $L(s,\chi)$ - this being the notorious problem of the exceptional, or "Landau-Siegel", zero. In this note we will only be concerned with zero free regions for a fixed L-function (ie what is called the t-aspect). For a recent discussion of the exceptional zero problem in general, see the paper [Ho-Ra].

The Poussin method generalizes to automorphic L-functions. Let K be a number field, $m\geq 1$ and let π be an automorphic cusp form on $GL_m(\mathbb{A}_k)$. The standard (finite part) L-function associated with π namely $L(s,\pi)$, has an analytic continuation to $\mathbb{C}$ and a functional equation $s\longrightarrow 1-s$, $\pi\longrightarrow\tilde{\pi}$ [Go-Ja]. Also well-known by now are the analytic properties (ie continuation and functional equation) of the Rankin-Selberg L-functions $L(s,\pi\times\pi')$, where π and π' are cusp forms on $GL_m(\mathbb{A}_k)$ and $GL_{m'}(\mathbb{A}_k)$ respectively. This follows from [Ja -PS-Sh],[Sh1], [Mo-Wa].

[‡]That is to say $L(\sigma,\chi)\neq 0$ for $\sigma\geq 1-\frac{c}{\log q}$ and some $c>0$.

Apply Poussin's method with

$$\begin{aligned} D(s) &= \zeta(s)\, L^2(s,\pi\times\tilde{\pi}) L^2(s+it_0,\pi)\, L^2(s-it_0,\tilde{\pi})\,\cdot \\ &\qquad\qquad\qquad\qquad L(s+2it_0,\pi\times\pi)\, L(s-2it_0,\tilde{\pi}\times\tilde{\pi}) \qquad (3) \\ &= L(s,\Pi\times\widetilde{\Pi}) \end{aligned}$$

where $\Pi := \mathbf{1} \boxplus \pi \otimes \alpha^{it_0} \boxplus \tilde{\pi} \otimes \alpha^{-it_0}$, (we are tacitly assuming that we have normalized π so that $\tilde{\pi} \neq \pi \otimes \alpha^{\pm 2it_0}$).

This yields a standard zero free region for $L(s,\pi)$ - that is a zero free region as in (1) but with $c = c(\pi)$.

(4)

As mentioned in the abstract, the nonvanishing of such an $L(s,\pi)$ for $\sigma = 1$ was first established (before the Rankin-Selberg theory was developed) by Jacquet-Shalika who used the method discussed in Section 2. The advantage of Poussin's method here is that it yields a standard zero free region (4). According to the general functoriality conjectures of Langlands, any automorphic L-function should be a (finite) product of such standard L-functions. If so, we would be in good shape at least in that the zero free region for the general L-function would be of the same quality as for $\zeta(s)$.

One can apply Poussin's method to certain Rankin-Selberg L-functions. If $\pi \neq \pi'$, take for D,

$$D(s) \,=\, L(s,\pi\times\tilde{\pi})\, L(s,\pi'\times\tilde{\pi}')\, L(s+it_0,\pi\times\pi')\, L(s-it_0,\tilde{\pi}\times\tilde{\pi}')\,. \qquad (5)$$

From this it follows that $L(1+it_0,\pi\times\pi') \neq 0$. This nonvanishing result was established in this way by Ogg [Og] for $m = 2 = m'$. The general case of any m and m' was first proven by Shahidi [Sh1] using the Eisenstein series method discussed in Section 2. If π and π' are self-dual, Moreno [Mo] established a standard zero free region for $L(s,\pi\times\pi')$. To see this, one can take $D(s) = L(s,\Pi\times\widetilde{\Pi})$ where

$$\Pi \,=\, \pi \boxplus \pi \otimes \alpha^{it_0} \boxplus \pi \otimes \alpha^{-it_0} \boxplus \pi' \boxplus \pi' \otimes \alpha^{it_0} \boxplus \pi' \otimes \alpha^{-it_0}\,. \qquad (6)$$

In this case, $D(s)$ has a pole of order 6 at $s = 1$ and a zero of order 8 at $s = \sigma$ if $L(\sigma+it,\pi\times\pi') = 0$.

These L-functions $L(s,\pi)$ and $L(s,\pi\times\pi')$ and any others that can be expressed in terms of these by known cases of functoriality are more or less all that can be handled by Poussin's method. We turn now to the Eisenstein series method.

§2. Nonvanishing via Eisenstein Series

This method is based on the spectral theory of locally homogeneous spaces and in particular Eisenstein series. In as much as this method works effectively in all cases where the methods of Section 1 apply as well as in many other cases, it must at least at the present time, be considered a principal method. There have been a number of suggestions as well as evidence for useful spectral interpretations of the zeroes of L-functions [Od],[Ka-Sa],[Co], [Za]. However it does not seem to be widely appreciated that the spectral interpretation of the zeroes of L-functions through poles of Eisenstein series (ie as resonances) has already proven to be very powerful.

To illustrate this, consider the symmetric power L-functions $L(s, \pi, \mathrm{sym}^{\mathrm{k}})$, where π is a cusp form on $GL_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character and $k \geq 1$ (see [Sh2] for definitions). The recently established functiorial lifts, $\mathrm{sym}^3 : \mathrm{GL}_2 \longrightarrow \mathrm{GL}_4$ and $\mathrm{sym}^4 : \mathrm{GL}_2 \longrightarrow \mathrm{GL}_5$ due to Kim and Shahidi [K-S1],[Ki] (see also [He]) allow one to study $L(s, \pi, \mathrm{sym}^{\mathrm{k}})$ for $1 \leq k \leq 8$. By decomposing $L(s, \mathrm{sym}^{\mathrm{i}}\pi \times \mathrm{sym}^{\mathrm{j}}\pi)$ $1 \leq i \leq 4$, $1 \leq j \leq 4$ into a product of primitive L-functions (see [K - S2]) and using (6) of Section 1 one sees that for $1 \leq k \leq 8$, $L(s, \pi, \mathrm{sym}^{\mathrm{k}})$ satisfies a standard zero free region. However, the next symmetric power $L(s, \pi, \mathrm{sym}^9)$ falls outside of the range of this approach. It is not known whether the Euler product for $L(s, \pi, \mathrm{sym}^9)$ converges absolutely for $\Re(s) > 1$. In particular $L(s, \pi, \mathrm{sym}^9)$ might even have zeroes in $\Re(s) > 1$! Thus an application of Poussin's method appears to be problematic. Concerning the absolute convergence, if we use the best bounds known towards the Ramanujan Conjectures [Ki-Sa] one sees that $L(s, \pi, \mathrm{sym}^9)$ converges absolutely for $\Re(s) > \frac{71}{64}$. Given the above comments, it is remarkable that the theory of Eisenstein series on E_8 together with the Langlands-Shahidi (see [Mi] for a recent summary and outline) method allows one to show that (see [K-S2]): $L(s, \pi, \mathrm{sym}^9)$ is meromorphic in the plane and is analytic and nonvanishing in $\Re(s) \geq 1$ except possibly for a finite number of simple zeros or poles in $\left[1, \frac{71}{64}\right]$. The same arguments give the analyticity and nonvanishing in $\{s | \Re(s) \geq 1\} \backslash [1, \infty)$, for a quite general class of L-functions - see [Ge-Sh]. Given the success of this technique the question arises as to what zero free region it yields. The proof of nonvanishing, though very simple (even magical) is quite indirect and it is not clear how to make it effective so as to yield zero free regions. We show below that in the simplest case of $\zeta(s)$, one can with some effort make the proof effective and it gives zero free regions which are almost as good as the standard zero free regions - see (53) below. At the end of this section we indicate how one might proceed in the general case.

In order to deal with $\zeta(s)$ we consider the Eisenstein series for the modular quotient X of the upper half plane $\mathbb{H}$, that is, $X = SL(2,\mathbb{Z})\backslash\mathbb{H}$. It is instructive in this analysis to consider more general Fuchsian groups $\Gamma \leq SL(2,\mathbb{R})$ for this allows us to separate the arithmetic and analytic features. So assume that $\Gamma\backslash\mathbb{H}$ has one cusp at infinity and it is normalized so that the stabilizer of infinity is $\Gamma_\infty = \left\{\pm\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) | m \in \mathbb{Z}\right\}$.

The corresponding Eisenstein series is defined by

$$E_\Gamma(z,s) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (y(\gamma z))^s\,,\ \text{for } \Re(s)\;>\;1 \tag{1}$$

and $z \;=\; x+iy \;\in\; \mathbb{H}$.

The spectral theory of Eisenstein series due to Selberg in this setting [Se1] and Langlands in general [La], asserts that $E_\Gamma(z,s)$ is meromorphic in s. Moreover, it is analytic in $\Re(s) \geq \frac{1}{2}$ except possibly for simple poles in $(\frac{1}{2},1]$. These general properties when applied to $\Gamma = SL(2,\mathbb{Z})$ imply that $\zeta(s) \neq 0$ for $\Re(s) = 1$.

One can formulate this in a very explicit way (see(15) below) which I learned from a lecture of Selberg (at Stanford $\pm$ 1980). Let $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$ be set of representative for these double cosets for such a $\Gamma \leq PSL(2,\mathbb{R})$. The set of c appearing in $\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right] \in \Gamma_\infty\backslash\Gamma/\Gamma_\infty$ can be chosen so that $c \geq 0$ and $c = 0$ corresponds to the identity coset. They form a discrete subset in $[0,\infty)$. For $m \in \mathbb{Z}$ and $c > 0$ as above set

$$r_{m,\Gamma}(c) \;=\; \sum_{\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty} e\left(\frac{md}{c}\right)\,. \tag{2}$$

Here $e(z) = e^{2\pi iz}$ and the sum is easily seen to be well defined. A standard calculation [Ku] shows that the $m-^{\text{th}}$ coefficient of $E_\Gamma(z,s)$, $\int_0^1 E_\Gamma(z,s)\,e(-mx)\,dx$, is given by

$$\begin{aligned} & y^s \;+\; \left(\sum_{c>o}\frac{r_{0,\Gamma}(c)}{c^{2s}}\right)\,\tfrac{\pi^{1/2}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,y^{1-s}\,,\ \text{if } m=0 \\ & := \; y^s \;+\; \phi_\Gamma(s)\,y^{1-s}\,, \end{aligned} \tag{3}$$

and

$$=\;\left(\sum_{c>0}\frac{r_{m,\Gamma}(c)}{c^{2s}}\right)\,\frac{2\pi^s|m|^{s-\frac{1}{2}}y^{1/2}\,K_{s-\frac{1}{2}}(2\pi|m|y)}{\Gamma(s)}\,,\ \text{if } m\neq 0\,. \tag{4}$$

Using (4) and the theory of Eisenstein series we can meromorphically continue the functions $D_m(s)$, where

$$D_m(s) \;:=\; \sum_{c>0}\frac{r_{m,\Gamma}(c)}{c^{2s}}\,. \tag{5}$$

In order to give growth bounds on $D_m(s)$ for $\Re(s) \geq \frac{1}{2}$ we use the technique in [Go-Sa]. For $m > 0$ set

$$P_m(z,s) \;=\; \sum_{\gamma \in \Gamma\infty \backslash \Gamma} (y(\gamma z))^s \, e(m\gamma z)\,. \tag{6}$$

This series converges absolutely for $\Re(s) > 1$ and for $\Re(s) \geq \frac{3}{2}$ we have the bound

$$P_m(z,s) \ll y^{1-\sigma} \;\text{ for } y \geq \frac{1}{2}\,. \tag{7}$$

Hence for $\sigma \geq \frac{1}{2}$ we may form

$$\begin{aligned} &\langle E_\Gamma(\cdot, s)\,,\; P_m(\cdot\,, \bar{s}\,+\,1)\rangle \\ &=\; \int_{\Gamma\backslash\mathbb{H}} E_\Gamma(z,s)\,\overline{P_m(z,\bar{s}+1)}\,\frac{dx\,dy}{y^2} \\ &=\; a\,b^s\,\tfrac{\Gamma(2s)}{\Gamma(s)\,\Gamma(s+1)}\;D_m(s)\,, \end{aligned} \tag{8}$$

for suitable constants a and b (depending on m).

We assume that $E_\Gamma(z,s)$ has no poles in $\left(\frac{1}{2}\,,1\right)$. For a given Γ this can be checked and anyway if such poles are present, they will enter in the asymptotics below in an explicit way. From (8) and the properties of Eisenstein series we see that $D_m(s)$ is holomorphic in $\Re(s) \geq \frac{1}{2}$ (there being no pole at $s = 1$ since $\langle 1, P_m\rangle = 0$). Also from (8) and Stirling's series we see that for $\Re(s) = \frac{1}{2}$ and $|t|$ large,

$$|D_m(s)| \;\sim\; |t|^{1/2}|<\; E(\cdot\,,s)\,, P_m(\cdot\,,\bar{s}+1)\rangle|\,. \tag{9}$$

For T large and $T \leq t \leq T+1$ write

$$P_m\left(\frac{3}{2}\,+it\right) \;=\; P_m\left(\frac{3}{2}\,+\,iT\right)\,+\,\frac{1}{i}\int_T^t P_m'\left(\frac{3}{2}\,+\,i\,\tau\right)\,d\tau\,.$$

Hence

$$\begin{aligned} &\int_T^{T+1}\left|\left\langle E\left(\frac{1}{2}+it\right)\,,P_m\left(\frac{3}{2}-it\right)\right\rangle\right|\,dt \\ &\quad\leq\; \int_T^{T+1}\left\{\left|\left\langle E\left(\frac{1}{2}+it\right)\,,P_m\left(\frac{3}{2}\;-iT\right)\right\rangle\right|\right. \\ &\qquad\qquad \left.+\int_T^t\left|\left\langle E\left(\frac{1}{2}+it\right)\,,P_m'\left(\frac{3}{2}-i\tau\right)\right\rangle\right|\,d\tau\right\}dt \end{aligned} \tag{10}$$

The spectral theory of $L^2(\Gamma\backslash\mathbb{H})$ and in particular Bessel's inequality yields

$$\int_{-\infty}^{\infty} \left| \left\langle E\left(\frac{1}{2}+it\right), P_m'\left(\frac{3}{2}-i\tau\right)\right\rangle \right|^2 dt \leq \left\| P_m'\left(\frac{3}{2}-i\tau\right)\right\|_2^2 \ll 1\,. \tag{11}$$

Putting this in (10) and applying Cauchy's inequality yields,

$$\int_{T}^{T+1} \left| \left\langle E\left(\frac{1}{2}+it\right), P_m\left(\frac{3}{2}-it\right)\right\rangle \right| dt \ll 1\,.$$

Combined with (9) we get

$$\int_{T}^{T+1} \left| D_m\left(\frac{1}{2}+it\right)\right| dt \ll T^{1/2}\,. \tag{12}$$

Equipped with (12) we return to (5) and apply Perron's formula. For $x > 0$

$$\sum_{c\leq x} \left(1-\frac{c}{x}\right) r_{m,\Gamma}(c) \;=\; \frac{1}{2\pi i}\int_{\Re(s)=2} D_m(s)\, x^{2s}\, \frac{ds}{s(2s+1)} \tag{13}$$

Now shift the contour in the last integral to $\Re(s) = \frac{1}{2}$. The estimate (12) easily justifies this shift and moreover we don't pick up any poles. Thus

$$\sum_{c\leq x} \left(1-\frac{c}{x}\right) r_{m,\Gamma}(c) \;=\; \frac{x}{2\pi}\int_{-\infty}^{\infty} \frac{D_m\left(\frac{1}{2}+it\right) e^{it\log x}}{\left(\frac{1}{2}+it\right)(1+2it)}\, dt \tag{14}$$

(12) ensures that the integral in (14) is absolutely convergent. Hence by the Riemann-Lebesgue lemma we conclude that as $x \longrightarrow \infty$

$$\sum_{c\leq x} \left(1-\frac{c}{x}\right) r_{m,\Gamma}(c) \;=\; o(x)\,. \tag{15}$$

This general result holds for any Γ for which $E(z,s)$ has no poles in $(\frac{1}{2},1)$. In particular it holds if $\lambda_1(\Gamma\backslash\mathbb{H}) \geq \frac{1}{4}$ where λ_1 is the second smallest eigenvalue of the Laplacian on $\Gamma\backslash\mathbb{H}$. For $SL(2,\mathbb{Z})\backslash\mathbb{H}$ as well as other low genus Riemann surfaces one can use geometric methods, see for example [S2, pp34], to show that $\lambda_1 \geq \frac{1}{4}$. Also, for $\Gamma = SL(2,\mathbb{Z})$ the c's run over integers and the sums $r_{m,SL(2,\mathbb{Z})}(c)$ are Ramanujan sums which may be evaluated explicitly in terms of the Mobius function μ.

$$r_{m,SL(2,\mathbb{Z})}(c) \;=\; \sum_{\substack{d|m\\ d|c}} \mu\left(\frac{c}{d}\right) d\,. \tag{16}$$

Thus for $m = 1$ and $\Gamma = SL(2,\mathbb{Z})$, (15) asserts that

$$M_1(x) \;:=\; \sum_{c \le x} \left(1 - \frac{c}{x}\right) \mu(c) \;=\; o(x)\,. \tag{17}$$

Now (17) is elementarily equivalent to the prime number theorem. Thus the above spectral analysis provides us with a nonarithmetic setting in which (15), a form of the prime number theorem, is valid in a family (the Beurling theory of generalized primes [Be] provides another such setting). However the standard zero free region for $\zeta(s)$ does not persist in the family and is apparently a rigid feature. To see this note that a zero free region is equivalent to a rate of decay in (17) or (15) and that this in turn is equivalent to a pole free region in $\beta < \frac{1}{2}$ for the poles $\rho = \beta + i\gamma$ of $D_m(s)$. From (8) this amounts to such pole free regions for $E_\Gamma(z,s)$ and according to the theory of Eisenstein series these poles occur at poles of $\phi_\Gamma(s)$. We apply the theory [P-S1], [P-S2] and see also [Wo1] and [Wo2], concerning the behavior of such poles under deformations of Γ. A consequence of the theory is that if the critical values of certain L-functions are nonzero then the corresponding eigenvalues of the Laplacian are dissolved into poles of Eisenstein series. For example using the recent nonvanishing results of Luo [Lu] for such critical values of Rankin-Selberg L-functions and assuming a suitable form of a standard multiplicity bound conjecture for the eigenvalues of the Laplacian on a congruence quotient of $\mathbb{H}$, we have that for the generic Γ:

> there is $c_\Gamma > 0$ such that the number of poles $\rho = \beta + i\gamma$ of $\phi_\Gamma(s)$ with $|\gamma| \le T$ and $0 \le \beta < \frac{1}{2}$ is at least $c_\Gamma T^2$.

(18)

On the other hand Selberg [Se2] has shown that for any Γ

$$\sum_{\substack{\rho \\ |\gamma| \le T \\ \beta < \frac{1}{2}}} \left(\frac{1}{2} - \beta\right) \;=\; O(T \log T)\,. \tag{19}$$

Combining (18) and (19) we see that for generic Γ there is a sequence of poles $\rho_j = \beta_j + i\gamma_j$ of $\phi_\Gamma(s)$ with

$$\frac{1}{2} \;>\; \beta_j \;>\; \frac{1}{2} - \frac{c'_\Gamma \log|\gamma_j|}{|\gamma_j|}\,. \tag{20}$$

In the case of $\Gamma = SL(2,\mathbb{Z})$, according to (2),(3) and (16) we have

$$\phi_{SL(2,\mathbb{Z})}(s) \;=\; \frac{\sqrt{\pi}\,\Gamma(s - \frac{1}{2})}{\Gamma(s)}\,\frac{\zeta(2s-1)}{\zeta(2s)} \tag{21}$$

and the $m-^{\text{th}}$ coefficient of $E(z,s)$ is

$$\frac{2\pi^s|m|^{s-1/2}\,y^{1/2}\,K_{s-\frac{1}{2}}\,(2\pi|m|y)}{\Gamma(s)\,\zeta(2s)}\,\sigma_{1-2s}\,(|m|) \tag{22}$$

where

$$\sigma_s(m)\;=\;\sum_{d|m} d^s\,. \tag{23}$$

Hence the standard zero free region for $\zeta(s)$ implies the pole free region for $SL_2(\mathbb{Z})$ of the form;

$$\beta\;\le\;\frac{1}{2}\;-\;\frac{c}{\log(|\gamma|+2)}\,. \tag{24}$$

Thus while the analogue of the nonvanishing of $\zeta(s)$ on $\sigma=1$ is valid for general Γ, the pole free region (24) is not. In particular this shows that one cannot apply this Eisenstein series method of showing nonvanishing of L-functions on $\sigma=1$ to get zero free regions at least if all one uses is the general spectral theory.

The key to effectivizing the nonvanishing proof in the case that the Fourier coefficients of $E(\cdot\,,s)$ along unipotents of parabolics are ratios of L-functions (as is the case in the Langlands-Shahidi method see [Ge-Sh]) is to exploit the inhomogeneous form of the Maass-Selberg-Langlands relation. For the rest we stick to the case $\Gamma=SL(2,\mathbb{Z})$. It is perhaps worth pointing out the simple magical and standard derivation of the nonvanishing of $\zeta(s)$ for $\sigma=1$ using $E(z,s)$. $E(z,s)$ is analytic for $\Re(s)=\frac{1}{2}$, thus if we look at (22) with $m\neq 0$ we see that such a coefficient is analytic for $\Re(s)=\frac{1}{2}$. Clearly, consideration of the denominator shows that $\zeta(1+2it_0)$ cannot be zero.

To continue our quantitative analysis we will use freely the analytic properties of $\zeta(s)$- that is the location of poles and the functional equation (all of which are essentially known for the general L-function that can be continued by the theory of Eisenstein series). We also use freely the asymptotic properties of the Whittaker functions $K_s(y)$ as well as the Gamma function.

The Maass-Selberg relation for $SL(2,\mathbb{Z})$ reads:

$$\int_X \left|E_A\left(z,\,\frac{1}{2}+it\right)\right|^2\,\frac{dxdy}{y^2}$$

$$=\;2\log A\;-\;\frac{\phi'}{\phi}\left(\frac{1}{2}+it\right)\;+\;\frac{\bar{\phi}\left(\frac{1}{2}+it\right)A^{2it}\;-\phi\left(\frac{1}{2}+it\right)A^{-2it}}{2it} \tag{25}$$

where $A \geq 1$ and where for $z \in \mathcal{F}$ the standard fundamental domain,

$$E_A(z,s) \; = \begin{cases} E(z,s)\,, & \text{for} \quad y \, \leq A \\ \\ E(z,s) - y^s \, - \phi(s) y^{1-s}\,, & \text{for} \quad y \, > \, A\,. \end{cases} \tag{26}$$

Normalizing the denominator in $\phi(s)$ (see (21)) gives

$$\int_X \left|\zeta\left(1+2it\right)\right|^2 \left|E_A\left(z,\, \frac{1}{2}\, + it\right)\right|^2 \frac{dxdy}{y^2}$$

$$= \; |\zeta(1+2it)|^2 \left| 2\log A \; - \frac{\phi'}{\phi}\left(\frac{1}{2} + it\right) \; + \; \frac{\overline{\phi\left(\frac{1}{2}+it\right)} A^{2it} \; - \phi\left(\frac{1}{2}+it\right)\; A^{-2it}}{2it} \right| \tag{27}$$

From (22), Parseval's inequality and the shape of $\mathcal{F}$ we have that

$$LHS \text{ of } (27) \; \gg \; \sum_{m=1}^{\infty} \int_1^{\infty} \left| \frac{K_{it}(2\pi|m|y)\, \sigma_{-2it}(m)}{\Gamma\left(\frac{1}{2}+it\right)} \right|^2 \frac{dy}{y} \tag{28}$$

On the other hand, (21) together with Stirling for $\frac{\Gamma'}{\Gamma}(s)$ and the functional equation for $\zeta(s)$ shows that for $t \geq 2$

$$\left|\zeta(1+2it)\,\frac{\phi'}{\phi}\,\left(\frac{1}{2}+it\right)\right| \ll \log\, t + |\zeta(1+2it)| \, + \, |\zeta'(1+2it)|\;. \tag{29}$$

It is elementary (and such upper bounds for L-functions can be derived generally) that for $t \geq 2$

$$\begin{aligned} |\zeta(1+2it)\,| \;\; &\ll \;\; \log\, t \\ |\zeta'(1+2it)| \;\; &\ll \;\; (\log\, t)^2\,. \end{aligned} \tag{30}$$

Hence

$$\left|\zeta(1+2it)\,\frac{\phi'}{\phi}\,\left(\frac{1}{2}+it\right)\right| \ll (\log\, t)^2 \tag{31}$$

Assuming that $|\zeta(1+2it)\,| \;\leq 1$, which we can in estimating this quantity from below; we have from (28) and (29) that

$$\sum_{m=1}^{\infty} |\sigma_{-2it}(m)|^2 \int_m^{\infty} \left|\frac{K_{it}(2\pi y)}{\Gamma\left(\frac{1}{2}+it\right)}\right|^2 \frac{dy}{y} \;\ll |\zeta(1+2it)|\,(\log\, t)^2 \tag{32}$$

Using the asymptotics [E page 87-88]; we have that for $y \; < \; \frac{t}{4}$

$$K_{it}(y) \;\sim\; \frac{e^{\frac{-\pi}{4}t}\sqrt{2\pi}}{\sqrt[4]{t^2-y^2}}\,\sin\left[\frac{\pi}{4}\; + \, th\,(y/t)\right]\,, \tag{33}$$

with h a fixed smooth function.

Hence

$$\int_{t/8}^{t/4} \left| \frac{K_{it}\left(y2\pi\right)}{\Gamma\left(\frac{1}{2}+it\right)} \right|^2 \frac{dy}{y} \gg \frac{1}{t}\,, \tag{34}$$

and so for $m \leq t/8$ we have

$$\int_{m}^{\infty} \left| \frac{K_{it}(2\pi y)}{\Gamma\left(\frac{1}{2}+it\right)} \right|^2 \frac{dy}{y} \gg \frac{1}{t}\,. \tag{35}$$

Applying (32) with $m = 1$ and using (35) and $|\sigma_{-2it}(1)| = 1$, we obtain

$$|\zeta(1+2it)| \gg \frac{1}{t(\log\, t)^2}\,. \tag{36}$$

We can do better by using many m's in (32). For example note that for $m = p$ a prime

$$|\sigma_{-2it}(p) - \sigma_{-2it}\left(p^2\right)| = 1\,.$$

Hence

$$|\sigma_{-2it}\left(p^2\right)|^2 + |\sigma_{-2it}\left(p\right)|^2 \gg 1\,. \tag{37}$$

Thus

$$\sum_{m \leq t/8} |\sigma_{-2it}(m)|^2 \gg \sum_{p \leq \frac{\sqrt{t}}{8}} 1\,.$$

The last sum is by elementary means (Chebyshev) $\gg \sqrt{t}\big/\log\, t$. Combining this with (32) and (35) yields

$$|\zeta(1+2it)| \gg \frac{1}{(\log\, t)^3 \sqrt{t}}\,. \tag{38}$$

To further improve this we examine intervals of primes p where $|1+p^{it2}|$ is small, ie where $\sigma_{+2it}(p)$ is small. To proceed we need more flexibility on the range of m's in (32). Set $\eta = t^{-\delta}$ with $\delta > 0$ (eg $\delta = 1$ will work) and consider instead of (27), the quantity

$$I = \int_{\eta}^{\infty} \int_{0}^{1} |\zeta(1+2it)|^2\, |E_A(z, \frac{1}{2}+it)|^2 \frac{dxdy}{y^2}\,. \tag{39}$$

If $N(z,\eta) = |\{\gamma \in \Gamma_\infty\backslash\Gamma |\, y(\gamma z) \geq \eta\}|$ then it is not difficult to see (Iwaniec [Iw] page 54) that for $\eta \leq 1$

$$N(z,\eta) \ll \frac{1}{\eta}\,. \tag{40}$$

Hence

$$I = \int_{\mathcal{F}} N(z,\eta) \left| E_A(z, \frac{1}{2}+it) \right|^2 |\zeta(1+2it)|^2 \frac{dxdy}{y^2}$$

$$\ll \frac{1}{\eta} \int_{\mathcal{F}} |\zeta(1+2it)|^2 \left| E_A(z, \frac{1}{2}+it) \right|^2 \frac{dxdy}{y^2} . \tag{41}$$

By (27) and (29) this gives

$$I \ll \frac{1}{\eta} |\zeta(1+2it)| \left[(\log t)^2 + 2\log A \right] . \tag{42}$$

Now if $\frac{1}{A} = \eta$ then we see that the nonzero Fourier coefficients of $E_A(z,s)$ coincide with those of $E(z,s)$, for $y \geq \eta$. So as in the discussion following (27) we deduce that

$$I \gg \sum_{m=1}^{\infty} |\sigma_{-2it}(m)|^2 \int_{\eta m}^{\infty} \left| \frac{K_{it}(2\pi y)}{\Gamma\left(\frac{1}{2}+it\right)} \right|^2 \frac{dy}{y}$$

$$\gg \frac{1}{t} \sum_{m \leq t/4\eta} |\sigma_{-2it}(m)|^2 . \tag{43}$$

On the other hand, for $A = 1/\eta$, $\log A = O(\log t)$ so that (42) and (43) yield

$$\frac{\eta}{t} \sum_{m \leq t/4\eta} |\sigma_{-2it}(m)|^2 \ll (\log t)^2 |\zeta(1+2it)| \tag{44}$$

This gives us the flexibility we need. If $\delta = 1$, ie $\eta = t^{-1}$ we have

$$\frac{1}{t^2} \sum_{\frac{t^2}{8} \leq m \leq \frac{t^2}{4}} |\sigma_{-2it}(m)|^2 \ll |\zeta(1+2it)|(\log t)^2 . \tag{45}$$

To give a lower bound for the left hand side of (45) we restrict m to primes $N \leq p \leq 2N$ with $N = \frac{t^2}{8}$. For integers m with

$$t \log N \leq 2\pi m \leq t \log(2N) \tag{46}$$

let I_m be the interval of p

$$|2t \log p - 2\pi m + \pi| < \frac{1}{100} \tag{47}$$

Note that for $p \notin I_m$

$$|\sigma_{2it}(p)| \gg 1 . \tag{48}$$

The length I_m satisfies

$$|I_m| \leq \frac{N}{50t}\,. \tag{49}$$

A well-known application of sieve theory (which is independent of the zeta function!) see for example [Bo-Da] who use the large sieve, asserts that for $N, M \geq 2$

$$\pi(M+N) - \pi(N) \leq \frac{3M}{\log M}\,, \tag{50}$$

where $\pi(x) = \sum\limits_{p \leq x} 1$.

Hence the number of primes in I_m is at most

$$\frac{3N}{50t\,\log(N/50t)} \leq \frac{3N}{25t\,\log N}\,. \tag{51}$$

The number of intervals I_m is less than t according to the set up (46). Thus the total number of primes in $\bigcup\limits_m I_m$ is at most

$$\frac{3N}{25\,\log N} \tag{52}$$

Again elementary arguments show that the number of primes p satisfying $N \leq p \leq 2N$ is at least $\frac{N}{4\log N}$. Hence there are at least $\frac{N}{8\log N}$ primes $N \leq p \leq 2N$ which are not in any I_m. Thus from (45) and (47) it follows that

$$|\,\zeta(1+2it)\,| \gg \frac{1}{(\log\, t)^3}\,. \tag{53}$$

This is the effective nonvanishing of ζ on $\sigma = 1$ that we sought to establish using $E(z,s)$. It leads immediately to a zero free region of the type; $\zeta(s) \neq 0$ for $\Re(s) > 1 - \frac{c}{(\log\, t)^5}$. This is not quite the standard zero free region but it is of the same general quality.

Note that once we arrive at (45) we are in a similar position to a proof of the nonvanishing of $\zeta(s)$ on $\sigma = 1$ due to Ingham [In]. He uses the identity

$$\sum_{n=1}^{\infty} |\sigma_{it_0}(n)|^2\, n^{-s} = \frac{\zeta^2(s)\,\zeta(s+it_0)\,\zeta(s-it_0)}{\zeta(2s)} \tag{54}$$

Indeed Balasubramanian and Ramachandra [Ba-Ra] in deriving zero free regions from (54) use similar sieving arguments to those used after (45). See also the comments by Heath-Brown on page 68 of [Ti]. Our point is that we arrive at (45) in a geometric way using the Maass-Selberg relation and hence our analysis can be generalized to the Langlands-Shahidi setting.

In that setting one would probably (at least at first) go as far as (36) in the above argument. That is to consider only one nonzero Fourier coefficient $E_\chi(s)$ (in the notation of [Ge-Sh]) of the Eisenstein series. This will require understanding the asymptotic behaviour in t of the archimedian local Whittaker function $W_{it}(e)$. Mckee [Mc] has made the first steps in this direction. We expect that these ideas will lead to effective lower bounds in general of the type;

$$|L_{\text{finite}}\,(1 + it,\, \pi, r)\,| \gg |t|^{-\delta} \tag{55}$$

for a constant δ depending the group on which the Eisenstein series lives. This falls short of a standard zero free region but it would be more than enough to establish the conjecture in [Ge-Sh].

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