

TOPOLOGIES OF NODAL SETS OF RANDOM BAND LIMITED FUNCTIONS

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ABSTRACT. It is shown that the topologies and nestings of the zero and nodal sets of random (Gaussian) band limited functions have universal laws of distribution. Qualitative features of the supports of these distributions are determined. In particular the results apply to random monochromatic waves and to random real algebraic hyper-surfaces in projective space.

1. INTRODUCTION

Nazarov and Sodin ([N-S, So]) have developed some powerful general techniques to study the zero (“nodal”) sets of functions of several variables coming from Gaussian ensembles. Specifically they show that the number of connected components of such nodal sets obey an asymptotic law. In [Sa] we pointed out that these may be applied to ovals of a random real plane curve, and in [L-L] this is extended to real hypersurfaces in \mathbb{P}^n . In [G-W] the barrier technique from [N-S] is used to show that “all topologies” occur with positive probability in the context of real sections of high tensor powers of a holomorphic line bundle of positive curvature, on a real projective manifold.

In this note we apply these techniques to study the laws of distribution of the topologies of a random band limited function. Let (M, g) be a compact smooth connected n -dimensional Riemannian manifold. Choose an orthonormal basis $\{\phi_j\}_{j=0}^\infty$ of eigenfunctions of its Laplacian

$$(1) \quad \Delta\phi_i + t_i^2\phi_i = 0,$$

$$0 = t_0 < t_1 \leq t_2 \dots$$

Fix $\alpha \in [0, 1]$ and denote by $\mathcal{E}_{M,\alpha}(T)$ (T a large parameter) the finite dimensional Gaussian ensemble of functions on M given by

$$(2) \quad f(x) = \sum_{\alpha T \leq t_j \leq T} c_j \phi_j(x),$$

where c_j are independent Gaussian variables of mean 0 and variance 1. If $\alpha = 1$, which is the important case of “monochromatic” random functions, we interpret (2) as

$$(3) \quad f(x) = \sum_{T-\eta(T) \leq t_j \leq T} c_j \phi_j(x),$$

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where $\eta(T) \rightarrow \infty$ with T , and $\eta(T) = o(T)$. The Gaussian ensembles $\mathcal{E}_{M,\alpha}(T)$ are our α -band limited functions, and they do not depend on the choice of the o.n.b. $\{\phi_j\}$. The aim is to study the nodal sets of a typical f in $\mathcal{E}_{M,\alpha}(T)$ as $T \rightarrow \infty$.

Let $V(f)$ denote the nodal set of f , that is

$$V(f) = \{x : f(x) = 0\}.$$

For almost all f 's in $\mathcal{E}_{M,\alpha}(T)$ with T large, $V(f)$ is a smooth $(n-1)$ -dimensional compact manifold. We decompose $V(f)$ as a disjoint union $\bigsqcup_{c \in \mathcal{C}(f)} c$ of its connected components.

The set $M \setminus V(f)$ is a disjoint union of connected components $\bigsqcup_{\omega \in \Omega(f)} \omega$, where each ω is a smooth compact n -dimensional manifold with smooth boundary.

The components ω in $\Omega(f)$ are called the nodal domains of f . The nesting relations between the c 's and ω 's are captured by the bipartite connected graph $X(f)$, whose vertices are the points $\omega \in \Omega(f)$ and edges e run from ω to ω' if ω and ω' have a (unique!) common boundary $c \in \mathcal{C}(f)$ (see Figure 2). Thus the edges $E(X(f))$ of $X(f)$ correspond to $\mathcal{C}(f)$.

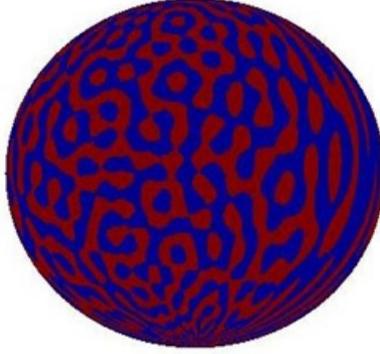


FIGURE 1. A nodal picture of a spherical harmonic. The blue and red are positive and negative domains respectively, and the nodal set is the interface between these.

As mentioned above, Nazarov and Sodin have determined the asymptotic law for the cardinality $|\mathcal{C}(f)|$ of $\mathcal{C}(f)$ as $T \rightarrow \infty$. There is a positive constant $\beta_{n,\alpha}$ depending on n and α (and not on M) such that, with probability tending to 1 as $T \rightarrow \infty$,

$$(4) \quad |\mathcal{C}(f)| \sim \beta_{n,\alpha} c_n \text{Vol}(M) T^n; \text{ here } c_n = \text{Vol}(S^n, \text{round}).$$

We call these constants $\beta_{n,\alpha}$ the Nazarov-Sodin constants. Except for $n = 1$ when the nodal set is a finite set of points and (4) can be established by the Kac-Rice formula ($\beta_{1,\alpha} = \frac{1}{2\sqrt{3\pi^2}} \cdot \sqrt{1 + \alpha + \alpha^2}$), these numbers are not known explicitly.

In order to study the distribution of the topologies of $\mathcal{C}(f)$ and $\Omega(f)$ and the graph $X(f)$ we need certain discrete spaces as well as their one-point compactifications. Let $\widetilde{H}(n-1)$ denote the one-point compactification of the discrete countable set of diffeomorphism classes of compact connected manifolds of dimension $(n-1)$. Similarly, let $\widetilde{B}(n)$ be the one-point compactification of discrete countable set of diffeomorphism classes of n -dimensional manifolds with boundary, and $\widetilde{\mathcal{T}}$ be

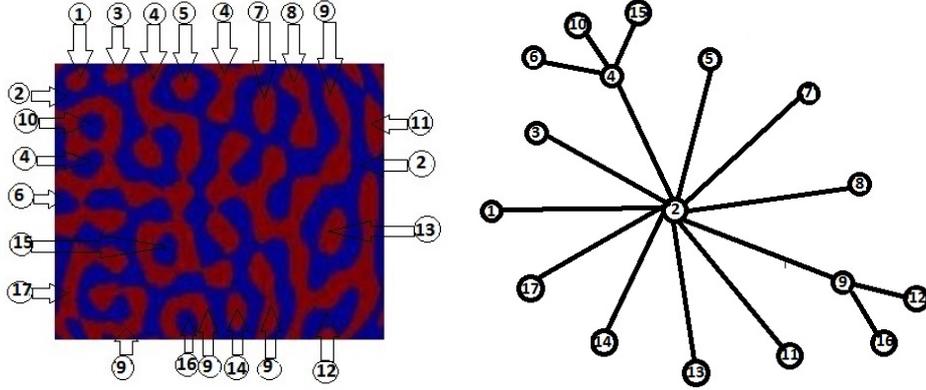


FIGURE 2. To the right: the nesting tree $X(f)$ corresponding to a fragment of the nodal picture in Figure 1, to the left, containing 17 nodal domains (where we neglected some small ones lying next to the boundary). Figure 1 is essential for deciding which components merge on the sphere outside of the fragment.

the one-point compactification of the (discrete countable) set of connected rooted finite graphs (i.e. graphs together with a marked node, referred to as the “root”). Note that each $c \in \mathcal{C}(f)$ and $\omega \in \Omega(f)$ clearly determine points in $\widetilde{H(n-1)}$ and $\widetilde{B(n)}$, which we denote by $t(c)$ and $t(\omega)$ respectively.

To each (or at least almost each) edge c in $\mathcal{C}(f) = E(X(f))$ we associate an end $e(c)$ in $\widetilde{\mathcal{T}}$ as follows: Removing c from $X(f)$ leaves either two components or one component. The latter will happen asymptotically very rarely and in this case we ignore this edge c (or we could make an arbitrary definition for $e(c)$). Otherwise the two components are rooted connected graphs and we define the end $e(c)$ to be smaller (in size) of these two rooted graphs (again, the event that they are of the same size is very rare and can be ignored). With these spaces and definitions we are ready to define the key distributions (they are essentially probability measures) on $\widetilde{H(n-1)}$, $\widetilde{B(n)}$ and $\widetilde{\mathcal{T}}$ by:

$$(5) \quad \mu_{\mathcal{C}(f)} = \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{t(c)},$$

$$(6) \quad \mu_{\Omega(f)} = \frac{1}{|\Omega(f)|} \sum_{\omega \in \Omega(f)} \delta_{t(\omega)}$$

$$(7) \quad \mu_{X(f)} = \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{e(c)},$$

where δ_ξ is a point mass at ξ . These measures give the distribution of topologies of nodal sets, nodal domains and ends of nestings for our given f .

Our first result asserts that as $T \rightarrow \infty$ and for a typical f in $\mathcal{E}_{M,\alpha}(T)$, the above measures converge w -star to universal measures which depend only on n and α . Let $\widetilde{H(n-1)}$ consist of all elements of $H(n-1)$ which can be embedded in \mathbb{R}^{n-1} , $\widetilde{B(n)}$ of those elements of $B(n)$ that can be embedded in \mathbb{R}^n , and $\widetilde{\mathcal{T}}$ the set of all finite rooted trees.

Theorem 1.1. *There are probability measures $\mu_{\mathcal{C},n,\alpha}$, $\mu_{\Omega,n,\alpha}$ and $\mu_{X,n,\alpha}$ supported on $H(n-1) \cup \{\infty\}$, $B(n) \cup \{\infty\}$ and $\mathcal{T} \cup \{\infty\}$ respectively, such that for any given $H \in \widetilde{H(n-1)}$, $B \in \widetilde{B(n)}$ and $G \in \widetilde{\mathcal{T}}$ and $\epsilon > 0$,*

$$\mathcal{P}\{f \in \mathcal{E}_{M,\alpha}(T) : \max(|\mu_{\mathcal{C}(f)}(H) - \mu_{\mathcal{C},n,\alpha}(H)|, |\mu_{\Omega(f)}(B) - \mu_{\Omega,n,\alpha}(B)|, |\mu_{X(f)}(G) - \mu_{X,n,\alpha}(G)|) > \epsilon\} \rightarrow 0,$$

as $T \rightarrow \infty$.

While the above ensures the existence of a law of distribution for the topologies, it gives little information about these universal measures. A central issue is the support of these measures and in particular:

- (1) Do any of the $\mu_{\mathcal{C},n,\alpha}$, $\mu_{\Omega,n,\alpha}$, $\mu_{X,n,\alpha}$ charge the point ∞ , that is, does some of the topology of $V(f)$ escape in the limit?
- (2) Are the supports of these measures equal to $H(n-1)$, $B(n)$ and \mathcal{T} respectively, i.e. do these measures charge each singleton in these sets, positively?

Remarks: (i). We expect that the answer to (1) is NO and to (2) is YES (see below). If the answer to (1) is no, then these measures capture the full distribution of the topologies and Theorem 1 can be stated in the stronger form

$$\mathcal{P}\{f \in \mathcal{E}_{M,\alpha}(T) : \max(D(\mu_{\mathcal{C}(f)}, \mu_{\mathcal{C},n,\alpha}), D(\mu_{\Omega(f)}, \mu_{\Omega,n,\alpha}), D(\mu_{X(f)}, \mu_{X,n,\alpha})) > \epsilon\} \rightarrow 0,$$

as $T \rightarrow \infty$, where the discrepancy is defined by

$$D(\mu_{\mathcal{C}(f)}, \mu_{\mathcal{C},n,\alpha}) = \sup_{F \subseteq \widetilde{H(n-1)}} |\mu_{\mathcal{C}(f)}(F) - \mu_{\mathcal{C},n,\alpha}(F)|,$$

the supremum being over all finite subsets F , and similarly for the other discrepancies.

(ii). The answer to (2) is only problematic in the monochromatic case $\alpha = 1$ (see Section 2).

(iii). Once (1) and (2) are answered the qualitative universal laws for topologies are understood. To get quantitative information the only approach that we know is to do Monte-Carlo (numerical) experiments (see below).

Our main Theorems resolve (1) and (2) in low dimensions. For $n = 2$, $H(1)$ is a point, namely the circle, and the measure $\mu_{\mathcal{C}(f)}$ is trivial. However $B(2)$ consists of all planar domains ω and these are parameterized by their connectivity $b(\omega) \in \mathbb{N}$, that is, we can identify $B(2)$ and \mathbb{N} .

Theorem 1.2. (1) *We have*

$$\mu_{\Omega,2,\alpha}[\{\infty\}] = 0$$

and the support of $\mu_{\Omega,2,\alpha}$ is all of \mathbb{N} , moreover the mean of $\mu_{\Omega,2,\alpha}$ is at most 2 (as a measure on \mathbb{N}).

- (2) *The support of $\mu_{X,2,\alpha}$ contains all of \mathcal{T} (but we don't know if $\mu_{X,2,\alpha}(\{\infty\}) = 0$).*

For $n = 3$, $H(2)$ consists of all orientable compact connected surfaces S , and these are determined by their genus $g(S) \in G = \{0, 1, 2, \dots\}$, that is, we can identify $H(2)$ with G .

Theorem 1.3. *The support of $\mu_{\mathcal{C},3,\alpha}$ is all of G , and $\mu_{\mathcal{C},3,\alpha}(\{\infty\}) = 0$, moreover the mean of $\mu_{\mathcal{C},3,\alpha}$ (as a measure on G) is finite.*

Applications: The extreme values of α , namely 0 and 1 are the most interesting. The case $\alpha = 1$ is the monochromatic random wave (and also corresponds to random spherical harmonics) and it has been suggested by Berry [Be] that it models the individual eigenstates of the quantization of a classically chaotic Hamiltonian. The examination of the count of nodal domains (for $n = 2$) in this context was initiated by [B-G-S], and [B-S], and the latter suggest some interesting connections to exactly solvable critical percolation models.

The law $\mu_{\Omega,2,1}$ gives the distribution of connectivities of the nodal domains for monochromatic waves. Barnett’s numerical experiments [Ba] give the following values for its mass on atoms.

connectivity	1	2	3	4	5	6	7	8	9
$\mu_{\Omega,2,1}$.906	.055	.010	.006	.003	.002	.001	.0008	.0004

The case $\alpha = 0$ corresponds to the algebro-geometric setting of a random real projective hypersurface. Let $W_{n+1,t}$ be the vector space of real homogeneous polynomials of degree t in $n + 1$ variables. For $f \in W_{n+1,t}$, $V(f)$ is a real projective hypersurface in $\mathbb{P}^n(\mathbb{R})$. We equip $W_{n+1,t}$ with the “real Fubini-Study” Gaussian coming from the inner product on $W_{n+1,t}$ given by

$$(8) \quad \langle f, g \rangle = \int_{\mathbb{R}^{n+1}} f(x)g(x)e^{-|x|^2/2} dx$$

(the choice of the Euclidian length $|x|$ plays no role [Sa]). This ensemble is essentially $\mathcal{E}_{M,0}(t)$ with $M = (\mathbb{P}^n(\mathbb{R}), \sigma)$ the projective sphere with its round metric (see [Sa]). Thus the laws $\mu_{\mathcal{C},n,0}$ describe the universal distribution of topologies of a random real projective hypersurface in \mathbb{P}^n (w.r.t. the real Fubini-Study Gaussian).

If $n = 2$ the Nazarov-Sodin constant $\beta_{2,0}$ is such that the random oval is about 4% Harnack, that is it has about 4% of the maximal number of components that it can have ([Na], [Sa]). The measure $\mu_{\Omega,2,0}$ gives the distribution of the connectivities of the nodal domains of a random oval. Barnett’s Monte-Carlo simulation for these yields:

connectivity	1	2	3	4	5	6	7	8	9
$\mu_{\Omega,2,0}$.937	.027	.009	.003	.002	.002	.001	.001	.0005

The measure $\mu_{\mathcal{C},3,0}$ gives the law of distribution of the topologies of a random real surface in $\mathbb{P}^3(\mathbb{R})$. It would be very interesting to Monte-Carlo this distribution and get some quantitative information beyond Theorem 1.3.

Remark 1.4. A P -valued topological invariant F is a map $F : H(n - 1) \rightarrow P$ (everything discrete). One defines the F distribution of $f \in \mathcal{E}_{M,\alpha}(T)$ to be

$$\mu_{F(f)} = \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{F(c)},$$

$\mu_{F(f)}$ is the pushforward of $\mu_{\mathcal{C}(f)}$ to P . Theorem 1.1 can be proven for $\mu_{F(f)}$ in the same way yielding universal laws $\mu_{F,n,\alpha}$ on $P \cup \{\infty\}$. Clearly, for any $y \in P$,

$$\mu_{F,n,\alpha}(Y) \geq \mu_{\mathcal{C},n,\alpha}(F^{-1}(Y)),$$

and if the answers to (1) and (2) are as expected, then we have equality in the above, and $\mu_{F,n,\alpha}$ has exact support equal to $F(H(n-1))$.

For example, let

$$F(c) = \mathbf{Betti}(c) = (\beta^{(1)}(c), \dots, \beta^{(k)}(c))$$

in $P_n := (\mathbb{Z}_{\geq 0})^k$, where $n = 2k + 2$ or $2k + 1$ with $k > 0$, and $\beta^{(j)}(c)$ is the j -th Betti number of c (the other Betti numbers are determined from the connectedness of c and Poincaré duality). In the case $\alpha = 0$ (that is the random projective hypersurface) one can show that $\mu_{\mathbf{Betti},n,0}$ is a probability measure on P_n if n is even, and on the subset $(\mathbb{Z}_{\geq 0})^{k-1} \times (2\mathbb{Z}_{\geq 0})$ of P_n , if n is odd. Moreover, each atom in these sets is charged positively by $\mu_{\mathbf{Betti},n,0}$ and its total mean (or total Betti number)

$$(9) \quad \sum_{y \in P_n} \left(\sum_{j=1}^k y_j \right) \mu_{\mathbf{Betti},n,0}(\{y\})$$

is finite. Thus $\mu_{\mathbf{Betti},n,0}$ describes the full law of distribution of the Betti numbers of a random real algebraic hypersurface in projective space.

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2. OUTLINE OF PROOFS

2.1. The covariance function for $\mathcal{E}_\alpha(T)$. Most probabilistic calculations with the Gaussian ensemble $\mathcal{E}_\alpha(T)$ (we fix M) start with the covariance function (also known as *covariance kernel*)

$$(10) \quad K_\alpha(T; x, y) := \mathbb{E}_{\mathcal{E}_\alpha} [f(x)f(y)] = \sum_{\alpha T \leq \lambda_j \leq T} \phi_j(x)\phi_j(y)$$

(with suitable changes if $\alpha = 1$). The function K_α is the reproducing kernel for our α -band limited functions. Note that

$$\int_M K_\alpha(T; x, x) d\nu(x) = \dim \mathcal{E}_\alpha(T) := \text{Vol}(M)D_\alpha(T),$$

where $D_\alpha(T)$ is the local dimension. The behaviour of K_α as $T \rightarrow \infty$ is decisive in the analysis and it can be studied using the wave equation on $M \times \mathbb{R}$ and constructing a smooth parametrix for the fundamental solution as is done in [Lax, Horm], see [L-P-S] for a recent discussion.

Let

$$\widetilde{K}_\alpha(T; x, y) = \frac{1}{D_\alpha(T)} K_\alpha(T; x, y),$$

then uniformly for $x, y \in M$,

$$(11) \quad \widetilde{K}_\alpha(T; x, y) = \begin{cases} B_{n,\alpha}(Td(x, y)) + O(T^{-1}) & \text{if } d(x, y)T \ll 1 \\ O(T^{-1}) & \text{otherwise} \end{cases},$$

where $d(x, y)$ is the distance from x to y in M , and for $w \in \mathbb{R}^n$

$$(12) \quad B_{n,\alpha}(w) = B_{n,\alpha}(\|w\|) = \frac{1}{|\Omega_\alpha|} \int_{\Omega_\alpha} e(\langle w, \xi \rangle) d\xi$$

and

$$\Omega_\alpha = \{w : \alpha \leq \|w\| \leq 1\}.$$

Moreover, the derivatives of the left hand side of (11) are also approximated by the corresponding derivatives of the right hand side.

Thus for points y within a neighbourhood of $1/T$ of x the covariance is given by (12) while if y is further away the correlation is small. This is the source of the universality of the distribution of topologies, since the quantities we study are shown to be local in this sense.

Let $H_{n,\alpha}$ be the infinite dimensional isotropic (invariant under the action of the group of rigid motions, i.e. translations and rotations) Gaussian ensemble (“field”) defined on \mathbb{R}^n as follows:

$$f(x) = \sum_{j=1}^{\infty} c_j \widehat{\psi}_j(x),$$

where c_j 's are i.i.d. standard (mean zero unit variance) Gaussian variables, and ψ_j are an orthonormal basis of $L^2(\Omega_\alpha, dv)$, where dv is the normalized Haar measure. The covariance function of $H_{n,\alpha}$ is given by

$$\mathbb{E}_{H_{n,\alpha}}[f(x)f(y)] = B_{n,\alpha}(x - y).$$

The typical element in the ensemble $H_{n,\alpha}$ is C^∞ , and the action by translations on $H_{n,\alpha}$ is ergodic by the classical Fomin-Grenander-Maruyama theorem (see e.g. [Gr]). As in [So] we show that the probability distributions that we are interested in are encoded in this ensemble $H_{n,\alpha}$.

2.2. On the proof of existence of limiting measures (Theorem 1.1). For the existence of the measures in Theorem 1.1 we follow the method in [N-S] and [So] closely. They examine the expectation and fluctuations of the (integer valued) random variable $N(f, T)$ on $\mathcal{E}_\alpha(T)$ which counts the number of connected components c of $V(f)$. We examine the refinements of these given as: for $S \in H^{n-1}$, $N(f, S, T)$ is the (integer valued) random variable which counts the number of such components c which are topologically equivalent to S ; for $\omega \in B(n)$, $N(f, \omega, T)$ counts the number of components c whose ‘inside’ is homeomorphic to ω , and for $e \in \mathcal{T}$ a rooted tree $N(f, e, T)$ counts the number of components c whose end is e . The fact that our random variables are all dominated pointwise by $N(f, T)$ allows us for the most part to simply quote the bounds for rare events developed in [So], and this simplifies our task greatly. The basic existence result for each of our random variables is the following, which we state for $N(f, S, T)$: There is a constant $\widetilde{\mu}_{n,\alpha}(S) \geq 0$ such that

$$(13) \quad \lim_{T \rightarrow \infty} \mathbb{E}_{\mathcal{E}_\alpha(T)} \left[\left| \frac{N(f, S, T)}{T^n} - \widetilde{\mu}_{n,\alpha}(S) \right| \right] = 0.$$

The constant $\widetilde{\mu}_{n,\alpha}(S)$ is determined from the Gaussian $H_{n,\alpha}$ as follows: For $f \in H_{n,\alpha}$ and $R \geq 1$ let $N(f, S, R)$ be the number of components of $V(f)$ which are homeomorphic to S and which lie in $B(R)$ the ball about 0 of radius R . This

function of f is in L^1 , and after suitable generalizations¹ of the sandwich estimates ([So], page 6) for our refined variables one shows that the following limit exists and yields $\widetilde{\mu}_{n,\alpha}(S)$:

$$\widetilde{\mu}_{n,\alpha}(S) = \lim_{R \rightarrow \infty} \frac{1}{\text{Vol}(B(R))} \mathbb{E}_{n,\alpha}[N(f, S, R)].$$

Hence in terms of μ ,

$$\mu_{\mathcal{C},n,\alpha}(S) = \frac{\widetilde{\mu}_{n,\alpha}(S)}{c_n \beta_{n,\alpha}}.$$

The proof of (13) is in two steps. The first is a localization in which one scales everything by a factor of T in $1/T$ -neighbourhoods of points in M , and reduces the problem to that of the limit ensemble $H_{n,\alpha}$. This process, called ‘‘coupling’’ in [So], can be carried out in a similar way for $N(f, S, T)$ (as well as our other counting variables) after relativizing the various arguments and inequalities. The second step concerns the study of the random variable $N(f, S, R)$ (and again the other counts) on $H_{n,\alpha}$, asymptotically as $R \rightarrow \infty$. A key point is that this latter variable is firstly measurable (it is locally constant) and is in $L^1(H_{n,\alpha})$. As in [So] this allows one to apply the ergodic theorem for the group of translations of \mathbb{R}^n to ensure that the counts in question converge when centred at different points. This provides the ‘soft’ existence for the limits at hand while providing little further information. As a ‘‘by-product’’ this approach implies that a typical nodal domain or a tree end of f lies in a geodesic ball of radius R/T in M for R large (but fixed); this ‘‘semi-locality’’ is the underlying reason for the ergodic theory to be instrumental for counting nonlocal quantities.

As was mentioned above, to infer information on f from $H_{n,\alpha}$ one needs to construct a coupling, that is, a copy of $H_{n,\alpha}$ defined on the same probability space as f , so that with high probability a random element g of $H_{n,\alpha}$ is merely a small perturbation of (the scaled version of) f in $C^1(\overline{B(2R)})$ (that is, both the values and the partial derivatives of g approximate those of f); this is possible thanks to (11). Moreover, in this situation, with high probability both f and g are ‘‘stable’’, i.e. the set of points where both f and $\|\nabla f\|$ are small is *negligible* (the same holding for g). Nazarov and Sodin used the ingenious ‘‘nodal trap’’ idea, showing that each of the nodal components c of g is bounded between the two hypersurfaces $g^{-1}(\pm\epsilon)$, to prove that under the stability assumption c corresponds to a *unique* nodal component of f . This allowed them to infer that the nodal count of g is approximating the nodal count of f (neglecting the unstable regions). We refine their argument by observing that the topological class of a nodal component inside the ‘‘trap’’ cannot change while perturbing from g to f , as otherwise, by Morse Theory, one would have encountered a low valued critical point; this is readily ruled out by the stability assumption. The same approach shows that neither the diffeomorphism class of the corresponding nodal domain nor the local configuration graph can change by such a perturbation. This completes the outline of the proof of Theorem 1.1.

2.3. The measures $\mu_{\Omega,2,\alpha}$ and $\mu_{\mathcal{C},3,\alpha}$ do not charge ∞ . To establish the claims Theorems 1.2 and 1.3 about the supports of the measures $\mu_{\mathcal{C},n,\alpha}$ (respectively $\mu_{\Omega,n,\alpha}$, $\mu_{X,n,\alpha}$) one needs to input further topological and analytic arguments.

¹The case of tree ends is the most subtle.

For part (1) of Theorem 1.2 take $M = \mathcal{S}^2$ with its round metric. For almost all $f \in \mathcal{E}_{M,\alpha}(T)$, $V(f)$ is nonsingular and the graph $X(f)$ is a tree (by the Jordan curve Theorem). Hence

$$\sum_{\omega \in \Omega(f)} b(\omega) = 2|\Omega(f)| - 2,$$

and hence the mean (over \mathbb{N}) of $\mu_{\Omega}(f)$ is equal to $2 - \frac{2}{|\Omega(f)|}$. It follows that the limit measures $\mu_{\Omega,2,\alpha}$ do not charge $\{\infty\}$, and that their means are at most 2. From the data in Barnett's tables (see Section 1) it appears that the means for $\alpha = 0$ and $\alpha = 1$ may well be less than 2. If this is so it reflects a nonlocal feature of "escape of topology" at this more quantitative level.

The proof of Theorem 1.3 uses the Kac-Rice formula (see [C-L, A-T]) and some topology. The expected value of the integral of over M of any local quantity, such as the curvature $\kappa(x)$ of the surface $V(f)$ ($n = 3$), over $\mathcal{E}_{M,\alpha}(T)$ can be computed. For example, if $M = \mathbb{R}\mathbb{P}^3$ with its round Fubini-Study metric, then by Gauss-Bonnet

$$\mathbb{E}_{\mathcal{E}_{M,\alpha}(T)} \left[\sum_{c \in \mathcal{C}(f)} 2(1 - g(c)) \right] \sim -\gamma_{\alpha} T^3$$

(here $\gamma_{\alpha} > 0$ and for $\alpha = 0$ it is computed explicitly in [Bu]). Hence as $T \rightarrow \infty$,

$$\mathbb{E}_{\mathcal{E}_{M,\alpha}(T)} \left[\sum_{c \in \mathcal{C}(f)} g(c) \right] \sim (\gamma_{\alpha} + 2 \text{Vol}(\mathbb{R}\mathbb{P}^3) c_3 \beta_{3,\alpha}) T^3.$$

From this one deduces that $\mu_{\mathcal{C},3,\alpha}(\{\infty\}) = 0$, and that the mean of $\mu_{\mathcal{C},3,\alpha}$ (over G) is at most

$$2 + \frac{\gamma_{\alpha}}{\text{Vol}(\mathbb{R}\mathbb{P}^3) \cdot c_3 \cdot \beta_{3,\alpha}}.$$

2.4. The measures $\mu_{\Omega,2,\alpha}$ and $\mu_{\mathcal{C},3,\alpha}$ charge every finite atom. The proof that the measures $\mu_{\mathcal{C},n,\alpha}$, $\mu_{\Omega,n,\alpha}$ and $\mu_{X,n,\alpha}$ charge every topological atom reduces to producing an $f \in H_{n,\alpha}$ for which the corresponding $V(f)$ (respectively $\Omega(f)$, $X(f)$) has the sought atomic configuration (since the g 's in a suitably small C^k neighbourhood of f have the same local configuration level and such a neighbourhood has positive measure in $H_{n,\alpha}$). For $\Sigma \subseteq \mathbb{R}^n$ a compact set let

$$\widehat{\Sigma} = \left\{ f : f(x) = \sum_{\xi \in \Sigma} a_{\xi} e(\langle x, \xi \rangle), \text{ with } a_{\xi} = 0 \text{ for all but finitely many } \xi \right\}.$$

For our purposes it suffices to find a $g \in \widehat{\Omega}_{\alpha}$ (real valued) with the desired topological atom. If $0 \leq \alpha < 1$ then one can show that for any compact ball B , $\widehat{\Omega}_{\alpha}|_B$ is dense in $C^k(B)$ (for any k). Hence constructing a function g of the type that we want is straightforward. However for $\alpha = 1$, the closure of $\widehat{\Omega}_1|_B$ in $C(B)$ is of infinite codimension. Nevertheless the following much weaker statement holds:

Lemma 2.1. *For $n \geq 2$ and $K \subseteq \mathbb{R}^n$ a finite set, $\widehat{\Omega}_1|_K = C(K)$, i.e. there is no restriction on the values attained by a function in $\widehat{\Omega}_1$ on a finite set.*

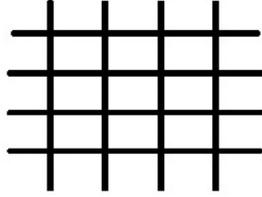
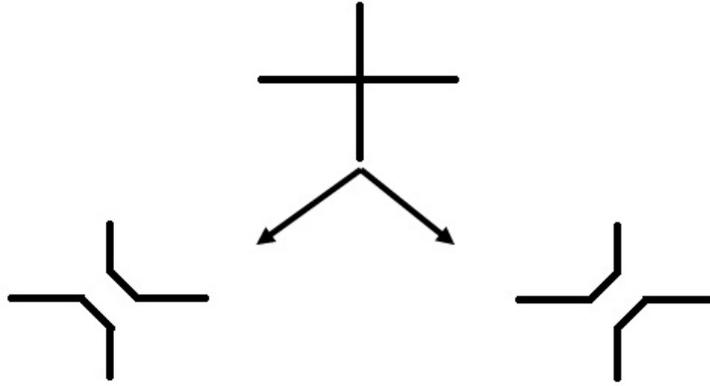
FIGURE 3. A grid, the nodal set of f_0 .

FIGURE 4. A singularity resolution.

Our proof uses asymptotics of Bessel functions and L^∞ bounds for spherical harmonics. One can also deduce Lemma 2.1 for K 's which are subsets of \mathbb{Z}^n (and this is sufficient for our purposes) from Ax's "function field Schanuel Theorem" [Ax]. In fact, one can deduce a much more general result which is useful in this context: If Σ is an r -dimensional ($r \geq 1$) real algebraic subvariety of \mathbb{R}^n which is not special in the sense of [Pila], and $K \subseteq \mathbb{Z}^n$ is finite, then $\widehat{\Omega}_1|_K = C(K)$.

To produce an $f \in \widehat{\Omega}_1$ (for $n = 2$) with $X(f)$ having a given end $e \in \mathcal{T}$, start with

$$f_0(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2),$$

whose nodal set $V(f)$ is a grid (see Figure 3) with conic singularities at the points of \mathbb{Z}^2 . For any finite $K \subseteq \mathbb{Z}^2$ we can choose $\psi(x_1, x_2)$ in $\widehat{\Omega}_1$ with $\psi(k) = \epsilon_k \in \{-1, 1\}$ for $k \in K$, where ϵ_k is any assignment of signs. Set

$$f(x_1, x_2) = f_0(x_1, x_2) + \epsilon \psi(x_1, x_2),$$

where ϵ is a small positive number. The singularity at k will resolve in either of the forms as in Figure 4, according to the sign of ϵ_k . One shows that this gives enough flexibility by choosing K and ϵ_k to produce any rooted tree in \mathcal{T} . This completes the outline of Theorem 1.2, part (2).

For a proof of Theorem 1.3 one uses the above Lemma in a similar way starting with the function

$$f_0(x_1, x_2, x_3) = \sin(\pi x_1) \sin(\pi x_2) + \sin(\pi x_1) \sin(\pi x_3) + \sin(\pi x_2) \sin(\pi x_3).$$

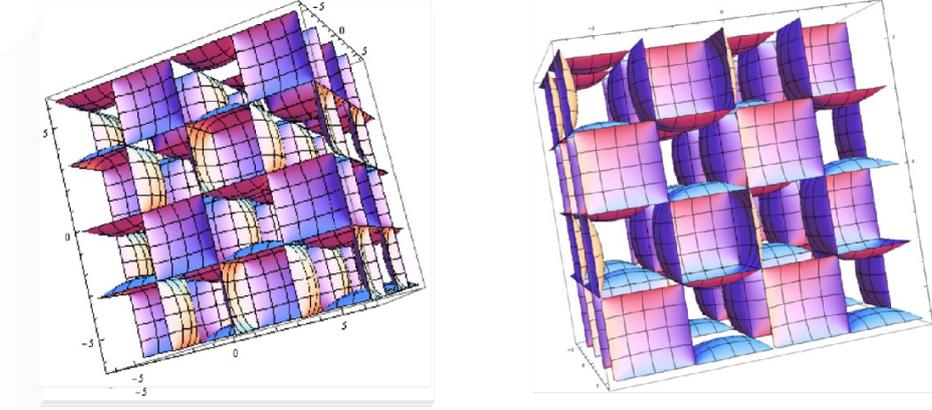


FIGURE 5. A fragment of the zero set of

$$f_0(x_1, x_2, x_3) = \sin x_1 \sin x_2 + \sin x_1 \sin x_3 + \sin x_2 \sin x_3$$

shown from two different perspectives. It consists of a lattice arrangement of infinitely many layers each containing boxes arranged on half of the black cells of an infinite chessboard, diagonally connected to the upper and lower layers.

The singularities of $V(f_0)$ are at integral lattice points and are conic (see Figure 5). Perturbing f_0 near such a point k resolves $V(f_0)$ to a 1-sheeted or 2-sheeted hyperboloid depending on the sign of ϵ_k . Again, one shows by examining the components of $\mathbb{R}^3 \setminus V(f_0)$ (which consists of infinitely many alternating cubes, and the complement, which is connected), that perturbing f_0 by a suitable $\epsilon\psi \in \widehat{\Omega}_1$ is enough to produce any element of $H(2)$ as a component of $V(f)$.

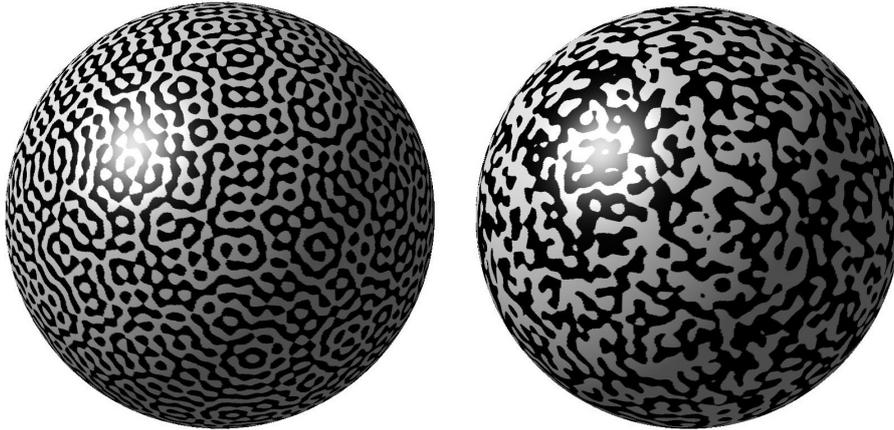


FIGURE 6. Nodal line and domains for a random element in $\mathcal{E}_{S^{1,2},\alpha}$ with $\alpha = 1$ (left) and $\alpha = 0$ (right), $T = \sqrt{80 \cdot 81}$, pictures produced by Alex Barnett. The nodal domains are the black and white connected components, and the nodal line is the interface between these.

We end with comments on Remark 1.4. According to [Mi], the total Betti number $\sum_{j=1}^{n-1} \beta^j$ of the zero set of a nonsingular real homogeneous polynomial of degree t in (x_0, x_1, \dots, x_n) is at most t^n . This together with (4) implies the finiteness assertion (9) which in turn ensures that $\mu_{\text{Betti},n,0}(\{\infty\}) = 0$. That the image of $H(n-1)$ under Betti is restricted as claimed follows from our c 's bounding a compact n -manifold so that $\chi(c)$ is even. On the other hand, starting from \mathcal{S}^{n-1} and applying suitable p -surgeries which increase β^{p+1} by 1 if $p+1$ is not in the middle dimension and by 2 if it is, shows that the image of Betti is as claimed. An interesting question about the Betti numbers raised in [G-W] page 4 in the context of their ensembles, is whether the limits

$$\frac{1}{\beta_{n,0} T^n} \mathbb{E}_{\mathcal{E}_{\mathbb{P}^n,0}(T)}[\beta^j(V_f)], \quad T \rightarrow \infty$$

exist for each $1 \leq j \leq k$? If so a natural question is whether these are equal to the corresponding mean for $\mu_{\text{Betti},n,0}$? These appear to be subtle questions related to the possible non-locality of these quantities (escape of mass) and it is unclear to us what to expect.

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