INTEGER POINTS ON AFFINE CUBIC SURFACES

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JOINT WORK WITH
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AND

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GAUSS DIARY JULY/10/1796

XX EYPHKA NUM = $\Delta + \Delta + \Delta$ — (*)

$$\left(k > 0 \;,\; k = \frac{x(x+1)}{2} + \frac{y(y+1)}{2} + \frac{z(z+1)}{2}\right)$$

THERE ARE NO LOCAL CONGRUENCE OBSTRUCTIONS TO (*) AND HE PROVED THERE IS ALWAYS A GLOBAL SOLUTION

"HASSE PRINCIPLE"

THE GENERAL PROBLEM OF A QUADRIC

$$f(x_1, x_2, \ldots, x_n) = k$$

f A QUADRATIC FORM AND x_j, k INTEGERS IN A NUMBER FIELD WAS INITIATED BY GAUSS IN HIS DISQUESTIONES ARITHMERICAE AND IS HILBERT'S ELEVENTH PROBLEM. IT WAS ONLY RESOLVED COMPLETELY IN 2000 AFTER CONTRIBUTIONS BY MANY PEOPLE.

THERE IS A LOCAL TO GLOBAL PRINCIPLE FOR FORMS IN THREE OR MORE VARIABLES WITH FINITELY MANY EXCEPTIONS IN k.

CUBIC FORMS

- AN AFFINE CUBIC f IS A POLYNOMIAL IN $\mathbb{Z}[x_1, \ldots, x_n]$ WITH LEADING HOMOGENEOUS PART f_0 OF DEGREE 3 AND NON-DEGENERATE, WE ALSO ASSUME THAT f AND $f-k$ ARE IRREDUCIBLE.

$$(**) \quad V_{k,f} = \{x : f(x) = k\}$$

AFFINE-HYPERSURFACE

- k IS ADMISSIBLE IF THERE ARE NO LOCAL CONGRUENCE OBSTRUCTIONS TO $(**)$ (THESE HAVE A SIMPLE DESCRIPTION)

RICHNESS OF $V_{k,f}(\mathbb{Z})$:

FOR k ADMISSIBLE IS $V_{k,f}(\mathbb{Z})$ NON-EMPTY (IE HAS A HASSE PRINCIPLE), ZARISKI-DENSE IN $V_{k,f}$, SATISFY A FORM OF STRONG-APPROXIMATION?

$\underline{n = 2}$ (SUPER-CRITICAL) THUE/SIEGEL

$$|V_{k,f}(\mathbb{Z})| < \infty$$

SCHMIDT SHOWS THAT FOR VERY FEW ADMISSIBLE k'S IS $V_{k,f}(\mathbb{Z}) \neq \phi$.

$n \geq 10$ (SUBCRITICAL) BROWNING-HEATH BROWN

f_0 NON-SINGULAR THEN FOR k ADMISSIBLE $V_{k,f}(\mathbb{Z}) \neq \phi$, IT IS ZARISKI DENSE AND SATISFIES STRONG APPROXIMATION.

$n \geq 4$ (SUBCRITICAL) HOOLEY

f HOMOGENEOUS AND NON-SINGULAR AND ASSUMING THE RIEMANN HYPOTHESIS FOR CERTAIN ASSOCIATED HASSE-WEIL ZETA FUNCTIONS THEN $V_{k,f}(\mathbb{Z}) \neq \phi$ FOR ALMOST ALL ADMISSIBLE k'S.

$n=3$ (CRITICAL) AFFINE CUBIC SURFACE, VERY LITTLE IS KNOWN.

EXAMPLE

$$f = S(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$$

k IS ADMISSIBLE IFF $k \neq 4$ or $5 \pmod 9$

IT IS POSSIBLE THAT FOR EVERY ADMISSIBLE k, $V_{S,k}(\mathbb{Z}) \neq \phi$ AND IS ZARISKI DENSE IN $V_{S,k}$.

- THE SMALLEST k FOR WHICH NO SOLUTIONS HAVE BEEN FOUND IS 33.

- LEHMER, BEUKERS SHOW THAT $V_{S,1}(\mathbb{Z})$ IS ZARISKI DENSE IN $V_{S,1}$.

USING CUBIC RECIPROCITY ONE CAN SHOW THAT STRONG APPROXIMATION FAILS FOR $V_{S,k}(\mathbb{Z})$.

<u>E.G.</u> $x \in V_{S,3}(\mathbb{Z}) \Rightarrow x_1 \equiv x_2 \equiv x_3 \pmod 9$

(CASSELS, HEATH-BROWN, COLLIOT-THELENE/WITTENBERG)

HOWEVER IN THE SLIGHTLY WEAKER FORM

$$V_{S,k}(\mathbb{Z}) \longrightarrow V_{S,k}(\mathbb{Z}/p\mathbb{Z}),$$ BEING ONTO

FOR p A LARGE PRIME, MAY HOLD.

A DIOPHANTINE THEORY FOR INTEGRAL POINTS ON SOME SPECIAL CUBIC SURFACES CAN BE DEVELOPED

A. GHOSH / S, J. BOURGAIN/A. GAMBURD/S

THESE START WITH MARKOFF'S SURFACES.

MARKOFF'S CUBIC SURFACES

$$M(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2 - x_1x_2x_3$$

$$V_k = V_{k,M} = \{x \mid M(x)=k\}.$$

$k=0$: IS MARKOFF'S SURFACE

$k=4$: IS THE CAYLEY CUBIC
(IT IS SPECIAL IN WHAT FOLLOWS)

- $V_k(\mathbb{Z})$ ARISES IN MANY CONTEXTS

DIOPHANTINE APPROXIMATION (MARKOFF)

SIMPLE CLOSE GEODESICS ON THE MODULAR SURFACE (H. COHN)

EXCEPTIONAL VECTOR BUNDLES OVER $\mathbb{P}^2$ (GORODENSTEV / RUDAKOV)

SMOOTHABLE DEL-PEZZO SURFACES (HACKING / PROKHOROV)

SYMPLECTIC 4-MANIFOLDS VIA LEFSCHETZ FIBRATIONS (AUROUX)

⋮

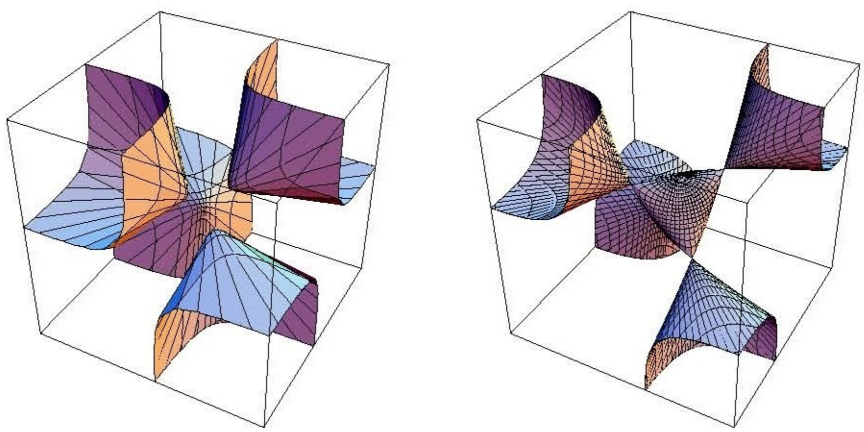
- V_k IS ALSO THE RELATIVE CHARACTER VARIETY OF REPRESENTATIONS OF $\pi_1(\Sigma_{1,1}) \to SL_2$

- IT ALSO ARISES ~~AS~~ IN THE NON-LINEAR MONODROMY GROUP OF PAINLEVÉ VI.

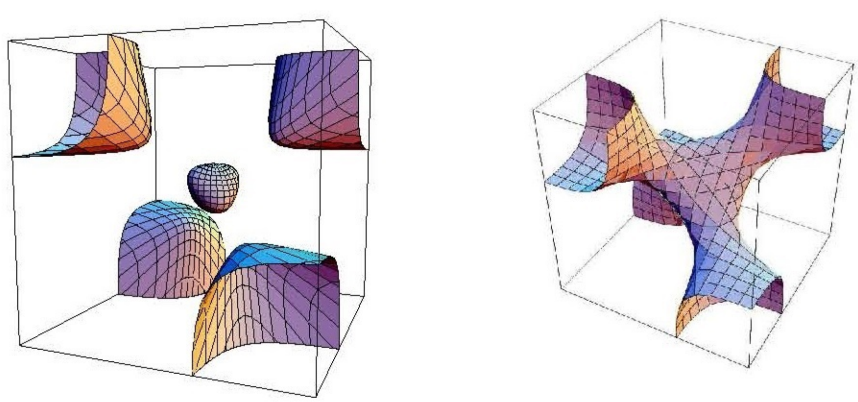
V_0 Markoff's cubic surface

V_4 Cayley's cubic surface

$V_k(\mathbb{R})$ for different k:



$k = 0$ and $k = 4$



$k = 2$ and $k = 8$

THE REASON ONE CAN STUDY $V_k(\mathbb{Z})$ IS THAT IT IS ACTED ON BY A NON-LINEAR GROUP OF MORPHISMS ALLOWING DESCENT.

Γ, THE GROUP IN $\mathrm{AUT}(\mathbb{A}^3)$ GENERATED BY PERMUTATIONS OF THE COORDINATES AND SWITCHING THE SIGNS OF TWO COORDINATES, AND THE VIETA INVOLUTIONS R_1, R_3, R_3

$$R_3(x_1, x_2, x_3) = (x_1, x_2, x_1 x_2 - x_3)$$

PRESERVES V_k AND $V_k(\mathbb{Z})$. ($\Gamma \cong PGL_2(\mathbb{Z})$)

- FOR $k \neq 4$, $V_k(\mathbb{Z})$ CONSISTS OF A FINITE NUMBER $h(k)$ OF Γ-ORBITS (MARKOFF, HURWITZ, MORDELL).

<u>CLASSICAL QUESTIONS</u>:

(i) WHEN IS $V_k(\mathbb{Z}) \neq \phi$ IE $h(k) > 0$.

(ii) IF $h(k) > 0$, IS $V_k(\mathbb{Z})$ INFINITE, ZARISKI DENSE, SATISFY A FORM OF STRONG APPROXIMATION ?

HASSE PRINCIPLE

LOCAL CONGRUENCE OBSTRUCTIONS:

$$V_k(\mathbb{Z}_p) \neq \phi \quad \text{FOR ALL } p \text{ IFF } k \not\equiv 3(4) \text{ OR } \pm 3 \pmod 9.$$

WE RESTRICT TO k'S WHICH HAVE LOCAL INTEGRAL POINTS AND SAY THAT V_k FAILS HASSE'S PRINCIPLE IF $V_k(\mathbb{Z}) = \phi$.

FOR $|k| \geq 5$ CALL k SPECIAL IF $V_k(\mathbb{Z})$ CONTAINS A POINT x WITH $|x_j| = 0, 1, 2$. THE SPECIAL k'S ARE EASY TO DESCRIBE AND ANALYZE, THEY ARE OF ZERO DENSITY. REMAINING k'S ARE CALLED GENERIC.

- FOR $k>0$ GENERIC A POINT $x \in V_k(\mathbb{Z})$ IS GHOSH REDUCED IF IT IS OF THE FORM $(-x_1, x_2, x_3)$ WITH $3 \leq x_1 \leq x_2 \leq x_3$ AND
$$x_1^2 + x_2^2 + x_3^2 + x_1 x_2 x_3 = k$$

- (GHOSH) FOR $k>0$ GENERIC
$$\Gamma \backslash V_k(\mathbb{Z}) \cong \text{GHOSH REDUCED POINTS.}$$

COR: (a) $h(k) \ll |k|^{1/3}$

(b) $$\sum_{0<k\leq K} h(k) \sim \frac{K(\log K)^2}{36}, \quad K\to\infty$$

$$\sum_{-K\leq k<0} h(k) \sim \frac{K(\log K)^2}{48}, \quad K\to\infty.$$

THE EXPLICIT FUNDAMENTAL DOMAINS ALLOW FOR THE NUMERICAL COMPUTATIONS OF THE $h(k)$'s AND THESE INDICATE THAT

$$\left|\{ |k|\leq K : V_k \text{ FAILS HASSE}\}\right| \sim cK^{\theta}$$

WITH $c\neq 0$ AND $\theta \approx 0.887\ldots$

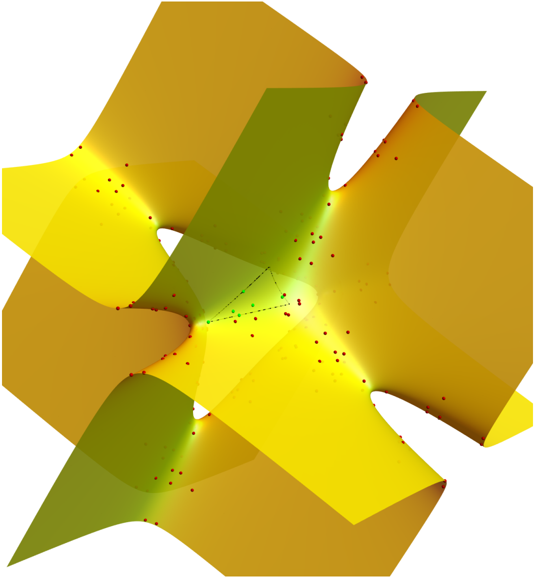


Figure: Lattice points and fundamental set (triangular) for $k = 3685$.

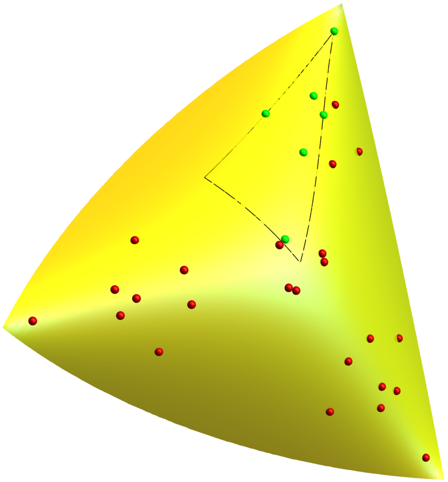


Figure: Closeup of fundamental set (triangular) for $k = 3685$.

THEOREM 1 (GHOSH / S 2017):

(i) THERE ARE INFINITELY MANY k's WHICH FAIL THE HASSE PRINCIPLE. THE NUMBER OF SUCH WITH $|k| \leq K$ IS AT LEAST $K^{1/2}/\log K$.

(ii) FIX $t \geq 0$

$$\#\{|k| \leq K : k \text{ ADMISSIBLE}, h(k) = t\} = o(K) \quad K \to \infty$$

$\Rightarrow$ ALMOST ALL k's SATISFY HASSE AND ALSO THESE $V_k(\mathbb{Z})$'s ARE ZARISKI DENSE.

COMMENTS:

(a) THE HASSE FAILURES ARE PRODUCED BY AN OBSTRUCTION VIA QUADRATIC RECIPROCITY AND FACTORIZATIONS THAT ARISE FROM $V_k \pmod p$ BEING THE CAYLEY CUBIC IF $k \equiv 4 \ (p)$.

FOR EXAMPLE IF

$$k = 4 + 2\nu^2$$

WITH ν HAVING ALL OF ITS PRIME FACTORS $\equiv \pm 1 \pmod 8$ AND $\nu \equiv 0, \pm 3, \pm 4 \pmod 9$, THEN k IS ADMISSIBLE BUT $V_k(\mathbb{Z}) = \phi$.

(b) IS PROVED BY COMPARING THE NUMBER OF POINTS ON $V_k(\mathbb{Z})$ IN CERTAIN TENTACLED REGIONS GOTTEN BY SPECIAL PLANE SECTIONS, WITH THE EXPECTED NUMBER OF SOLUTIONS ACCORDING TO A PRODUCT OF LOCAL DENSITIES. THE KEY IS THAT THE VARIANCE OF THIS COMPARISON GOES TO ZERO ON AVERAGING $|k| \leq K$. THIS MOVING PLANE QUADRIC METHOD APPLIES TO MORE GENERAL CUBIC SURFACES INCLUDING ONES THAT DON'T CARRY MORPHISMS.

STRONG APPROXIMATION

WHEN $V_k(\mathbb{Z}) \neq \phi$ HOW RICH IS IT BEYOND ZARISKI DENSITY?

WE DISCUSS THE MARKOFF CASE $k=0$.

THE GENERAL CASE IS SIMILAR BUT FIRST REQUIRES A STUDY OF <u>ALL</u> THE FINITE Γ-ORBITS IN $\mathbb{A}^3(\overline{\mathbb{Q}})$.

• THIS IS CLOSELY RELATED TO THE DETERMINATION (DUBROVIN–MAZZACO) OF THE ALGEBRAIC PAINLEVÉ VI'S.
OUR TREATMENT OF THE LATTER USES THE RESOLUTION (EFFECTIVE) OF LANG'S $\mathbb{G}_m$ CONJECTURE.

$$Y: \quad x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0$$

Γ AS BEFORE EXCEPT THAT THE R_j'S ARE

$$R_3((x_1,x_2,x_3)) = (x_1, x_2, 3x_1x_2 - x_3)$$

$h=2$, WITH ONE ORBIT BEING $\{0\}$
AND THE OTHER $Y^*(\mathbb{Z}) = \Gamma.(1,1,1)$.

STRONG APPROXIMATION CONJECTURE FOR Y:

REDUCTION MOD p

$$Y^*(\mathbb{Z}) \longrightarrow Y^*(\mathbb{Z}/p\mathbb{Z})$$ IS ONTO FOR ALL PRIMES p.

- Γ ACTS ON REDUCTION mod p AS A PERMUTATION GROUP ON $Y^*(\mathbb{Z}/p\mathbb{Z})$.

- STRONG APPROXIMATION $\Longleftrightarrow$ Γ ACTING TRANSITIVELY ON $Y^*(\mathbb{Z}/p\mathbb{Z})$.

NOTE: $$|Y^*(\mathbb{Z}/p\mathbb{Z})| \sim p^2 \quad \text{AS } p \to \infty.$$

AS LONG AS $p^2 - 1$ IS NOT VERY SMOOTH (EG $= k!$) WE CAN PROVE STRONG APPROXIMATION.

THEOREM 2 (BOURGAIN-GAMBURD-S, 2014)

FOR $\varepsilon > 0$ AND p LARGE THERE IS A Γ ORBIT $\mathcal{O}(p)$ IN $Y^*(\mathbb{Z}/p\mathbb{Z})$ SUCH THAT

$$|\mathcal{O}^c(p)| = |Y^*(\mathbb{Z}/p\mathbb{Z}) \setminus \mathcal{O}(p)| \ll_\varepsilon p^\varepsilon$$

AND EVERY Γ-ORBIT $t(p)$ IN $Y^*(\mathbb{Z}/p\mathbb{Z})$ SATISFIES $|t(p)| \gg (\log p)^{1/3}$.

THESE HAVE BEEN RECENTLY IMPROVED BY KONYAGIN / MAKARYCHEV / SHPARLINSKI / VYUGIN (2017) TO

$$|\mathcal{O}^c(p)| \le \exp\left((\log p)^{\frac{2}{3}+o(1)}\right)$$

$$|t(p)| \gg (\log p)^{7/9}.$$

THEOREM 3 (B-G-S 2015)

$$|\{p \le T : \text{STRONG APPROXIMATION FAILS FOR } p\}| \ll_\varepsilon T^\varepsilon, \quad \varepsilon > 0.$$

SO IN GENERAL THERE IS ALWAYS A GIANT COMPONENT AND THE STRONG APPROXIMATION CONJECTURE HOLDS EXCEPT PERHAPS FOR VERY FEW P'S.

THEOREM 4 (MEIRI-PUDER 2017):

IF $p \equiv 1(4)$ AND p SATISFIES STRONG APPROXIMATION FOR $Y^*(\mathbb{Z}/p\mathbb{Z})$ THEN THE ACTION OF Γ ON $Y^*(\mathbb{Z}/p\mathbb{Z})$ IS EITHER THE FULL ALTERNATING OR SYMMETRIC GROUP.

- THE ABOVE ALLOWS ONE TO SHOW THAT

$$Y^*(\mathbb{Z}) \longrightarrow Y^*(\mathbb{Z}/q\mathbb{Z}) \quad \text{IS ONTO}$$

FOR $q = p_1 p_2 \cdots p_\ell$, $p_j \equiv 1(4)$ AND SATISFY STRONG APPROXIMATION. (GOURSAT LEMMA).

WITH THESE WE CAN EXECUTE SOME SIMPLE SIEVING AND COUPLE IT WITH SOME ANALYSIS USING TEICHMULLER DYNAMICS (MIRZAKHANI) TO ANSWER SOME OLD QUESTIONS ABOUT MARKOFF NUMBERS:

$\mathcal{M}$: MARKOFF NUMBERS, THAT IS CO-ORDINATES OF A MARKOFF TRIPLE $x \in \underline{Y}(\mathbb{Z})$ WITH $x_j > 0$.

$\mathcal{M}$: $1, 2, 5, 13, 29, 34, 89, 169, 194, \dots$

- FROBENIUS: $m \in \mathcal{M} \Rightarrow m \not\equiv 0, \pm 2/3 \mod p$, IF $p \equiv 3(4)$ AND $p \neq 3$.
- STRONG APPROXIMATION $\Rightarrow$ THESE ARE THE ONLY CONGRUENCE OBSTRUCTIONS.

$\mathcal{M}$ IS LACUNARY:

$$|\{m \leq T : m \in \mathcal{M}\}| \sim c(\log T)^2,$$

ZAGIER (1982)
MIRZAKHANI (2016)

THEOREM 5: (B-G-S 2015)

ALMOST ALL $m \in \mathcal{M}$ ARE COMPOSITE

$$|\{p \leq T : p \in \mathcal{M}, p \text{ prime}\}| = o(|\{m \in \mathcal{M} : m \leq T\}|)$$

AS $T \to \infty$.

REMARKS

TOOLS ARE ELEMENTARY COMING FROM ANALYTIC NUMBER THEORY, CURVES OVER FINITE FIELDS AND COMBINATORICS. ONE INTERESTING FEATURE BEING:

$$C_\lambda : \begin{cases} y + y^{-1} = x + \lambda x^{-1}, & x, y \in \mathbb{F}_p \\ x \in H_1,\ y \in H_2 \end{cases} \qquad \lambda \neq 1$$

$$H_1, H_2 \leq \mathbb{F}_p^*, \quad |H_1| \leq |H_2|.$$

NEED AN UPPER BOUND FOR $|C|$ OF THE FORM

$$|C| \leq |H_2|^\delta$$ FOR SOME $\delta < 1$ INDEPENDENT OF p.

- IF $|H_2| \geq \sqrt{p}$ ONE CAN USE THE RIEMANN HYPOTHESIS FOR CURVES OVER FINITE FIELDS TO PROVE THIS.
- FOR $|H_2|$ SMALL THIS IS OF NO USE AND WE USE STEPANOV'S ELEMENTARY PROOF OF WEIL'S THEOREM (SPECIFICALLY AUXILIARY POLYNOMIALS) TO ESTABLISH SUCH A BOUND.

- THE RICHNESS RESULTS FOR MARKOFF SURFACES EXTEND TO OTHER CUBIC SURFACES, THOUGH STILL SPECIAL

  f_0 THE HOMOGENEOUS CUBIC PART SHOULD BE REDUCIBLE.

UNIVERSAL PERFECT FORMS

A CUBIC FORM IN THREE VARIABLES IS UNIVERSAL AND PERFECT IF IT REPRESENTS EVERY k AND RICHLY ($V_k(\mathbb{Z})$ IS ZARISKI DENSE AND A FORM OF STRONG APPROXIMATION HOLDS)

A POSSIBLE EXAMPLE ?:

EVERY k IS ADMISSIBLE FOR

$$x_1^3 + x_2^3 + 2x_3^3$$

AND PERHAPS IT IS UNIVERSAL AND PERFECT.

GHOSH / S (2017):

$$U(x_1, x_2, x_3) = x_2(x_3 - x_1) + x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3$$

IS UNIVERSAL AND PERFECT.

CHARACTER VARIETIES

V_k IS THE RELATIVE CHARACTER VARIETY OF REPRESENTATIONS OF THE FUNDAMENTAL GROUP OF A SURFACE OF GENUS ONE WITH ONE PUNCTURE, TO SL_2. THE ACTION OF THE MAPPING CLASS GROUP IS THAT OF Γ.

MORE GENERALLY THE (AFFINE) RELATIVE CHARACTER VARIETY X_k OF REPRESENTATIONS OF $\pi_1(\Sigma_{g,n})$ INTO SL_2 IS DEFINED OVER $\mathbb{Z}$.

$\Sigma_{g,n}$ IS A SURFACE OF GENUS g WITH n PUNCTURES.

ONE CAN STUDY THE DIOPHANTINE PROPERTIES OF $X_k(\mathbb{Z})$.

J.H. WHANG (PRINCETON THESIS 2018) HAS MADE BIG STEPS IN THIS DIRECTION.

(i) X_k HAS A PROJECTIVE COMPACTIFICATION RELATIVE TO WHICH X_k IS "LOG CALABI YAU". ACCORDING TO CONJECTURES OF VOJTA THIS PLACES X_k AS BEING IN THE SAME THRESHOLD SETTING AS AFFINE CUBIC SURFACES.

(ii) $X_k(\mathbb{Z})$ HAS A FULL DESCENT IN THAT THE MAPPING CLASS GROUP ACTS VIA NON-LINEAR MORPHISMS ON $X_k(\mathbb{Z})$ WITH FINITELY MANY ORBITS.

- THESE AND MORE GENERAL CHARACTER VARIETIES CONNECTED WITH HIGHER TEICHMULLER THEORY OFFER A RICH FAMILY OF THRESHOLD AFFINE VARIETIES FOR WHICH ONE CAN APPROACH THE STUDY OF INTEGRAL POINTS.

REFERENCES CAN BE FOUND IN

"INTEGRAL POINTS ON MARKOFF TYPE CUBIC SURFACES"

A. GHOSH + P. SARNAK

arXiv: 1706.06712

AND

"NONLINEAR DESCENT ON MODULI OF LOCAL SYSTEMS"

J. H. WHANG ARXIV: 1710:01848.