Chapter IV

ASYMPTOTIC BEHAVIOR OF A VARIATION OF HODGE STRUCTURE

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§1. *Nilpotent Orbit Theorem*

Consider a variation of Hodge structure over a punctured disk,

$$\phi : \Delta^* \to \{T^k\}\backslash D ,$$

where T is the image under the monodromy representation of the generator of $\pi_1(\Delta^*)$. By the monodromy theorem T is quasi-unipotent:

$$(T^N - I)^{m+1} = 0 \quad \text{for some} \quad N, \quad m \in \mathbf{Z} .$$

Let s be the coordinate on Δ^*. Replacing s by s^N, we may assume that T is unipotent. If the variation of Hodge structure arises from a degenerating family, this amounts to pulling the family back to an N-fold cover of Δ^*. Because the upper half plane $\mathfrak{h}$ is simply connected, the local liftability of the period map ϕ implies its global liftability to $\mathfrak{h}$:

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\tilde{\phi}} & D \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\phi} & \{T^k\}\backslash D . \end{array}$$

Here

$$\mathfrak{h} = \{\omega = u + iv \mid v > 0\}$$

and

$$s = \pi(\omega) = \exp(2\pi i\omega)\ .$$

Note that

$$\tilde{\phi}(\omega+1) = T\tilde{\phi}(\omega)\ .$$

In terms of the data $\mathcal{O} = \{\mathcal{H}_{\mathbf{Z}}, \mathcal{F}^p, \nabla, S = \Delta^*\}$ we think of T as an action on the lattice $H_{\mathbf{Z}} = (\mathcal{H}_{\mathbf{Z}})_{s_0}$, which of course induces an action on $H_{\mathbf{C}} = (\mathcal{H}_{\mathbf{C}})_{s_0}$. Analytic continuation around $s = 0$ gives

$$H_{e^{2\pi i}s} = TH_s\ .$$

Because T is unipotent, its logarithm can be defined as a finite sum:

$$N = \log T = (T-I) - \frac{(T-I)^2}{2} + \cdots + (-1)^{m+1}\frac{(T-I)^m}{m}\ .$$

It is an elementary fact that every holomorphic vector bundle over the punctured disk is trivial. Hence the cohomology bundle $\mathcal{H} \to \Delta^*$ is trivial. Each trivialization gives rise to an extension of a bundle over the disk Δ. Of the many trivializations possible, we single out one, called the *privileged extension*, defined as follows.

Using the lattice bundle $\mathcal{H}_{\mathbf{Z}}$ inside $\mathcal{H}$, it is possible to speak of the *horizontal displacement* of an element e in the fiber $\mathcal{H}_{s_0}$. By horizontally displacing e, we get a multi-valued flat global section $e(s)$ of $\mathcal{H}$ over Δ^*; $e(s)$ is multi-valued because $e((\exp 2\pi i)s) = Te(s)$. Define

$$\sigma_e(s) = \exp\left(-\frac{\log s}{2\pi i}N\right)e(s)\ .$$

Because

$$\sigma_e((\exp 2\pi i)s) = \exp\left(-\frac{\log s}{2\pi i}N\right)T^{-1}\,Te(s) = \sigma_e(s)\ ,$$

$\sigma_e(s)$ is a *single-valued* holomorphic section of $\mathcal{H}$ over Δ^*.

DEFINITION 1. The *privileged extension* of $\mathcal{H} \to \Delta^*$ to $\overline{\mathcal{H}} \to \Delta$ is given by taking $\{\sigma_e\}$ to be a holomorphic frame, as e ranges over a basis of the fiber $\mathcal{H}_{s_0}$.

Return now to the period map $\phi: \Delta^* \to D\backslash\Gamma$ of a degenerating family. As before, we have the diagram

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\tilde{\phi}} & D \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\phi} & D\backslash\Gamma\ . \end{array}$$

Set

$$\tilde{\psi}(w) = \exp(-wN)\tilde{\phi}(w) \in \check{D}\ .$$

Then

$$\tilde{\psi}(w+1) = \tilde{\psi}(w)$$

so that $\tilde{\psi}$ descends to a *single-valued* map $\psi: \Delta^* \to \check{D}$ given by

$$\begin{aligned} \psi(s) &= \tilde{\psi}(w) \\ &= \tilde{\psi}\left(\frac{\log s}{2\pi i}\right) \\ &= \exp\left(-\frac{\log s}{2\pi i}N\right)\phi(s)\ . \end{aligned}$$

THEOREM 2. *The map* $\psi: \Delta^* \to \check{D}$ extends across the origin to a map $\psi: \Delta \to \check{D}$.

For a proof see Cornalba and Griffiths [1, p. 89] or Griffiths and Schmid [2, p. 104]. The idea is as follows. We view ψ as a map into a product of Grassmannians, represented by a matrix whose entries are the periods. Composing ψ with the Plücker embedding P gives a map $P\circ\psi$ into a projective space. By the theorem on regular singular points, the periods have at most poles at the origin. Since $P \circ \psi$ is given by the minors of ψ, it is meromorphic at the origin. By factoring out the common factors of $P \circ \psi$, it follows that $P \circ \psi$ and hence ψ can be extended across the origin.

DEFINITION 3. The filtration $\psi(0) \in \check{D}$ will be called the *limiting filtration* and will be denoted by $\{F_\infty^p\}$.

We remark that $\{F_\infty^p\}$ may well lie outside $\bar{D}$. It arises in two contexts; one is the nilpotent orbit theorem (to be discussed momentarily), and the other is as the Hodge filtration in the limiting mixed Hodge structure (to be discussed below).

DEFINITION 4. The *nilpotent orbit* of a degenerating family over Δ^* is the map $\mathcal{O}: h \to \check{D}$ given by

$$\mathcal{O}(w) = \exp(wN)\psi(0)$$

The nilpotent orbit satisfies

$$\mathcal{O}(w+1) = T\,\mathcal{O}(w) .$$

Schmid's nilpotent orbit theorem says that this nilpotent orbit is a very good approximation of the original period map.

THEOREM 5 (Nilpotent orbit theorem). a) *The nilpotent orbit is horizontal.* b) *For* $\mathrm{Im}\, w >> 0$, *the nilpotent orbit assumes values in* D. c) *The nilpotent orbit osculates to the period map to very high order; more precisely, there are constants* A *and* B *such that for* $\mathrm{Im}\, w \geq A > 0$,

$$\rho_D(\mathcal{O}(w), \tilde{\phi}(w)) \leq (\mathrm{Im}\, w)^B e^{-2\pi \mathrm{Im} w} .$$

For a proof see Cornalba and Griffiths [1, p. 90]. The original proof is in [3].

§2. *The* SL_2*-orbit Theorem*

The nilpotent orbit theorem is just the first step. For example, using it we still do not see that $N^{n+1} = 0$, where n is the weight of the Hodge structure in question — this is the strong form of the monodromy theorem, giving $n+1$ as a bound on the Jordan blocks of the monodromy matrix. Moreover, we do not see from it the answer to the main question:

QUESTION. Is the Hodge length $\|e\|$ of an invariant cohomology class $e \in H$ bounded on Δ^* ?

The affirmative answer to this is what is needed to extend the theorem on the fixed part and its consequences to a variation of Hodge structure with an arbitrary algebraic base space. The point is that a *bounded* plurisubharmonic function on an algebraic variety (possibly noncomplete) is constant.

These results are consequences of the SL_2-orbit theorem of W. Schmid [3]. Roughly speaking, given a nilpotent orbit $\mathcal{O}: \mathfrak{h} \to D$, the SL_2-orbit theorem enables us to construct a variation of Hodge structure

$$\phi: \mathfrak{h} \to D$$

which lifts to a homomorphism of Lie groups

$$\psi: SL_2(R) \to G_R$$

such that

$$\rho_D(\mathcal{O}(w), \phi(w)) \leq C(\mathrm{Im}\, w)^{-1}$$

for some constant C. Thus the nilpotent and SL_2-orbits are asymptotic to each other. A stronger but more technical form of this asymptotic behavior is possible (see Schmid [3]). Rather than stating it, we will now discuss the consequences.

First, given a variation of Hodge structure of weight n over Δ^*, we get

$$N^{n+1} = 0 .$$

Furthermore, there exists a unique ascending filtration $\{W_\ell\}$ of H_Q, called the *monodromy filtration,*

$$0 \subset W_0 \subset \cdots \subset W_{2n} = H_Q ,$$

satisfying

$$N : W_i \to W_{i-2}$$

$$N^k : W_{n+k}/W_{n+k-1} \xrightarrow{\sim} W_{n-k}/W_{n-k-1} .$$

One should think of the monodromy operator N as the analogue of the operator "cup product with the Kähler class" in the Hard Lefschetz theorem.

The monodromy filtration is uniquely characterized by these properties. For by taking $k = n$, we get

$$N^n : W_{2n}/W_{2n-1} \xrightarrow{\sim} W_0 ,$$

from which it follows that

$$W_{2n-1} = \ker N^n$$

and

$$W_0 = \operatorname{im} N^n .$$

The other terms of the monodromy filtration can now be defined by induction, as follows. Set

$$H'_Q = W_{2n-1}/W_0 .$$

Then N induces an operator N' on H'_Q satisfying $(N')^n = 0$. The filtration on H'_Q is

$$\begin{array}{ccccc} 0 \subset W'_0 \subset \cdots \subset & W'_{2n-2} = H'_Q \\ \| & \| \\ W_1/W_0 & W_{2n-1}/W_0 \end{array} .$$

So W_{2n-2} and W_1 are the inverse images of W'_{2n-3} and W'_0 respectively under the projection $W_{2n-1} \to H'_Q$. This process continues and uniquely constructs the monodromy filtration on H_Q.

REMARK. Ron Donagi points out the following picture of the monodromy weight filtration: Set

$$N^{p,q} = \operatorname{im} N^p \cap \ker N^{n-q} .$$

These are the obvious spaces that can be constructed from the pair (H,N). Note that $N^{p,q} \supset N^{p+1,q}$ and $N^{p,q} \supset N^{p,q+1}$. Then

$$W_q = \operatorname{span}\Big(\sum_{r+s \leq n-q} N^{r,s} \Big) .$$

DEFINITION 6. A *mixed Hodge structure* $\{H_Q, F^p, W_\ell\}$ of weight n consists of an ascending weight filtration, defined over Q,

$$0 \subset W_0 \subset \cdots \subset W_{2n} = H_C ,$$

such that the Hodge filtration induces a pure Hodge structure of weight m on the graded piece $Gr_m = W_m/W_{m-1}$ of the weight filtration for each $m = 0, \cdots, 2n$. The induced Hodge filtration on Gr_m is

$$F^p(Gr_m) = (F^p \cap W_m)/(F^p \cap W_{m-1}) .$$

For $r \in \mathbf{Z}$, a *morphism of type* (r,r) of mixed Hodge structures is a rationally defined map $f : H_Q \to H'_Q$ such that

$$f(W_\ell) \subset W'_{\ell+2r}$$

and

$$f(F^p) \subset (F')^{p+r} .$$

The main consequence of the proof of the SL_2-orbit theorem is that given a variation of Hodge structure over Δ^*, the limiting filtration $\{F^p_\infty\}$

together with the monodromy weight filtration $\{W_\ell\}$ gives a mixed Hodge structure on the vector space H_Q. This is called the *limiting mixed Hodge structure*. Relative to the limiting mixed Hodge structure N is a morphism of type $(-1,-1)$.

We also get a characterization of the monodromy filtration in terms of the growth of the Hodge length, namely,

$$W_\ell = \left\{ e \in H : \|e\| = 0\left(\left(\log \frac{1}{|s|}\right)^{\frac{\ell-n}{2}}\right)\right\}.$$

COROLLARY 7. *Every local invariant cohomology class has bounded Hodge length.*

Proof. Since

$$N = (T-I) - \frac{(T-I)^2}{2} + \cdots + (-1)^{n+1} \frac{(T-I)^n}{n}$$

and

$$T-I = N + \frac{N^2}{2!} + \cdots + \frac{N^n}{n!},$$

$T-I$ and N have the same kernel. So the invariant cohomology classes are precisely $\ker N$. Since $\ker N \subset W_n$, the characterization of W_n above proves the corollary. q.e.d.

REMARK. There is an interesting, and also *confusing*, point concerning the limiting mixed Hodge structure. At the risk of making matters worse, we shall attempt to clarify it.

Given a variation of Hodge structure over the punctured disk, we lift to the upper-half-plane and consider the VHS as a holomorphically varying filtration

$$\begin{cases} F_w^p \subset H & p = 0, \cdots, n \quad \text{and} \quad \operatorname{Im} w > 0 \\ F_{w+1}^p = TF_w^p \end{cases}$$

on the fixed vector space H. Set $h_p = \dim F_w^p$ and $h = h^0 = \dim H$.

LEMMA 8. *There exists a holomorphic basis* $f_i(w) \in F_w^p$ *satisfying*

$$\begin{cases} \text{(i)} & f_i(w+1) = Tf_i(w) \\ \text{(ii)} & f_i(w) = \sum_{\alpha=0}^{k} f_{i\alpha}(w) w^\alpha, \quad \textit{where} \\ \text{(iii)} & f_{i\alpha}(w+1) = f_{i\alpha}(w) \quad \textit{and} \quad f_{i\alpha}(w) = O(1). \end{cases}$$

Here, $f_i(w)$ *and* $f_{i\alpha}(w)$ *are vectors in the fixed vector space* H.

Proof. Set $s = e^{2\pi i w}$ and

$$\tilde{F}_s^p = \exp(-wN) F_w^p.$$

Then $\{\tilde{F}_s^p\}_{s \in \Delta^*}$ gives a holomorphically varying and single-valued filtration on the vector space. By our discussion above, the holomorphic mapping $\Delta^* \xrightarrow{\psi} G(h_p, H)$ extends across $s = 0$; we let $\tilde{f}_i(s) \in \tilde{F}_s^p$ be a holomorphically varying basis. Then

$$f_i(w) = \exp(wN) \tilde{f}_i(e^{2\pi i w}) \in F_w^p \subset H$$

satisfies the requirements of the lemma. q.e.d.

We now consider the non-zero vector

$$\Lambda_p(w) = f_1(w) \wedge \cdots \wedge f_{h_p}(w) \in \Lambda^{h_p} H$$

Clearly we have

$$\begin{cases} \Lambda_p(w+1) = (\Lambda^p T) \cdot \Lambda_p(w) \\ \Lambda_p(w) = w^{h_p} \tilde{\Lambda}_p(w) + O(w^{h_p - 1}) \end{cases}$$

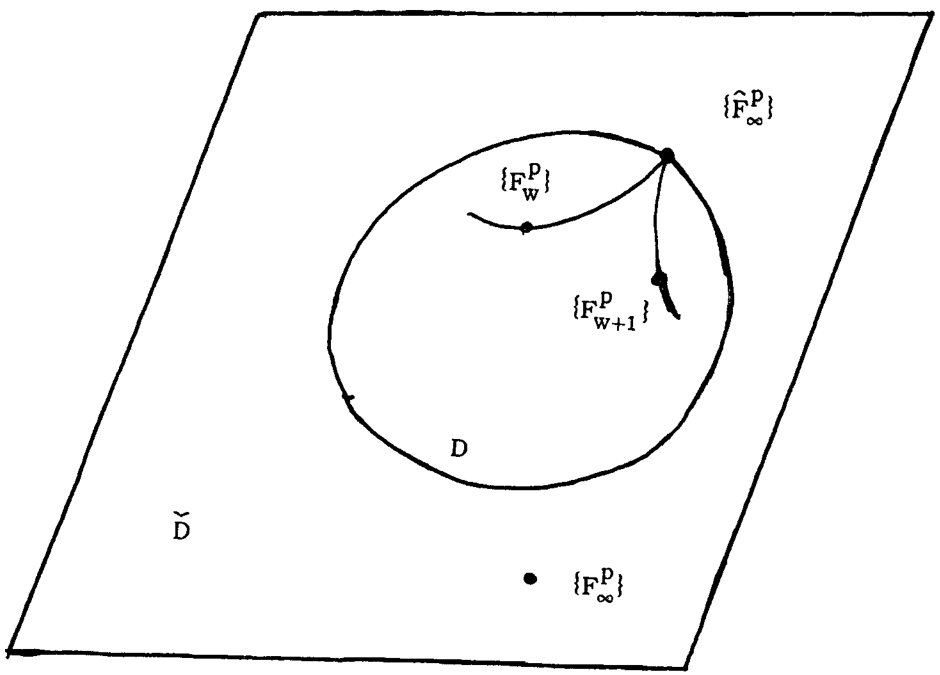
where $\tilde{\Lambda}_p(w+1) = \tilde{\Lambda}_p(w)$, and where $\tilde{\Lambda}_p\left(\frac{\log s}{2\pi i}\right)$ is a holomorphic and non-vanishing function for $|s| < \varepsilon$. Then

$$\begin{aligned}\hat{\Lambda}_p(w) &= w^{-h_p}\Lambda_p(w)\\ &= \tilde{\Lambda}_p(w) + O(w^{-1})\end{aligned}$$

gives the Plücker coordinate of F_w^p. In particular, $\hat{\Lambda}_p(i\infty) = \lim_{w\to i\infty} \hat{\Lambda}_p(w)$ exists as a point in $G(h_p,H) \subset P\Lambda^{h_p}H$; in this way we determine a filtration $\{\hat{F}_\infty^p\}$ on H. Clearly,

$$\begin{cases} \text{(i)} \quad \lim_{w\to i\infty} F_w^p = \hat{F}_\infty^p \in \bar{D} \subset \check{D} \\ \text{(ii)} \quad T\hat{F}_w^p = \hat{F}_w^p . \end{cases}$$

The second statement implies that $\{\hat{F}_w^p\} \in \partial D = \bar{D} - D$ in case $T \neq I$, and so we have a picture like



In the naive sense $\{\hat{F}_\infty^p\}$ is the limiting Hodge filtration, so we call it the *naive limiting Hodge filtration.*

The points we wish to make are

a) *The naive limiting Hodge filtration is not the same as the limiting filtration* $\{F_\infty^p\}$ *discussed above*; and

b) *The limiting filtration* $\{F_\infty^p\}$ *is the "correct" object.*

Point a) is clear since $N\hat{F}_\infty^p \subseteq \hat{F}_\infty^p$ while this is certainly false for F_∞^p. Concerning point b), a preliminary remark is that this same property $(N\hat{F}_\infty^p \subseteq \hat{F}_\infty^p)$ makes it unlikely that the limiting Hodge filtration should give a mixed Hodge structure. Before giving a deeper reason for b), we remark that the relation between the two filtrations is obviously

$$\lim_{w\to i\infty} \exp(wN)\cdot F_\infty^p = \hat{F}_\infty^p .$$

Concerning b), let us first agree that the privileged extension $\bar{\mathcal{H}} \to \Delta$ is the "correct" extension for the cohomology bundle $\mathcal{H} \to \Delta^*$. (This claim will be justified algebro-geometrically in Chapter VII below.) Then, since $\mathcal{F}_s^p$ gives the Hodge filtration on $\mathcal{H}_s$ for $s \neq 0$ and $\lim_{s\to 0} \mathcal{F}_s^p =: \bar{\mathcal{F}}_0^p$ exists (by our discussion above), the "correct" limiting Hodge filtration must be given by $\{\bar{\mathcal{F}}_0^p\}$ on $\bar{\mathcal{H}}_0$ (= the fibre of $\bar{\mathcal{H}} \to \Delta$ over $s = 0$).

Now let $e_i \in H$ be any basis, and denote by $e_i(w) \in \mathcal{H}_s$ $(s = e^{2\pi i w})$ the multi-valued horizontal section of $\mathcal{H} \to \Delta$ determined by e_i. Setting $f_i(s) = \exp(-wN)\,e_i(w) \in \mathcal{H}_s$ $(s = e^{2\pi i w})$ we obtain a *single-valued* holomorphic framing of $\mathcal{H} \to \Delta^*$. The definition of the privileged extension $\bar{\mathcal{H}} \to \Delta$ is that a holomorphic section $g(s) = \Sigma g^i(s) f_i(s) \in \mathcal{H}_s$ extends across $s = 0$ if, and only if, the holomorphic functions $g^i(s)$ extend across $s = 0$. In other words, via the frame field $\{f_i(s)\}$ we have an isomorphism

$$\bar{\mathcal{H}} \cong \mathcal{O}^{(h)} .$$

Intrinsically this is

(9) $$\bar{\mathcal{H}} \cong \mathcal{O}(H)$$

where $\mathcal{O}(H) \to \Delta$ is the trivial bundle with fibre H. Under this isomorphism the subspaces $\mathcal{F}_s^p \subset \mathcal{H}_s$ must go to subspaces of H that are holomorphic *and single-valued* functions of $s \in \Delta^*$. The only such possibility is that $\mathcal{F}_s^p$ maps to $\tilde{F}_s^p$, and therefore $\overline{\mathcal{F}}_0^p$ maps to F_∞^p. More formally we may state this as:

PROPOSITION 10. *Under the isomorphism (9) the subspace* $\mathcal{F}_s^p \subset \mathcal{H}_s$ *maps to* $\tilde{F}_s^p \subset H$. *In particular, at* $s = 0$ *the subspace* $\overline{\mathcal{F}}_0^p$ *maps to* F_∞^p.

The proof consists in unwinding the definitions, and is perhaps therefore best left as a private matter.

Because of the proposition we see that $\{F_\infty^p\}$ is the correct limiting Hodge filtration granted that $\overline{\mathcal{H}} \to \Delta$ is the correct extension of $\mathcal{H} \to \Delta^*$. To make this completely convincing we need to see that $\{F_\infty^p\}$ gives the right answer in the geometric case, and that the limiting mixed Hodge structure has to do with the mixed Hodge structure on the central fibre. This will be done in Chapters VI and VII.

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Chapter V

MIXED HODGE STRUCTURES, COMPACTIFICATIONS AND MONODROMY WEIGHT FILTRATION

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§0. *Introduction*

In his survey paper [10], Griffiths conjectured the existence of partial compactifications for the arithmetic quotients of classifying spaces for polarized Hodge structures, that would generalize the Satake-Baily-Borel compactification for arithmetic quotients of Hermitian symmetric spaces. The richness of the problem becomes clear in the fundamental work of Schmid [16], on the asymptotic behavior of the period mapping (see Chapter IV); in particular, and as a consequence of his Nilpotent and SL_2-orbit theorems, Schmid was able to show—as conjectured by Deligne—the existence of a limiting mixed Hodge structure associated to a one-parameter variation of polarized Hodge structures. (This was also done independently by Steenbrink [18] for the geometric case. His approach will be discussed in Chapter VII.)

Schmid's work is at the core of the topological partial compactification constructed in [5] for the case of Hodge structures of weight two. However, this compactification as well as Satake's in the Hermitian symmetric case contain only part of the "information" in the limiting mixed Hodge structure; namely the Hodge structures on the graded pieces of the weight filtration. On the other hand, Carlson's work [3] has shown that the extension data of the limiting mixed Hodge structure contains significant geometric information. It is then natural to attempt the construction of partial compactifications which incorporate in the boundary the full limiting mixed Hodge structure. When this is done in the Hermitian symmetric case, one