LOGARITHMIC HODGE STRUCTURES: (REPORT ON THE WORK OF KATO-USUI)*

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Outline

I. Introduction

Given $(H_{\mathbb{Z}}, Q, \mathbf{h}, \Gamma)$ where

- $H_{\mathbb{Z}}$ is a lattice and $Q : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$ non-degenerate and $Q(u,v) = (-1)^n Q(v,u)$;
- $\mathbf{h} = (h^{n,0}, \dots, h^{0,n})$, $h^{p,q} = h^{q,p}$ is a set of Hodge numbers;
- Γ is an arithmetic subgroup of $G_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q)$.

Since the late 1960's it is known that there exists

$$\mathcal{M}_{\mathbf{h}} = \left\{ \begin{array}{l} \text{moduli space } \Gamma\text{-equivalence} \\ \text{classes of } \textit{polarized Hodge} \\ \textit{structures} \text{ on } H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}. \end{array} \right\}$$

This moduli space is

$$\mathcal{M}_{\mathbf{h}} = \Gamma \setminus D_{\mathbf{h}}$$

where $D_{\mathbf{h}} = G_{\mathbb{R}}/V$ is a homogeneous complex manifold on which Γ acts properly discountinuously. In the *classical case* $n = 1$ (polarized abelian varieties) or $n = 2$, $h^{2,0} = 1$, D is a bounded symmetric domain and $\Gamma \setminus D$ is a quasi-projective variety defined over a number field. In the non-classical case the situation is quite different.

Given a smooth projective family $\mathcal{X} \xrightarrow{\pi} S$ over a quasi-projective base, there is a *period map*

$$\varphi : S \to \mathcal{M}_{\mathbf{h}}$$

where, setting $X_s = \pi^{-1}(s)$ for $s \in S$,

$$\varphi(s) = \{\text{Hodge structure on } H^n(X_s)_{\text{prim}}\} .$$

*Notes from a seminar given at MSRI on March 18, 2009.

Here we assume that the *monodromy group* of the family is contained in Γ.

Since the beginning it has been desired to "enlarge" $\mathcal{M}_{\mathbf{h}}$ to $\mathcal{M}_{\mathbf{h},\Sigma}$ so that, among other things, if $\mathcal{X} \to S$ is completed to a family over $\bar{S}$ where $S = \bar{S} \setminus Z$ for Z a reduced normal crossing divisor in $\bar{S}$, the period mapping extends to

$$\varphi : \bar{S} \to \mathcal{M}_{\mathbf{h},\Sigma} .$$

Here, Σ is a *fan* compatible with Γ. Such enlargements have been done in the classical case (Baily-Borel, Mumford et. al.,…). One may take $\mathcal{M}_{\mathbf{h},\Sigma}$ to be essentially smooth and $\mathcal{M}_{\mathbf{h},\Sigma} \setminus \mathcal{M}_{\mathbf{h}}$ a reduced normal crossing divisor. The problem of extending this story to the non-classical case has been around for about 50 years; it has now been resolved by Kato-Usui:

Theorem. *There exists $\mathcal{M}_{\mathbf{h},\Sigma} = \Gamma \setminus D_{\mathbf{h},\Sigma}$ which is a fine moduli space of polarized logarithmic Hodge structures with a Γ-level structure whose local monodromies are in the directions of cones $\sigma \in \Sigma$.*[1]

Moreover, period mappings extend *in the log-analytic category*; $\mathcal{M}_{\mathbf{h},\Sigma}$ is essentially smooth as a log-analytic variety; $\mathcal{M}_{\mathbf{h},\Sigma} \setminus \mathcal{M}_{\mathbf{h}}$ is a (reduced) normal crossing divisor, etc.

In the $n = 1$ case, this is almost certainly related to Olsson's work.

In the non-classical case, $\mathcal{M}_{\mathbf{h},\Sigma}$ is a new type of object.

- It is *not compact* — it is only compact relative to period mappings.
- It is an *analytic variety with slits.*

There are reflections of the fact that in the non-classical case there is a $G_{\mathbb{R}}$-invariant distribution

$$W \subset TD_{\mathbf{h}}$$

such that period mappings satisfy

$$\varphi_x : TS \to W \subset T\mathcal{M}_{\mathbf{h}} ;$$

i.e. there are *universal differential constraints* on how the Hodge structures vary.

The theory has already had a nice application.

Theorem (Usui)**.** *The global Torelli theorem holds for Calabi-Yau threefolds of mirror quintic type.*

[1] To have a fine moduli space, one must assume that Γ is neat. There are also some further technical aspects to the full statement of the Kato-Usui result.

These are smooth varieties in the family

$$x_0^5 + \cdots + x_4^5 + t x_0 \dots x_4 = 0$$

factored by a finite group of 5^{th} roots of unity acting on the x_i's and permutations of the coordinates. Such an X has

$$\begin{cases} h^{3,0}(X) = h^{2,1}(X) = 1 \\ \#\ \text{moduli} \ = 1 \\ \dim D_{\mathbf{h}} = 4 \text{ but integrals of } W \text{ have } \dim = 1.^2 \end{cases}$$

They have some of the flavor of elliptic curves. Usui's proof uses the period map at the boundary, and is the first global Torelli theorem I am aware of in non-classical cases. One of the proofs (Friedman) of the Torelli theorm for polarized K3 surfaces also uses the period map at the boundary. One may suspect there will be more such results using the Kato-Usui spaces.

In this over-view talk I will attempt to

- describe $D_{\mathbf{h}}$
- describe $D_{\mathbf{h},\Sigma}$ as a set
- define polarized logarithmic Hodge structures.

The construction of $\mathcal{M}_{\mathbf{h},\Sigma}$ is a combination of

(i) log-analygic geometry
(ii) algebraic group theory/toroidal geometry
(iii) degenerations of Hodge structures (Catalani-Kaplan-Schmid and Kashiwara)

I will not have time to go into detail but will try to indicate how (i) enters using the results from (iii).

Bottom Line. *One may enlarge the moduli space of polarized Hodge structures to a moduli-space of polarized logarithmic Hodge structures in the category of log analytic varieties such that*

(A) *period maps extend;*
(B) *one may "do geometry" on the enlarged space.*[3]

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[2]In this case, $W \subset TD_{\mathbf{h}}$ is a 2-plane field given in suitable local coordinates (x, y, y', y'') by

$$\begin{cases} dy - y'dx = 0 \\ dy' - y''dx = 0. \end{cases}$$

[3]e.g., the differential of the period map along the discriminant locus parametrizing the singular varieties, after semi-stable reduction as per Abramovich and Karu.

II. Moduli spaces of polarized Hodge structures

Given $H_{\mathbb{Z}}, Q$ as above, we set $H = H_{\mathbb{Q}}$ and $G = \mathrm{Aut}(H, Q)$ (it is more convenient to do Hodge structures and mixed Hodge structures over $\mathbb{Q}$ instead of $\mathbb{Z}$).

Definition. A *polarized Hodge structure* (PHS) *of weight* n (H, F, Q) is given by a filtration

$$F^n \subset F^{n-1} \subset \cdots \subset F^0 = H_{\mathbb{C}}$$

that satisfies (*Hodge filtration conditions*)

$$F^p \oplus \bar{F}^{n-p+1} \xrightarrow{\sim} H_{\mathbb{C}}$$

for each p, as well as (*polarization conditions*)

$$\begin{cases} Q(F^p, F^{n-p+1}) = 0 \\ \qquad Q(Cu, \bar{u}) > 0, \qquad u \neq 0\ . \end{cases}$$

Here, C is the *Weil operator*, defined to be multiplcation by $\sqrt{-1})^{p-q}$ on $H^{p,q} =: F^p \cap \bar{F}^q$ $(p+q = n)$. The *Hodge numbers* are $h^{p,q} = \dim H^{p,q}$. Henceforth we fix the sequence of $h^{p,q}$'s and omit reference to them.

The set D of PHS's is acted on by the real group $G_{\mathbb{R}}$. It is elementary that this action is transitive and that the isotropy group V of a reference point $F \in D$ is compact. Thus

$$D = G_{\mathbb{R}}/V$$

is a homogeneous complex manifold. The complex structure arises as follows: Set $f^p = h^{n,0} + \cdots + h^{p,n-p}$ and let

$$\check{D} \subset \prod \mathrm{Grass}(f^p, H_{\mathbb{C}})$$

be the flags defined by $Q(F^p, F^{n-p+1}) = 0$. Then

$$\check{D} = G_{\mathbb{C}}/B$$

is a homogeneous projective variety, where B is a parabolic subgroup of $G_{\mathbb{C}}$. The *period domain* D is an open set in $\check{D}$, hence is a complex manifold. One refers to $\check{D}$ as the *compact dual* of D. It plays a central role in degenerations of Hodge structures (see section III below). Polarized Hodge structures are functorial for all linear algebra constructions. In particular, the Lie algebra of G

$$\mathcal{G} = \mathrm{End}(H, Q) \subset \mathrm{Hom}(V, V)$$

has a Hodge structure of weight zero

$$\begin{cases} \mathcal{G}_{\mathbb{C}} = \bigoplus_i \mathcal{G}^{-i,i} \\ \mathcal{G}^{-i,i} = \{A \in \mathcal{G} : A(H^{p,q}) \subset H^{p+i,q-i}\}\ . \end{cases}$$

There is a natural identification

$$\begin{array}{ccc} T_F D & = & \bigoplus_{i>0} \mathcal{G}^{-i,i} \\ \cup & & \cup \\ W_F & = & \mathcal{G}^{-1,1} \, . \end{array}$$

The arithmetic group Γ, assumed to be commensurable with $G_{\mathbb{Z}}$, acts properly discontinuously on D and we set

$$\mathcal{M} = \left\{ \begin{array}{l} \text{moduli space of} \\ \Gamma\text{-equivalence classes} \\ \text{of PHS's} \end{array} \right\} = \Gamma \setminus D \, .$$

Let now S be a smooth, quasi-projective variety. We assume $S = \bar{S} \setminus Z$ where $\bar{S}$ is smooth and Z is a reduced NCD.

Definition. A *period mapping* is given by a locally liftable holomorphic mapping

$$\varphi : S \to \mathcal{M}$$

that satisfies the differential constraint

$$(*) \qquad \varphi_* : TS \to W \, .$$

Much is known about period mappings. For example, the image $\varphi(S) \subset \mathcal{M}$ is an *algebraic* subvariety of the analytic variety $\mathcal{M}$. Assuming for simplicity that φ is finite, and denoting by $\mathcal{F}^p \to \mathcal{M}$ the obvious vector bundles and setting

$$\mathcal{L} = \prod_p \det(\mathcal{F}^{n-p}/\mathcal{F}^{n-p+1})^{n-p} \, ,$$

then

$\varphi^{-1}(\mathcal{L})$ extends naturally to a line bundle $\bar{\mathcal{L}} \to \bar{S}$, and $\bar{\mathcal{L}}$ is ample modulo Z.

In the classical case, *automorphic forms* give sections of $\mathcal{L}^{\otimes m} \to \mathcal{M}$.

Question. *Extend $\mathcal{L}$ to a log-analytic line bundle $\mathcal{L}_\Sigma \to \mathcal{M}_\Sigma$.*

This seems to be a subtle issue. I am not sure of its status in even the classical case.

For later reference we give the geometric interpretation of the differential constraint $(*)$. For a period mapping as above we have the *monodromy representation*

$$\rho : \pi_1(S) \to \Gamma \, .$$

This gives a flat bundle $(\mathcal{H}, \nabla)$, or in more common parlance a *local system* $\mathcal{H}_{\mathbb{Z}}$ on S, such that $\mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_S \cong \mathcal{H}$. Then the Gauss-Manin connection

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_S$$

is defined by $\nabla(\mathcal{H}_{\mathbb{Z}} \otimes \mathbb{C}) = 0$. There are the Hodge filtration bundles $\mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \subset \mathcal{F}^0 = \mathcal{H}$, and the condition $(*)$ translates to

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S \; .$$

III. Degeneration of polarized Hodge structures

Given a period mapping $\varphi : S \to \mathcal{M}_{\mathbf{h}} = \Gamma \setminus D$ where $S = \bar{S} \setminus Z$ as above, the behavior of $\varphi(s)$ as s tends to Z is a very rich story. Localizing in the analytic topology around a point $s_0 \in Z$, we obtain

$$\varphi : (\Delta^*)^k \times \Delta^l \to \Gamma \setminus D$$

where $s_0 = (0, \ldots, 0)$. We take first the case $k = 1$, $l = 0$. Then the image of $\pi_1(\Delta^k) \cong \mathbb{Z}$ is generated by the *monodromy transformation* $T \in G_{\mathbb{Z}}$. It is known that the eigenvalues of T are roots of unity, so that by base change we may assume that T is *unipotent* — in fact $(T - I)^{n+1} = 0$ — and we set

$$N = \log T = (T - I) - \frac{(T - I)^2}{2} + \cdots \pm \frac{(T - I)^n}{n} \; .$$

We then have a diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\tilde{\varphi}} & D \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\varphi} & \{T^m\} \setminus D \end{array}$$

where $\mathcal{H} = \{\operatorname{Im} z > 0\}$ and $\mathcal{H} \to \Delta^*$ is given by $s = \exp(2\pi\sqrt{-1}z)$. Note that

$$\tilde{\varphi}(z + 1) = T\tilde{\varphi}(z) \; .$$

Because of monodromy, periods given by φ are multi-valued on Δ^*. We can make φ single-valued by setting

$$\tilde{\psi}(z) = \exp(-zN)\tilde{\varphi}(z) \in \check{D} \; ,$$

where $\exp(-zN) \in G_{\mathbb{C}}$. We then have from the above that $\tilde{\psi}(z + 1) = \tilde{\psi}(z)$ so that there is an induced map

$$\psi = \Delta^* \to \check{D} \; .$$

Theorem. (i) *ψ extends across the origin to give a holomorphic map $\psi : \Delta \to \check{D}$. We set $\psi(0) = F_0 \in \check{D}$.* (ii) *The original period mapping is strongly approximated by the* **nilpotent orbit** $\exp(\mathbb{C}N) \cdot F_0$.

Setting $l(s) = \frac{\log s}{2\pi\sqrt{-1}}$, we may replace φ by

$$\exp(l(s)N)F_0 \ .$$

This has the properties
(**)

$$\begin{cases} \text{(i)} \quad \exp(l(s)N)F_0 \in D \text{ for } 0 < |s| < \epsilon \\ \\ \text{(ii)} \quad N(F_0^p) \subseteq F_0^{p-1} \text{ (this reflects the differential constraint } (*)). \end{cases}$$

Historical Remark. The interaction between logarithmic analytic geometry and Hodge theory may be said to have at least in part begun with the result[4] that the differential equations (*Picard-Fuchs equations*) satisfied by periods of integrals of algebraic differential forms have regular singular points. This is the *regularity of the Gauss-Manin connection*, as reported on in Deligne's book. In the geometric case, this is basically (i) above. The result (ii) and its implications are due to Schmid (1972). They have the following consequence: Associated to any nilpotent endomorphism N there is a unique *weight filtration* $W_\bullet(N)$, defined over $\mathbb{Q}$,

$$0 \subset W_0(N) \subset \cdots \subset W_{2n}(N) = H$$

satisfying

$$\begin{cases} N(W_m(N)) \subset W_{m-2}(N) \\ N^k : \mathrm{Gr}_{n+k} W_\bullet(N) \xrightarrow{\sim} \mathrm{Gr}_{n-k} W_\bullet(N) \ . \end{cases}$$

Theorem. *The data $(H, Q, W_\bullet(N), F_0)$ gives the polarized mixed Hodge structure. Moreover, any such polarized mixed Hodge structure arises in this way.*

Thus, in a very precise way Hodge structures degenerate into a special type of mixed Hodge structures.

In several variables, one has commuting monodromies T_i and their logarithms N_i. These generate a cone

$$\sigma = \{\Sigma \lambda_i N_i, \quad \lambda_i > 0\} \ .$$

Cattani-Kaplan proved the beautiful result that the monodromy weight filtations are the same for all $N \in \sigma$. Schmid's results were then extended to the several variable case by Cattani-Kaplan-Schmid and Kashiwara.

[4]Proved in a seminar here in Berkeley in the mid 1960's.

The appearance of the monodromy cones suggests — correctly — that toroidal geometry will enter the picture when one seeks to enlarge $\mathcal{M}_{\mathbf{h}}$ to $\mathcal{M}_{\mathbf{h},\Sigma}$. In fact, a *fan* Σ is just a collection of nilpotent cones satisfying certain conditions, including compatibility with Γ.

Definition. A *nilpotent orbit* (σ, F) is given by a nilpotent cone as above and $F \in \check{D}$ such that the analogues of $(**)$

$$\begin{cases} \text{(i)} \quad \exp(\sum_i z_i N_i) F \in D \text{ for } \operatorname{Im} z_i \gg 0 \\ \text{(ii)} \quad N(F^p) \subseteq F^{p-1} \text{ for } N \in \sigma \end{cases}$$

are satisfied.

Definition. As a set

- $D_{h,\Sigma} = D \cup \{\text{set of } \sigma\text{-nilpotent orbits for } \sigma \in \Sigma\}$
- $\mathcal{M}_{\mathbf{h},\Sigma} = \Gamma \setminus D_{\mathbf{h},\Sigma}$.

The issue now is to put a structure of a log analytic variety on $D_{\mathbf{h},\Sigma}$ and $\mathcal{M}_{\mathbf{h},\Sigma}$.

Remark that in the classical case, the above coincides with those constructed using toiroidal compactifications. According to Kato-Usui, even in the classical case the log-analytic structure is new.

IV. Logarithmic Hodge structures and Kato-Usui spaces

Review of log structures — notations and terminology.

• A *logarithmic structure* on a local ringed space is given by a sheaf of monoids M_X and $\alpha : M_X \to \mathcal{O}_X$ such that $\alpha : \alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}_X^*$.

• A *standard example* is given by a complex manifold X, a reduced normal crossing divisor $Z \subset X$ defined locally by $q_1 \dots q_k = 0$, and after setting $U = X \setminus Z$ defining

$$M_X = \{f \in \mathcal{O}_X : f \text{ is invertible on } U\}$$

with the obvious map $M_X \to \mathcal{O}_X$.

• Let R be an integral, saturated, finitely generated monoid, and $h : R \to \mathcal{O}_X$ a homomorphism of sheaves of monoids, R being the constant sheaf. The associated logarithmic structure on X is the pushout $\tilde{R}$ in the diagram

$$\begin{array}{ccc} h^{-1}(\mathcal{O}_X^*) & \longrightarrow & R \\ \downarrow & & \\ \mathcal{O}_X^*. & & \end{array}$$

Definition. An *fs logarithmic* structure is one locally of this type.

For X, Z as above, $R = \mathbb{N}^k$ and

$$\begin{array}{ccc} \mathbb{N}^k & \to & \mathcal{O}_X \\ \Cup & & \Cup \\ (n_i) & \to & \prod_i q_i^{n_i} \end{array}$$

we obtain the standard example.

• Given a logarithmic structure, we set

$$X^{\log} = \left\{ \begin{array}{l} (x,h) : x \in X \text{ and } h : M_{X,x} \to S^1 \\ \text{such that } h(u) = \frac{u(x)}{|u(x)|} \text{ for } u \in \mathcal{O}^*_{X,x} \end{array} \right\}.$$

There is a natural topology on $X^{\log}$ and a continuous map $\pi : X^{\log} \to X$.

If $X = (\Delta^*)^k \times \Delta^l$ as in the standard example, then as a topological space

$$X^{\log} \cong (|\Delta| \times S^1)^k \times \Delta^l$$

where $|\Delta| = \{t : 0 \leqq \epsilon < 1\}$.

• There is a sheaf of rings $\mathcal{O}_X^{\log}$ on $X^{\log}$. For $X = (\Delta^*)^k \times \Delta^l$ and $y \in X^{\log}$ any point lying over the origin x, the stalk

$$\mathcal{O}^{\log}_{X,y} \text{ ``}\cong\text{''} \ \mathcal{O}_{X,x}[\log q_1, \ldots, \log q_k] \ .$$

$\mathcal{O}_X^{\log}$ is not a sheaf of local rings, but has a global ring-theoretic nature. Let $y \in X^{\log}$ and

$$sp(y) = \left\{ \begin{array}{l} \text{homomorphisms } s : \mathcal{O}^{\log}_{X,y} \to \mathbb{C} \\ \text{such that } s(f) = f(x) \text{ for } f \in \mathcal{O}_{X,x} \ . \end{array} \right\}$$

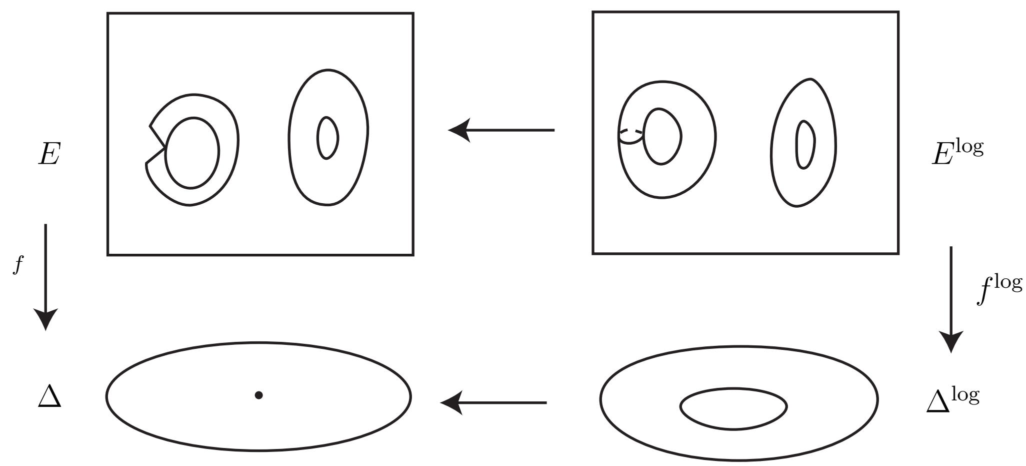
Then, if we fix $s_0 \in sp(y)$

$$\begin{array}{ccll} sp(y) & \cong & \mathrm{Hom}((M_X^{gp}|\mathcal{O}_X^*)_x, \mathbb{C}^{\mathrm{add}}) & \\ \Cup & & \quad\Cup & \\ s & \to & s(\log(f)) - s_0((\log f)) & \qquad f \in M^{gp}_{X,x} \ . \end{array}$$

• Logarithmic differential forms ω_X^q are defined on X. In the standard example

$$\omega_X^q = \Omega_X^q(\log(Z)) \ .$$

More precisely, it is a pullback of this sheaf to $X^{\log}$. The standard picture of a degenerating family of elliptic curves is



Discussion.

- $R^1_\pi\mathbb{Z}$ is a local system over Δ $\begin{cases} (R^1_f\mathbb{Z})_x \cong \mathbb{Z}^2, & x \neq 0 \\ R^1_f\mathbb{Z})_x \cong \mathbb{Z}, & x = 0. \end{cases}$
- there are canonical extensions $\mathcal{H}_e, \mathcal{F}^1_e$ of $\mathcal{H}, \mathcal{F}^1$ to Δ, and
$$\nabla : \mathcal{H}_e \to \mathcal{H}_e \otimes \Omega^1_\Delta(\log(0)).$$
- On the other hand $R^1_{f^{\log}}\mathbb{Z}$ is a local system over $\Delta^{\log}$; *all* stalks are isomorphic to $\mathbb{Z}^2$.

In fact, $E^{\log} \to \Delta^{\log}$ is locally topologically trivial over the base. Note that $\pi_1(\Delta^{\log}) \cong \mathbb{Z}$, so it is not globally topologically trivial. In fact, going around S^1 induces the monodromy transformation on the homology of the fibre. This picture extends in generality to several variable degenerations when we have the situation of semi-stable reduction.

Logarithmic variation of polarized Hodge structure (LVPHS). Let X be a logarithmically smooth fs logarithmic analytic space (think of the standard example).

Definition. A LVPHS on X is given by $(\mathcal{H}_\mathbb{Z}, Q, \mathcal{F}^p)$ where $\mathcal{H}_\mathbb{Z}$ is a local system of constant rank on $X^{\log}$, $Q : \mathcal{H}_\mathbb{Q} \otimes \mathcal{H}_\mathbb{Q} \to \mathbb{Q}$ is as in the definition of PHS, and $\mathcal{F}^p$ is a filtration of $\mathcal{H} = \mathcal{H}_\mathbb{Z} \otimes \mathcal{O}^{\log}_X$ by locally free submodules satisfying (1)–(3) below.

(1) There exists a locally free $\mathcal{O}_X$-module $\mathcal{E}$ and a filtration of $\mathcal{E}$ by locally free sub-modules $\mathcal{G}^p$ such that

$$\mathcal{F}^p = \mathcal{O}_X^{\log} \underset{\pi^{-1}(\mathcal{O}_X)}{\otimes} \mathcal{G}^p$$

where $\mathcal{F}^0 = \mathcal{H}$, $\mathcal{G}^0 = \mathcal{E}$.[5]

(2) Let $x \in X$ and $q_j \in M_{X,x}$ be such that the $q_j \bmod \mathcal{O}_{X,x}$ generate $(M_X/\mathcal{O}_X^*)_x$. Let $y \in \pi^{-1}(x) \subset X^{\log}$. Then if $s \in sp(y)$ and if $\exp(s(\log q_j))$ are near to 0, $(\mathcal{H}_{\mathbb{Z},y}, Q_y, \mathcal{F}_s^p)$ is a polarized Hodge structure. Here, $\mathcal{F}_s^p$ — the *specialization* of $\mathcal{F}^p$ at s — is defined by

$$\mathcal{F}_s^p = \mathbb{C} \underset{\mathcal{O}_{X,y}^{\log}}{\otimes} \mathcal{F}_y^p$$

where $\mathcal{O}_{X,y}^{\log} \xrightarrow{s} \mathbb{C}$; and

(3) $\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \underset{\mathcal{O}_X^{\log}}{\otimes} \omega^{1,\log}$ (differential constraint reflecting $(*)$).

Definition. A *polarized logarithmic Hodge structure* is the stalk of a LVPHS at a point $y \in X^{\log}$.

Theorem (informally stated)**.** *(i) There is a* 1-1 *correspondence, as sets, between* $D_{\mathbf{h},\Sigma}$ *and polarized logarithmic Hodge structures. (ii) Period mappings extend to* $\mathcal{M}_{\mathbf{h},\Sigma} = \Gamma \setminus D_{\mathbf{h},\Sigma}$.

Here, one has to say how PLHS's correspond to nilpotent orbits. This is somewhat implicit in the construction: Given $X = (\Delta^*)^k \times \Delta^l$, a cone σ is generated by the logarithms of monodromies around the S^1's where $X^{\log} \cong (|\Delta| \times S^1)^k \times \Delta^l$. Associated to a cone there is a monoid R as above, and this leads to the PLHS.

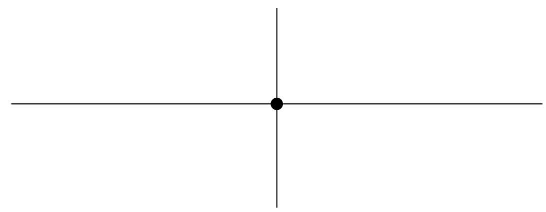
Differences with the classical case.

(a) $\mathcal{M}_{\mathbf{h},\Sigma}$ is non-compact and very non-algebraic (there are no non-constant meromorphic functions on it, except of course in the classical case). However, with respect to period maps, it behaves as if it were a projective variety. Completely missing so far are the arithmetic aspects — e.g., the field of definition.

(b) The underlying analytic variety — i.e., without being enriched by the log-analytic structure — is not an analytic variety in the usual sense. It has "slits" — something like

[5] That is to say that the Hodge filtration bundles are pulled back to $X^{\log}$ from their canonical extensions on X.

$$\{(x,y)\in\mathbb{C}^2\}\setminus\{(0,y):y\neq 0\}$$



(take out the y-axis but leave the origin in)

In the log-analytic world, however, this has a nice structure.

(c) In all cases, boundary components correspond to nilpotent cones σ. The strata of the boundary component correspond to the faces of σ. In the classical case, there is one filtration $W_\bullet(\sigma)$ such that all the monodromy weight filtrations for all the faces of σ (including σ itself) are sub-filtrations of $W_\bullet(\sigma)$. Then

$$P(\sigma) =: \{A\in G : A(W_\bullet(\sigma) = W_\bullet(\sigma)\}$$

is a parabolic subgroup of G. This leads to the description of boundary components in terms of orbits of parabolic groups, thence to the toroidal structure on the boundary components.

In the non-classical case, this is no longer true and the algebraic group aspects of the Kato-Usui spaces is much more subtle. This is portrayed schematically in the diagram

$$\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
D_\Sigma & \leftarrow & D^\sharp_\Sigma. & & & &
\end{array}$$

In this diagram "val" refers to various lattice-type structures of monodromy weight filtration arising from certain sub-monoids of the monoid associated to a nilpotent cone σ. Except in the classical case and when $\dim\sigma = 1$, the boundary components are not acted on transitively by a parabolic group. So far as I know, the geometry of these boundary components as log-analytic spaces remains to be studied. In the classical case, as ordinary algebraic varieties they have a very rich geometric and arithmetic structure. One may of course hope that much of this, including the arithmetic aspects, will be eventually extended to the higher weight case.

Example. For mirror quintic Calabi-Yau varieties, the boundary components all have dimension one and may be pictured as follows

$$N^3 \neq 0 \qquad \begin{array}{ll} \bullet & (3,3) \\ \downarrow & \\ \bullet & (2,2) \\ \downarrow & \\ \bullet & (1,1) \\ \downarrow & \\ \bullet & (0,0) \end{array}$$

This is an iterated extension of Hodge-Tate structures. The intrinsic part of the extension data has arithmetic meaning (essentially $\zeta(-3)$).

$$N^3 = 0, N^2 \neq 0 \qquad \text{doesn't occur}$$

$$\text{rank } N = 1 \left\{ \begin{array}{ccc} & (2,2) & \\ & \bullet & \\ (3,0)\bullet & \downarrow & \bullet(0,3) \\ & \bullet & \\ & (1,1) & \end{array} \right.$$

$$N^2 = 0, N \neq 0$$

$$\text{rank } N = 2 \left\{ \begin{array}{cc} (3,1) & (1,3) \\ \bullet & \bullet \\ \downarrow & \downarrow \\ \bullet & \bullet \\ (2,0) & (0,2) \end{array} \right.$$