COMPLEX ALGEBRAIC GEOMETRY

BROUWER LECTURE, JULY 14 (2008)

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Outline

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  (iii) A potential topological reformulation
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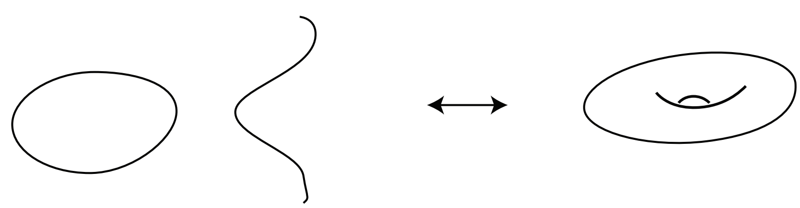
I. Introduction

This talk will be about the geometry of *complex algebraic varieties*. These are defined by "adding the points at infinity" to the solutions in $\mathbb{C}^N$ of polynomial equations

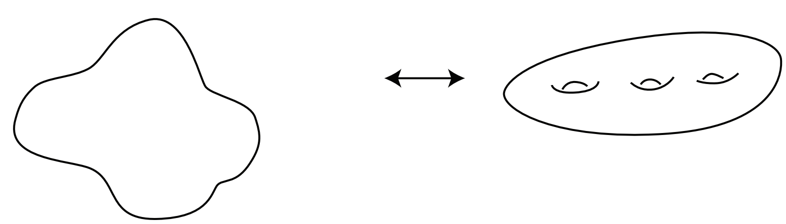
$$\begin{cases} f_1(x_1, \ldots, x_N) = 0 \\ f_m(x_1, \ldots, x_N) = 0\ . \end{cases}$$

Examples include the following

$y^2 = x^3 + ax + b$

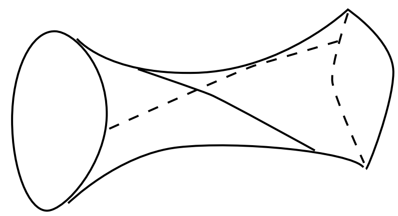


quartic curve

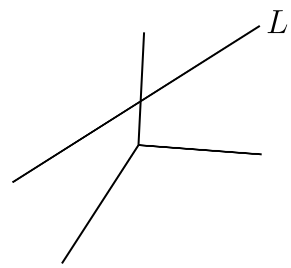


algebraic surfaces

quadric surface



Grassmannian $\mathbb{G}(1,3) =$



$\subset \mathbb{P}^5$

Threefolds such as the family in $\mathbb{P}^4$

$$\left(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5\right) - t(x_0x_1x_2x_3x_4) = 0$$

that has been much studied by physicists and mathematicians.

The geometry of an algebraic variety is especially well revealed by the configurations of algebraic subvarieties lying in it, together with equivalence relations among these subvarieties. For complex algebraic varieties Hodge theory provides the fundamental invariants for the variety and its configurations of subvarieties. Two of the main conjectures, the Hodge conjecture and the conjecture of Beilinson-Bloch, provisionally provide a general framework for understanding the structure of these configurations. In this talk we will seek to explain the two conjectures, together with a summary of their current status and a brief discussion of some of their implications.

This talk will be of a general "overview" nature. We will not be able to discuss important aspects of general Hodge theory, such as *mixed Hodge structures* (cf. the recent book by Peters-Steenbrink) or *variations of Hodge structure* (cf. the books by C. Voisin) or the more arithmetic aspects (*Mumford-Tate groups, endomorphism algebra*, etc. (cf. the lectures by B. Moonan). The title of this talk might better have been

Hodge theory and algebraic cycles

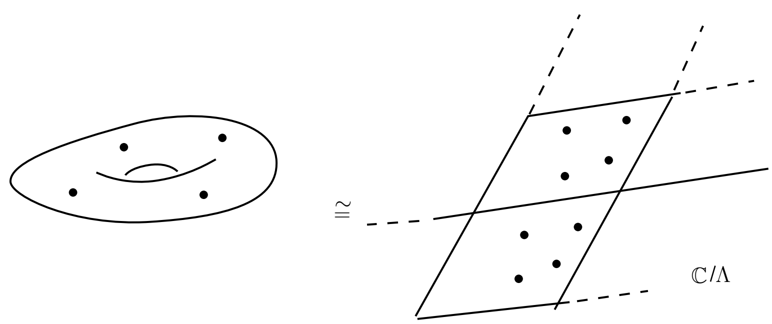
emphasizing in this way the historical roots of the subject as well as a (perhaps "the") central way in which Hodge theory has interacted with geometry. Somewhat in contrast with the classical development of the subject, one conclusion of what follows will be

> *In* **complex** *algebraic geometry, once one leaves the classical realm of codimension-one subvarieties (divisors), arithmetic considerations necessarily and centrally enter into purely* **complex** *algebro-geometric questions.*

II. Algebraic cycles

We consider a smooth, projective algebraic variety X over $\mathbb{C}$. The configurations of the algebraic subvarieties of X underlie questions such as:

— When $\dim X = 1$, X is an algebraic curve, which is the same as a compact Riemann surface



when is a configuration of points p_i, q_i the zeroes and poles of a rational, or equivalently a meromorphic, function on X? Recall that

$$\# p_i = \# q_i \; ;$$

i.e., the number of zeroes is the same as the number of poles.

— When a projective embedding

$$X \subset \mathbb{P}^N$$

is given,[1] the study of the lines lying in X is of interest:

18^{th} century:

- the ∞^2 family of lines (rulings) on the quadric surface

[1] Any complex submanifold of $\mathbb{P}^N$ is an algebraic variety (Chow).

mid 19^{th} century:

- the configuration 27 lines on a smooth cubic surface $X \subset \mathbb{P}^3$

early 20^{th} century:

- the Fano surface of lines on a smooth cubic threefold $X \subset \mathbb{P}^4$

late 20^{th} century:

- the 2875 lines on a (generic?) smooth quintic threefold $X \subset \mathbb{P}^4$

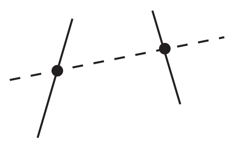
More generally, the configuration of general rational curves in an algebraic variety has surfaced as an important topic.

late 20^{th} and early 21^{st} centuries:

- Mori theory
- rationally connected varieties
- Gromov-Witten invariants (for rational curves)

Many other types of subvarieties enter into interesting geometric questions:

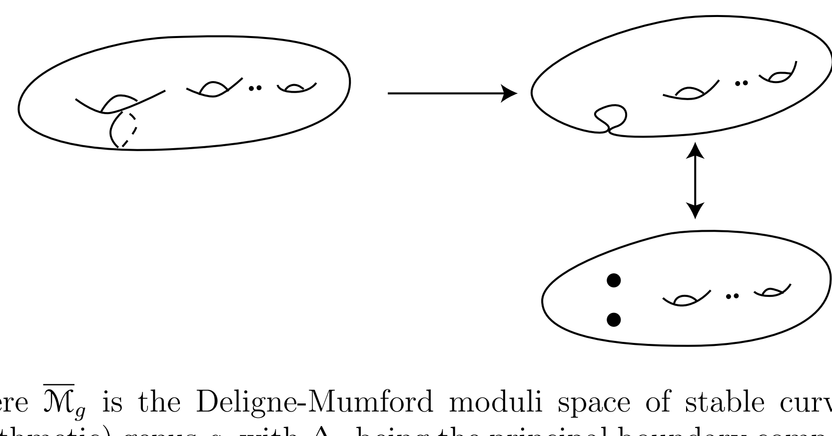
- In *linear geometry* — Schubert cycles in $\mathbb{G}(1,3)$ such as



$\left\{ \begin{array}{l} \text{lines in } \mathbb{P}^3 \\ \text{meeting two} \\ \text{skew lines} \end{array} \right.$

- In *moduli theory*, the subvariety giving the boundary component

$$\Delta_1 \subset \overline{\mathcal{M}}_g$$



where $\overline{\mathcal{M}}_g$ is the Deligne-Mumford moduli space of stable curves of (arithmetic) genus g, with Δ_1 being the principal boundary component corresponding to irreducible curves with a single node.

In the first example above, for $f \in \mathbb{C}(X)^*$ a rational function one defines the *divisor* of f

$$(f) = \sum_{p \in X} \nu_p(f) p$$

to be the formal linear combination of points of X with the integer coefficients $\nu_p(f) = \nu$ to where locally around p with coordinate z $f(z) = z^\nu g(z)$, $g(0) \neq 0$. Relations such as

$$\begin{cases} (fg) = (f) + (g) \\ (f^{-1}) = -(f) \end{cases}$$

suggest considering in general the group

$$Z^p(X) = \left\{ Z = \sum_i n_i Z_i, \quad n_i \in \mathbb{Z} \right\}$$

of *codimension-p algebraic cycles*, where Z_i is a codimension p irreducible subvariety of X. This group is too big, and again the first example suggests considering the *Chow group*

$$\boxed{\mathrm{CH}^p(X) = Z^p(X)/\sim_{\mathrm{rat}}}$$

where $\sim_{\mathrm{rat}}$ is the equivalence relation generated by

$$\begin{array}{l} Z \sim_{\mathrm{rat}} Z' \textit{ if there exists} \\ \mathcal{Z} \in Z^p(X \times \mathbb{P}^1) \textit{ with} \\ \begin{cases} \mathcal{Z} \cdot X \times \{0\} = Z \\ \mathcal{Z} \cdot X \times \{\infty\} = Z' . \end{cases} \end{array}$$

In example one the graph $\Gamma_f \subset X \times \mathbb{P}^1$ gives

$$f^{-1}(0) \sim_{\mathrm{rat}} f^{-1}(\infty) .$$

Cycles may be moved into general position in $\mathrm{CH}^p(X)$, and this leads to good functoriality properties. If we think of the above as associated to a boundary operator on

$$\begin{array}{c} Z^p(X \times (\mathbb{P}^1, \{0, \infty\})) \\ \big\downarrow \\ Z^p(X) , \end{array}$$

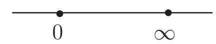
the consideration of similar boundary operators on

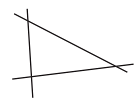
$$\begin{array}{c} Z^p(X \times (\mathbb{P}^1, \{0, \infty\})^{k+1}) \\ \big\downarrow \\ Z^p(X \times (\mathbb{P}^1, \{0, \infty\})^k) \end{array}$$

leads to Bloch's higher $\mathrm{CH}^p(X,k)$'s. It is perhaps fair to say that these form the basic algebro-geometric invariants for the study of the configurations of subvarieties of X, and therefore of X itself. Today we shall only be concerned with the $\mathrm{CH}^p(X) = \mathrm{CH}^p(X,0)$'s.

In addition, *relative Chow groups*

$$\begin{cases} \mathrm{CH}^1(\mathbb{P}^1, \{0,\infty\}) \\ \\ \mathrm{CH}^2(\mathbb{P}^2, T) \end{cases}$$





may also be defined. In fact, they already suggest the appearance of arithmetic phenomena in the geometry of higher codimension cycles *over* $\mathbb{C}$. First, anticipating the discussion below of a filtration on the Chow groups, for the first graded piece of $\mathrm{CH}^1(\mathbb{P}^1)$

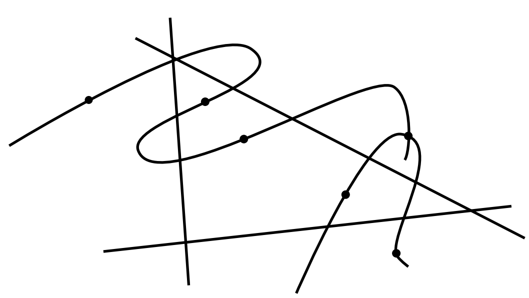
$$\mathrm{Gr}_1\mathrm{CH}^1(\mathbb{P}^1) = 0 \ .$$

This is because any $Z = \sum_i n_i z_i \in Z^1(\mathbb{P}^1)$ has $Z = (\prod_i (z-z_i)^{n_i})$ if $\sum_i n_i = 0$. But if all $z_i \neq 0, \infty$ and $\sum_i n_i = 0$ we have

$$\mathrm{Gr}_1\mathrm{CH}^1(\mathbb{P}^1, \{0,\infty\}) \cong \mathbb{C}^* \quad (= K_1(\mathbb{C}))$$

since for the relative Chow groups we can only use $f \in \mathbb{C}(\mathbb{P}^1)^*$ with $f(0) = f(\infty)$.

Turning to $(\mathbb{P}^2, T)$



one has (Bloch, Suslin,...)

$$\begin{cases} \mathrm{Gr}_1\mathrm{CH}^2(\mathbb{P}^2, T) \cong \mathbb{C}^* \times \mathbb{C}^* \\ \mathrm{Gr}_2\mathrm{CH}^2(\mathbb{P}^2, T) \cong K_2(\mathbb{C}) \end{cases}$$

and, for a field k, $K_2(k)$ is a "very arithmetic" object. Thus

$$\begin{cases} \dim K_2(\mathbb{C}) = \infty \\ \qquad K_2(\overline{\mathbb{Q}}) = 0 \ . \end{cases}$$

III. Hodge-theoretic invariants of algebraic cycles

The general story may be said to have begun with *Abel's theorem* (~ 1820). For X a compact Riemann surface and

$$Z = \sum_i n_i p_i \in Z^1(X)$$

the basic homological invariant is

$$\begin{array}{ccc} [Z] = \sum_i n_i [p_i] \in & H^2(X,\mathbb{Z}) \\ \updownarrow & \wr\| \\ \deg Z = \sum_i n_i \in & H_0(X,\mathbb{Z})\ . \end{array}$$

If $Z = (f)$, then

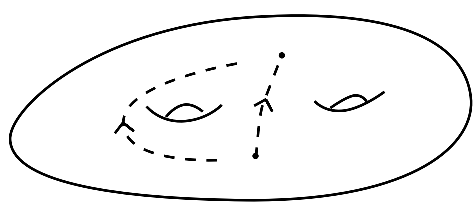
$$Z = \partial \Gamma$$

where $\Gamma = f^{-1}(\mathbb{R}^-)$, and since $\deg(f) = 0$ we have

$$0 \to \mathrm{CH}^1(X)_{\mathrm{hom}} \to \mathrm{CH}^1(X) \xrightarrow{\deg} H^2(X,\mathbb{Z}) \to 0$$
$$\wr\| \qquad$$
$$\mathbb{Z}\ . \qquad$$

If now $\omega \in H^0(\Omega^1_X) \cong \mathbb{C}^g$ denotes the space of holomorphic differentials on X, then

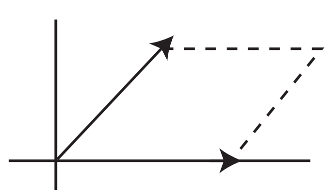
$$\int_\Gamma \omega \bmod \{\text{periods } \int_\delta \omega,\ \text{ where } \delta \in H_1(X,\mathbb{Z})\}$$



is well defined in the *Jacobian variety*

$$\begin{array}{c} J(X) = H^0(\Omega^1_X)^*/H_1(X,\mathbb{Z}) \\ \wr\| \\ \mathbb{C}^g/\Lambda \end{array}$$

where $\Lambda \cong \mathbb{Z}^{2g}$ is a lattice



$(g = 1)$

Abel proved

$$Z = (f) \Rightarrow \int_\Gamma \omega \equiv 0 \bmod \{\text{periods}\}$$

so that $\langle \mathrm{AJ}_X(Z), \omega\rangle =: \int_\Gamma \omega$ is well defined and we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}^1(X)_{\mathrm{hom}} & \longrightarrow & \mathrm{CH}^2(X) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \mathrm{AJ}_X} & & & & & & \\ & & J(X)\ . & & & & & & \end{array}$$

Abel and Jacobi proved that the mapping AJ_X is an isomorphism:

$$\boxed{\mathrm{AJ}_X : \mathrm{CH}^1(X)_{\mathrm{hom}} \xrightarrow{\sim} J(X)\ .}$$

Finally, the de Rham cohomology of X in degree one is a direct sum:

$$\begin{cases} H^1_{\mathrm{DR}}(X,\mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X) \\ H^{0,1}(X) = \overline{H^{1,0}(X)} \end{cases}$$

(*Hodge structure of weight one*) where

$$H^{1,0}(X) = H^0\left(\Omega^1_X\right)$$

so that

$$J(X) \cong H^{1,0}(X)^*/H_1(X,\mathbb{Z})\ .$$

This summarizes the relation between Hodge theory and algebraic cycles in the classical case. It may be expressed by:

There exists a filtration $F^k\mathrm{CH}^1(X)$ with $k = 0, 1$ and

$$\begin{cases} \mathrm{Gr}_0\mathrm{CH}^1(X) \cong H^2(X,\mathbb{Z}) \\ \mathrm{Gr}_1\mathrm{CH}^1(X) \cong J(X) = H^{1,0}(X)^*/H_1(X,\mathbb{Z})\ . \end{cases}$$

We will see in a moment that $H^2(X,\mathbb{Z}) = \mathrm{Hg}^1(X)$ and $J(X)$ have Hodge theoretic interpretations. In summary, *there is a filtration on the Chow group whose graded quotients have Hodge-theoretic interpretations.*

In general, a *Hodge structure of weight* r is given by a lattice $V_\mathbb{Z} \cong \mathbb{Z}^b$ together with a Hodge decomposition[2]

$$\begin{cases} V_\mathbb{C} = \bigoplus_{p+q=r} V^{p,q} \\ V^{q,p} = \bar{V}^{p,q}\ . \end{cases}$$

[2]The most important Hodge structures have the additional data of a *polarization*, which we shall not discuss here.

Hodge's theorem: *For X a smooth, projective variety, $H^r(X,\mathbb{Z})/torsion$ has a canonical Hodge structure of weight r.*

This Hodge structure on cohomology comes about as follows: By de Rham's theorem

$$\begin{aligned} H^r(X,\mathbb{C}) &\cong H^r_{\mathrm{DR}}(X) \\ &= \frac{\text{(closed forms)}}{\text{(exact forms)}} \,. \end{aligned}$$

Now X is a complex manifold with local holomorphic coordinates $z^1,\dots,z^n$

$$\begin{cases} A^r(X) = \underset{p+q=r}{\oplus} A^{p,q} \\ \bar{A}^{p,q} = A^{q,p} \end{cases}$$

and locally $\varphi \in A^r(X)$ is

$$\varphi = \sum_{I,J} \varphi_{I\bar{J}} dz^I \wedge d\bar{z}^J$$

where $I = (i_1,\dots,i_p)$, $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, etc. Hodge proved that for special types of compact, complex manifolds — those that have a *Kähler metric* — the decomposition on forms induces one on cohomology. Submanifolds of Kähler manifolds are Kähler using the induced metric. Since $\mathbb{P}^N$ is a Kähler manifold, smooth complex projective varieties are Kähler manifolds (of a special type). Thus their cohomology groups have canonical Hodge structures. For the study of algebraic cycles, their Hodge-theoretic invariants will lie in

— the group of *Hodge classes*

$$\boxed{\mathrm{Hg}^p(X) =: H^{p,p}(X) \cap H^{2p}(X,\mathbb{Z}) \,.}$$

— the *intermediate Jacobian*, which when $\dim X = n$ is given by

$$\boxed{J^p(X) =: F^{n-p+1}H^{2n-2p+1}(X)^*/H_{2n-2p+1}(X,\mathbb{Z})}$$

where

$$F^m V_{\mathbb{C}} = \underset{p \geqq m}{\oplus} V^{p,r-p} \,.$$

When $n = p = 1$

$$J^1(X) = H^{1,0}(X)^*/H_1(X,\mathbb{Z})$$

as above. We then have the basic diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{CH}^p(X)_{\mathrm{hom}} & \to & \mathrm{CH}^p(X) & \xrightarrow{[\ ]} & \mathrm{Hg}^p(X) & \xrightarrow{?} & 0 \\ & & \downarrow \mathrm{AJ}^p_X & & \downarrow [\ ]_{\mathcal{D}} & & \| & & \\ 0 & \to & J^p(X) & \to & H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) & \longrightarrow & \mathrm{Hg}^p(X) & \longrightarrow & 0 \end{array}$$

The map $[\ ]$ assigns to $Z \in \mathrm{CH}^p(X)$ its fundamental class

$$[Z] = \sum_i n_i [Z_i] \ .$$

If $[Z] = 0$, then $Z = \partial\Gamma$ and

$$\langle \mathrm{AJ}_X(Z), \omega \rangle = \int_\Gamma \omega \qquad \bmod \{\text{periods}\}$$

where

$$\left\{ \begin{array}{l} \dim_{\mathbb{R}} \Gamma = 2n - 2p + 1 \\ \omega \in F^{n-p+1} H^{2n-2p+1}(X, \mathbb{C}) \end{array} \right\} .$$

The middle mapping $[\ ]_{\mathcal{D}}$ is the *Deligne class*, which gives the proper way of formulating the above.

The above diagram summarizes the basic Hodge-theoretic invariants of an algebraic cycle. When $p = 1$ (i.e. divisors or codimension one cycles) it is exact on each end (Picard-Poincaré-Lefschetz — late 19th and early 20th centuries) and, using standard dualities

$$\begin{array}{rcl} \mathrm{AJ}^1_X : \mathrm{Pic}^\circ(X) & \xrightarrow{\sim} & H^1(\mathcal{O}_X)/H^1(X, \mathbb{Z}) \\ & & \| \\ & & J^1(X) \end{array}$$

is an isomorphism.

The (original form of) the *Hodge conjecture* (1954) is

$$[\ ] \textit{ is surjective} \ .$$

The *Beilinson-Bloch conjecture* ($\sim$ 1980) concerns the kernel of

$$\mathrm{AJ}^p_X : \mathrm{CH}^p(X)_{\mathrm{hom}} \to J^p(X) \ .$$

(here modulo torsion). For $p \geqq 2$ both conjectures involve new considerations — both geometric and especially arithmetic — that are not present for $p = 1$ when one is doing *complex* algebraic geometry.[3]

[3]They are, of course, present and interesting if one is working in *arithmetic* algebraic geometry.

Both conjectures ultimately involve an *existence* result; i.e. constructing something given Hodge theoretic and/or arithmetic assumptions.

IV. The Hodge conjecture[4]

(i) *Status*

- — True for $p = 1$ (Lefschetz, ~ 1920)
- — Beyond this and interesting but special examples, nothing general is known.
- — A few consequences of the HC have been verified and a few modifications/refinements have been found.

Specifically

— *It must be modified to be over* $\mathbb{Q}$ *for* $p \geqq 2$.

- Atiyah-Hirzerbruch example of a torsion, non-algebraic cohomology class (1960's)
- Kollar et al. example (1990's) of
  - — $\zeta \in H^4(X, \mathbb{Z}) \quad (\dim X = 3)$
  - — $\zeta = [Z]$ where $Z \in Z^2(X) \otimes \mathbb{Q}$ but where we cannot choose $Z \in Z^2(X)$ to be integral (must have denominators)

— *Kähler version is false.* For X a Kähler manifold and $\zeta \in \mathrm{Hg}^1(X)$, there exists a holomorphic line bundle $L \to X$ with $c_1(L) = \zeta$. Voisin ($\sim$ 2002) constructs a (non-algebraic) complex torus $X = \mathbb{C}^4/\Lambda$ and $\zeta \in \mathrm{Hg}^2(X)$ where X has no coherent sheaves $\mathcal{F}$ with $c_i(\mathcal{F}) \neq 0$ for $i \neq 0, 8$ (X is complex analytically "barren").

Conclusion: Any general construction of cycles must be modulo torsion and must make use of having an algebraic variety.

(ii) *An aritmetic implication*

Given a field k of characteristic zero and a smooth variety X defined over k (think of polynomial equations with coefficients in k), the *algebraic de Rham cohomology*

$$\mathbb{H}^r_{\mathrm{DR}}\left(\Omega^\bullet_{X(k)/k}\right)$$

is defined (for X affine think of the usual de Rham cohomology constructed from polynomial differential forms with coefficients in k and restricted to X). For any embedding

$$\sigma : k \hookrightarrow \mathbb{C}$$

[4]We shall not discuss the generalized Hodge conjecture (GHC).

there is a corresponding complex algebraic variety $X_\sigma(\mathbb{C})$, and for X smooth and projective Grothendieck proved that

$$\begin{cases} \mathbb{H}^r_{\mathrm{DR}}\left(\Omega^\bullet_{X(k)/k}\right) \otimes_k \mathbb{C} \cong H^r_{\mathrm{DR}}(X_\sigma(\mathbb{C}), \mathbb{C}) \\ \mathbb{H}^r_{\mathrm{DR}}\left(\Omega^{\geqq p}_{X(k)/k}\right) \otimes_k \mathbb{C} = F^p H^r(X_\sigma(\mathbb{C}), \mathbb{C})\,. \end{cases}$$

Given $\zeta \in \mathbb{H}^{2p}_{\mathrm{DR}}\left(\Omega^{\geqq p}_{X(k)/k}\right)$ we denote by

$$\zeta_\sigma \in F^p H^{2p}(X_\sigma(\mathbb{C}), \mathbb{C})$$

the corresponding class. On the face of it the rationality condition

$$\zeta_\sigma \in H^{2p}(X_\sigma(\mathbb{C}), \mathbb{Q})$$

or equivalently

$$\zeta_\sigma \in \mathrm{Hg}^p(X_\sigma(\mathbb{C})) \tag{$*$}$$

depends on the embedding σ. However

$$\boxed{\mathrm{HC} \implies (*) \quad \text{is independent of} \quad \sigma\,.}$$

Remarks: In concrete terms, on an affine open set $\mathcal{U}$ in X given by

$$f_\lambda(x_1, \dots, x_N) = 0$$

where the coefficients are in k, ζ is represented by a differential form

$$\varphi_\zeta = \sum_{\boldsymbol{\alpha}} g_{\boldsymbol{\alpha}}(x) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_m}$$

where $g_{\boldsymbol{\alpha}}(x) \in k[x_1, \dots, x_N]$. Setting

$$f_{\lambda,\sigma}(x) = \sigma \cdot (\text{coefficients of } f_\lambda(x))$$

the equations $f_{\lambda,\sigma}(x) = 0$ define a complex variety $\mathcal{U}_\sigma \subset \mathbb{C}^N$. Then $(*)$ is equivalent to the form $\sigma(\varphi_\zeta)$ having rational periods

$$\sum_{\boldsymbol{\alpha}} \int_\Gamma g_{\boldsymbol{\alpha},\sigma}(x) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_m} \in \mathbb{Q}$$

for $\Gamma \in H_m(\mathcal{U}_\sigma, \mathbb{Q})$. As for the implication $\implies$ we have:

$$Z \in Z^p(X(k)) \implies \left\{ \begin{array}{c} \text{the fundamental class} \\ [Z]_k \in \mathbb{H}^{2p}\left(\Omega^{\geqq p}_{X(k)/k}\right) \text{ is defined.} \end{array} \right\}$$

If $Z_\sigma \in Z^p(X_\sigma(\mathbb{C}))$ is the corresponding cycle using the embedding $\sigma : k \hookrightarrow \mathbb{C}$, then

$$[Z]_k = [Z_\sigma]$$

under the above isomorphism. Thus *all* ζ_σ *are Hodge classes.* If some $\zeta_\sigma = [Z']$ where $Z' \in Z^p(X_\sigma(\mathbb{C}))$, then by an algebraic equivalence we may move Z' to a cycle Z defined $/k$.

We say that

$$\zeta \in \mathbb{H}^{2p}\left(\Omega^{\geqq p}_{X(k)/k}\right)$$

is a *Hodge class* if $(*)$ holds for one σ, and that σ is an *absolute Hodge class* if $(*)$ holds for any embedding σ. Then

$$\boxed{(**) \qquad \text{HC} \implies \text{Hodge classes are absolute.}}$$

Deligne has proved the RHS of $(**)$ for abelian varieties ($\sim$ 1980), but it is not known in general.

If ζ is an absolute Hodge class, then consideration of all the embeddings leads to a family (smooth and projective total space and base space)

$$\mathcal{X} \to S$$

- defined over a number field
- having X as a fibre
- having $\mathcal{Z} \in \mathrm{Hg}^p(\mathcal{X})$ with

$$\mathcal{Z}\mid_X = \zeta\ .$$

Maillot and Soulé have asked the question: *Can the HC be reduced to the case of varities defined over* $\bar{\mathbb{Q}}$? This is related to the *field of definition of Noether-Lefschetz loci.* Without assuming that Hodge classes are absolute, Claire Voisin ($\sim$ 2006) has proved that in some interesting cases these loci are defined $/\bar{\mathbb{Q}}$.

(iii) *A potential topological reformulation*

Given (X^{2n}, L, ζ) where

- $\zeta \in \mathrm{Hg}^n(X)_{\mathrm{prim}}$
- $L \to X$ gives $X \hookrightarrow \mathbb{P}_L$

there exists an *admissable normal function*

$$\check{\mathbb{P}}_L \xrightarrow{\nu_\zeta} \tilde{J}_\Sigma$$

with *singular locus*

$$\operatorname{sing}\nu_\zeta \subset \check{\mathbb{P}}_L$$

such that

$$\boxed{\text{HC} \iff \operatorname{sing}\nu_\zeta \neq \emptyset \ \text{ for } \ L \gg 0\ .}$$

The proof in this strong form is due to deCataldo-Migliorini and Brosnan-Fang-Nie-Pearlstein (2007) — it uses the decomposition theorem (Beilinson-Bernstein-Deligne-Gabber) and results of M. Saito. The space $\tilde{J}_\Sigma$ is a *universal Néron model* for intermediate Jacobians. It's construction is "work in progress" by Andrew Young and others; it is based on the major work by Kato-Usui (2006) constructing partial compactifications of quotients of period domains by (suitable) arithmetic groups. There is hoped to be a subvariety

$$\Xi_\Sigma \subset \tilde{J}_\Sigma$$

such that

$$\operatorname{sing} \nu_\zeta = \nu_\zeta^{-1}(\Xi_\Sigma)\ .$$

Thus

$$\boxed{\nu_\zeta^*\left([\Xi_\Sigma]\right) \neq 0 \implies \operatorname{sing} \nu_\zeta \neq \emptyset\ .}$$

In a number of cases the converse has been proved, indicating that it is at least plausible to have such a topological reformulation of the HC.

V. The Beilinson-Bloch conjectures

In the late 1960's it was found that the wonderfully harmonious story for divisors breaks down in higher codimensions. For the fundamental class mapping, the (modified) HC provides a potential answer about the image. However, for AJ_X the situation was found to be radically different from the codimension one case.

Example: Let X be an algebraic surface with $H^0(\Omega_X^2) \neq 0$ (eg., $X \subset \mathbb{P}^3$ a smooth surface of degree $\geqq 4$). For 0-cycles $Z = \sum_i n_i p_i$ one has

—
$$\begin{array}{rcccc} [Z] & = & \sum_i n_i[p_i] & \in & H^4(X,\mathbb{Z}) \\ & & \updownarrow & & \updownarrow \\ \deg Z & = & \sum_i n_i & \in & H_0(X,\mathbb{Z})\ . \end{array}$$

— If $[Z] = 0$ then there is defined

$$\mathrm{AJ}_X(Z) \in H^0(\Omega_X^1)^*/H_1(X,\mathbb{Z})\ .$$

(*Albanese map* — same as for curves), and Mumford showed

$$\boxed{\dim\{\ker \mathrm{AJ}_X\} = \infty\ .}$$

In addition to $X \subset \mathbb{P}^3$ of degree $\geqq 4$, $(\mathbb{P}^2, T)$ gives a relative example with 2-form $\frac{dx}{x} \wedge \frac{dy}{y}$.

Example: Let $X \subset \mathbb{P}^4$ be a generic quintic threefold. Using the lines on it, Clemens and Voisin showed

$$\boxed{\mathrm{Im}\{\mathrm{AJ}_X\}\ \textit{is a countable, non-finitely generated group.}}$$

Thus the converse to Abel's theorem and the Jacobi inversion theorem are very false in higher codimension. Some conjectural order was brought to at least $\ker\{\mathrm{AJ}_X\}$ by the conjectures of Beilinson and Bloch (late 1970's, early 1980's). These conjectures relate to the structure of $\mathrm{CH}^p(X)$ and may be informally stated as: *Over $\mathbb{Q}$ there exists a filtration*

$$F^k\mathrm{CH}^p(X)$$

such that

$$\begin{cases} F^0\mathrm{CH}^p(X) = \mathrm{CH}^p(X) \\ \quad\cup \\ F^1\mathrm{CH}^p(X) = \mathrm{CH}^p(X)_{\mathrm{hom}} \\ \quad\cup \\ F^2\mathrm{CH}^p(X) = \mathrm{Ker}\,\mathrm{AJ}^p_X \\ \quad\vdots \\ F^{p+1}\mathrm{CH}^p(X) = 0\ . \end{cases}$$

Moreover, if X is defined over a number field k, then

$$\boxed{F^2\mathrm{CH}^p(X(\bar{k})) = 0}$$

(as noted above, $K_2(\bar{\mathbb{Q}}) = 0$ is a theorem). There is a conjectural formula for the graded quotients in terms of the category of mixed motives. In particular, over $\mathbb{C}$ and assuming the GHC, the graded quotients should have Hodge-theoretic interpretations. There have been a number of proposals (Murre, Jannsen, Nori,...) for the definition of the $F^m\mathrm{CH}^p(X)$.

To at least this complex algebraic geometer the last, boxed statement in Bloch-Beilinson seemed to come out of the blue — what could the field of definition have to do with rational equivalence in *higher* codimension? It turns out there were at least two hints:

- the formal tangent space to $F^2\mathrm{CH}^p(X(\bar{k}))$ is zero for $p \geqq 2$. ($T_f K_2(F) \cong \Omega^1_{F/\mathbb{Q}}$ (van der Kallen) and $\Omega^1_{\bar{\mathbb{Q}}/\mathbb{Q}} = 0$.)
- the natural generalization of the Birch and Swinnerton-Dyer conjecture concerning the order of vanishing of an L-function suggests the boxed statement above.

About three years ago it was found that:

> *The GHC plus the boxed part of conjecture imply the remaining B^2 conjectures. Moreover, the graded pieces have Hodge-theoretic interpretations .*

The kernel of the idea occurs already for 0-cycles on a surface: Given $Z = \sum_i n_i p_i \in Z^2(X)$ with

$$\text{0-forms:} \qquad \deg Z = \int_Z 1 = 0, \quad 1 \in H^0(\Omega_X^0)$$

$$\text{1-forms:} \qquad \mathrm{AJ}_X(Z)(\varphi) = \int_\Gamma \varphi \equiv 0 \quad \text{(mod periods)}$$

$$\text{where } \partial\Gamma = Z,\ \varphi \in H^0(\Omega_X^1)$$

can we construct a real 2-chain Λ such that

$$\text{2-forms:} \qquad \boxed{\int_\Lambda \omega \equiv 0 \ \text{ for } \ \omega \in H^2(\Omega_X^2) \implies Z \sim_{\text{rat}} 0?}$$

For simplicity we assume that X is defined over $\mathbb{Q}$ by equations

$$(*) \qquad f_\lambda(x_1, \dots, x_N) = 0$$

with coefficients in $\mathbb{Q}$. Then

$$Z = \sum_i n_i p_i$$

where

$$p_i = (x_{i,1}, \dots, x_{i,N})$$

has coefficients in a finitely generated extension k of $\mathbb{Q}$. Applying $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ to (X, Z) leaves X fixed and “spreads” Z to an algebraic family of 0-cycles. The component of this family containing Z may be assumed to be a smooth, projective variety S with

$$\dim S = \operatorname{tr} \deg_{\mathbb{Q}}(k)$$

and where S is defined over a number field, which for simplicity we take to be $\mathbb{Q}$. The *spread* of Z then gives

$$\mathcal{Z} \in Z^2(X \times S(\mathbb{Q}))$$

where we may think of

$$\mathcal{Z} = \{Z_s\}_{s \in S}$$

as a family of 0-cycles parametrized by S. The idea now is to write

$$Z_s = \partial\Gamma_s$$

and consider the integrals

$$(**) \qquad \int_\Lambda \omega, \qquad \omega \in H^0\left(\Omega^2_{X(\mathbb{Q})}\right)$$

where Λ is the 2-chain traced out by the 1-chains Γ_s over a closed loop $\lambda \in H_1(S, \mathbb{Z})$. There are *ambiguities* in this construction:

- choice of Z in $[Z] \in \mathrm{CH}^2(X)$
- choice of S in its birational equivalence class
- choices of Γ_s and λ.

Modding out these ambiguities one sees that one obtains

$$[\mathcal{Z}] \in F^1\mathrm{CH}^2(X \times S)/\text{ambiguities}$$

and that

$$\mathrm{AJ}_{X\times S}(\mathcal{Z}) \equiv 0 \iff \begin{cases} \mathrm{AJ}_X(Z) = 0 \\ (**) \equiv 0 \bmod \text{periods.} \end{cases}$$

By the boxed statement in Bloch-Beilinson one sees that the RHS gives the condition (modulo torsion)

$$\mathcal{Z} \sim_{\text{rat}} 0 \implies Z \sim_{\text{rat}} 0\ .$$

Remark: If one does the analogous construction for divisors on curves, one finds that there is *no new information* modulo the ambiguities.

One may informally express the idea behind the above as

> *In codimension $\geqq 2$, the invariantly defined part of the Hodge theory of the field of definition of the cycle must be used.*

In this way, one see again that arithmetic considerations necessarily enter into the study of cycles in *complex* geometry in higher codimension, which has been the main theme of this talk.

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