COMPLETE MAPS AND DIFFERENTIABLE COVERINGS

Joseph A. Wolf and Phillip A. Griffiths

INTRODUCTION

We define the notion of completeness for maps of Riemannian manifolds and prove that a complete differentiable map $f: M \to N$ of connected Riemannian n-manifolds is a covering provided that the Riemannian metric on M is complete. To apply this result to a differentiable map $g: X \to Y$ of connected differentiable n-manifolds, one uses a result of Nomizu and Ozeki [3] (stated here as Theorem 2) which ensures that X has a complete Riemannian metric if it is second countable. This application can be viewed as the extension to infinite coverings of a theorem of Nijenhuis and Richardson [2] on proper maps with nonvanishing Jacobians. Another application of our result consists of giving conditions on f which ensure that the Riemannian metric on N is complete.

STATEMENT OF RESULTS

Definition. Let $f: M \to N$ be a differentiable map of Riemannian manifolds. The map f is *complete* if there exists a continuous function λ on N with positive values which bounds shrinking of tangent vectors by f in the following sense: If $x \in N$ and X is tangent to M at a point of $f^{-1}(x)$, then $\|f_* X\| \geq \lambda(x) \|X\|$. The map f is *uniformly complete* if λ can be found with the property above and such that λ is bounded above zero on every bounded subset of N.

Note that λ must be defined on all of N, and thus cannot become infinite on the boundary of $f(M)$ in N.

Observe that a complete map has nonvanishing Jacobian, and that a complete map is uniformly complete if the image manifold is complete. Recall that a map is *proper* if the inverse image of every compact set is compact. If $h: U \to V$ is a proper differentiable map with nonvanishing Jacobian, then it follows easily from definitions that h is complete in every choice of Riemannian metrics for U and V. Thus the following theorems extend [2] in the case of nonvanishing Jacobian:

THEOREM 1. *Let* $f: M \to N$ *be a differentiable map of connected Riemannian* n-*manifolds, where both* f *and* M *are complete. Then* (1) $f(M) = N$; (2) $f: M \to N$ *is a differentiable covering; and* (3) N *is complete if and only if* f *is uniformly complete.*

THEOREM 2 (Nomizu and Ozeki [3]). *Every Riemannian manifold is conformally diffeomorphic to a complete Riemannian manifold. In particular, every paracompact differentiable manifold admits a complete Riemannian metric.*

As mentioned above, Theorems 1 and 2 yield this corollary:

COROLLARY 1. *Let* $f: X \to Y$ *be a proper differentiable map of connected paracompact* n-*manifolds. If the Jacobian of* f *never vanishes, then* $f(X) = Y$ *and* $f: X \to Y$ *is a differentiable covering.*

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We next state a useful special case of Theorem 1.

COROLLARY 2. *Let* $f: M \to N$ *be a local isometry of connected Riemannian* n-*manifolds where* M *is complete. Then* $f(M) = N$, $f: M \to N$ *is a Riemannian covering, and* N *is complete.*

Another useful consequence of Theorem 1 is as follows.

COROLLARY 3. *Let* $f: M \to N$ *be a differentiable map of connected Riemannian* n-*manifolds, where both* f *and* M *are complete. If* $\pi_1(N)$ *is finite, then* f *is proper. If* f *is proper, then* N *is complete.*

The first statement follows because the covering f must be finite, and the second statement is seen by observing that f is uniformly complete if it is proper, for the Jacobian of f never vanishes and the inverse image of a bounded set lies in a compact set.

A LEMMA ON OPEN SUBMANIFOLDS

LEMMA. *Let* U *be a proper open subset of a Riemannian manifold* N. *Then there exists a geodesic arc* $\{a_t\}$ *in* N, $0 \leq t \leq 1$, *such that* $a_t \in U$ *for* $0 \leq t < 1$ *and* a_1 *is a boundary point of* U. *In particular, there exists a Cauchy sequence* $\{y_n\}$ *in* U *that lies on a geodesic arc and converges to a boundary point of* U.

Proof. Choose z in the boundary ∂U. As U is open and cannot be disjoint from a normal coordinate neighborhood of z, there is a geodesic arc $\{z_t\}$, $0 \leq t \leq 1$, such that $z = z_0$ and $z_1 \in U$. Let $h(t)$ be the distance from z_t to ∂U; h is continuous, $h(1) > 0$, $h(0) = 0$, and $h(t) = 0$ is equivalent to $z_t \in \partial U$. The zeros of h form a closed subset of the segment $[0, 1]$. Let u be the least upper bound. Then $u < 1$, $z_u \in \partial U$, and $z_t \in U$ for $u < t \leq 1$. Now define $a_s = z_t$, where $s = (t - 1)/(u - 1)$ and $0 \leq t \leq 1$.

For the last remark, assume t is arclength in $\{a_t\}$, and define $y_n = a_{t(n)}$, where $t(n) = 1 - 1/n$. Q.E.D.

PROOF OF THEOREM 1

To prove $f(M) = N$, it suffices to prove that $f(M)$ is closed in N, because $f(M)$ is a nontrivial open subset of N since the Jacobian of f never vanishes, and because N is connected. If $f(M)$ is not closed in N, then since $f(M)$ is open in N, there exists a sequence $\{y_n\}$ in $f(M)$, with $\{y_n\} \to y \in N - f(M)$, of the type described in the Lemma above. We can thus choose $r > 0$ with the properties (a) $f(M) \cap B_r(y)$ has compact closure in N, where $B_r(y)$ is the open ball of radius r about y, (b) if $y_m, y_n \in B_r(y)$, then there is a unique shortest geodesic arc $\alpha_{m,n}(t)$, $0 \leq t \leq 1$, in N from y_m to y_n, and $\alpha_{m,n}(t) \in f(M) \cap B_r(y)$ for $0 \leq t \leq 1$. Without loss of generality we now assume that $y_n \in B_r(y)$ for $n \geq 1$, and we define $\alpha_n = \alpha_{1,n}$. Choose $x_1 \in f^{-1}(y_1)$. By pulling back the unit tangent field of α_n, we lift α_n to an arc $\beta_n(t)$ in M from x_1 to some point $x_n \in f^{-1}(y_n)$.

Let L be the infimum of the positive numbers $\lambda(z)$ as z ranges over the compact set which is the closure of $B_r(y) \cap f(M)$ and where λ is a continuous function on N whose existence is guaranteed by the definition of completeness of f. Let d_M and d_N denote the distance functions on M and N. Clearly, $d_N(y_m, y_n) \leq 2r$ for $m, n \geq 1$. Thus

$$2r \geq \int_0^1 \|\alpha_n'(t)\| \, dt = \int_0^1 \|f_* \beta_n'(t)\| \, dt \geq L \int_0^1 \|\beta_n'(t)\| \, dt \geq L d_M(x_1, x_n),$$

so $\{x_n\}$ lies in the ball of radius $2r/L$ about x_1 in M. As M is complete, it follows that $\{x_n\}$ has an accumulation point x in M. Now passing to a subsequence, we see that $\{x_n\} \to x$, and continuity of f implies $\{y_n\} \to f(x) \in f(M)$. This proves $f(M) = N$.

Any continuous arc in N can be lifted to M. For if the arc is differentiable, we lift its tangent vectorfield and then integrate. If the arc is sectionally smooth, we do the same thing by pieces. In general we approximate by a sectionally smooth arc. Now f is a local homeomorphism of connected, locally arcwise connected, locally simply connected spaces, and every arc can be lifted. A theorem of F. Browder [1] shows that $f: M \to N$ is a covering. This proves that $f: M \to N$ is a differentiable covering.

Let f be uniformly complete; we will prove that N is complete. Let $\{y_n\}$ be a Cauchy sequence in N. Then there is a number $r > 0$ and smooth arcs $\alpha_n(t)$, $0 \leq t \leq 1$, from y_1 to y_n, such that α_n has length at most r. The arcs α_n lie in the ball B of radius 2r about y_1, and uniform completeness of f implies a number $L > 0$ exists such that $\|f_* X\| \geq L \|X\|$ whenever X is a tangentvector to M at a point of $f^{-1}(B)$. Choose $x_1 \in f^{-1}(y_1)$ and lift α_n to an arc $\beta_n(t)$ in M from x_1 to some $x_n \in f^{-1}(y_n)$. As before, we calculate:

$$r \geq \int_0^1 \|\alpha_n'(t)\| \, dt = \int_0^1 \|f_* \beta_n'(t)\| \, dt \geq L \int_0^1 \|\beta_n'(t)\| \, dt \geq L d_M(x_1, x_n).$$

Now $\{x_n\}$ lies in a bounded subset of M. Since M is complete, we pass to a subsequence and conclude $\{x_n\} \to x \in M$. It follows that $\{y_n\} \to f(x)$. Thus N is complete.

Theorem 1 is now proved.

Added in proof. Corollary 2 was known to W. Ambrose [*Parallel translation of Riemannian curvature*, Ann. of Math. (2) 64 (1956), 337-363]; R. S. Palais later gave a proof which N. Hicks [*A theorem on affine connections*, Illinois J. Math. 3 (1959), 242-254] extended to the affine case.

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The University of California at Berkeley